On the distribution of the roots of polynomial $z^k - z^{k-1} - \cdots - z - 1$

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Abstract. We consider the polynomial $f_k(z) = z^k - z^{k-1} - \cdots - z - 1$ for $k \geq 2$ which arises as the characteristic polynomial of the k-generalized Fibonacci sequence. In this short paper, we give estimates for the absolute values of the roots of $f_k(z)$ which lie inside the unit disk.

Keywords: polynomial root distribution

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1. Introduction

For an integer $k \geq 2$, the polynomial $f_k(z) = z^k - z^{k-1} - \cdots - z - 1$ arises as the characteristic polynomial of the linear recurrence sequence $(u_n)_{n \geq 0}$ of recurrence $u_{n+k} = u_{n+k-1} + \cdots + u_n$. The classic study of the linear recurrence sequences, see [2], is based on knowledge of the roots of their characteristic polynomial. While studying the roots of $f_k(z)$, it is common to work with the polynomial:

$$g_k(z) = (z-1)f_k(z) = z^{k+1} - 2z^k + 1. \tag{1}$$

Except for the extra root at $z = 1$, $g_k(z)$ has the same roots as $f_k(x)$.

By Descartes' rule of signs, the polynomial $f_k(z)$ has exactly one positive real root, say $z = \alpha_1$. Since $f_k(1) = 1 - k$ and $f_k(2) = 1$, it follows that $\alpha_1 \in (1, 2)$. In fact, it is known that $2(1 - 2^{-k}) < \alpha_1 < 2$, see [4, Lemma 2.3] or [9, Lemma 3.6]. Moreover, given that $\alpha_1^{k+1} - 2\alpha_1^k + 1 = 0$, we obtain $\alpha_1 = 2 - \alpha_1^{-k} < 2 - 2^{-k}$. So, $2(1 - 2^{-k}) < \alpha_1 < 2(1 - 2^{-(k+1)})$ for all $k \geq 2$. Thus α_1 approaches 2 as k tends to infinity. E. P. Miles in [6] showed that the roots of $f_k(z)$ are distinct and the remaining $k-1$ roots of $f_k(z)$ different from α_1 lie inside the unit disk. He showed this by reducing the equation $f_k(z) = 0$ to a form where Rouché's theorem could be applied. This fact was reproved by M. D. Miller in [7] by an elementary argument. In particular, α_1 is a Pisot number and $f_k(z)$

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is an irreducible polynomial over $\mathbb{Q}[z]$. Properties of the roots of the similar but more general polynomial $f_{k,c}(z) = z^k - z^{k-1} - \cdots - z - c$ for positive values of the parameter c have been proved in [5].

Recently, we gave the following estimate on the ratio of roots of $f_k(z)$, see Lemma 2.2 in [3].

Theorem 1. *If α and β are roots of $f_k(z)$ with $|\alpha| > |\beta|$, then*

$$\frac{|\alpha|}{|\beta|} > 1 + 8^{-k^4}.$$

This result was used in [3] as one of the main ingredients in the study of the zero-multiplicity of a particular linear recurrence sequence with characteristic polynomial $f_k(z)$.

Here, we give upper and lower bounds on the absolute values of the roots of $f_k(z)$ which lie inside the unit disk.

Theorem 2. *Let z_0 be a root of $f_k(z)$ with $\varrho = |z_0| < 1$. Then*

$$1 - \frac{\log 3}{k} < \varrho < 1 - \frac{1}{2^8 k^3}.$$

2. Proof of Theorem 2

We put $z_0 = \varrho e^{i\theta}$ with $0 < \varrho < 1$ and $\theta \in (0, 2\pi)$. By (1), we obtain that $1 \le \varrho^k(2 + \varrho) < 3\varrho^k$, or

$$\varrho > 3^{-1/k} = e^{-\log 3/k} \ge 1 - \frac{\log 3}{k}.$$

Now, again by (1), we get

$$z_0^k(2 - z_0) = 1 \qquad \text{and} \qquad \overline{z_0}^k(2 - \overline{z_0}) = 1.$$

After multiplying the above identities, we obtain $\varrho^{2k}(2 - z_0)(2 - \overline{z_0}) = 1$. Then

$$\varrho^{-2k} = (2 - z_0)(2 - \overline{z_0}) = 4 - 4\mathrm{Re}(z_0) + \varrho^2,$$

which leads to

$$\varrho^2(\varrho^{-2k-2} - 1) = 4(1 - \mathrm{Re}(z_0)). \tag{2}$$

We assume that

$$1 - \varrho < \frac{c_0}{k^3} \qquad \text{with} \quad c_0 = 2^{-8}. \tag{3}$$

Then, $1 < \varrho^{-1} \le (1 - (c_0/k^3))^{-1} < 1 + (2c_0/k^3)$ for all $k \ge 2$. Hence,

$$\varrho^{-(2k+2)} < \left(1 + \frac{2c_0}{k^3}\right)^{2k+2} < \exp\left(\frac{2c_0(2k+2)}{k^3}\right) < 1 + \frac{4c_0(2k+2)}{k^3}.$$

In the above inequalities, we have used that $\mathrm{e}^x \ge 1+x$ for all real x and $\mathrm{e}^x \le 1+2x$ for all real x such that $|x| < 1/2$, as well as the fact that $2c_0(2k+2)/k^3 < 1/2$ for all $k \ge 2$. So, we obtain $\varrho^2\left(\varrho^{-2k-2} - 1\right) < 8\varrho^2 c_0(k+1)/k^3$ and by (2), we get

$$1 - \mathrm{Re}(z_0) < \frac{2\varrho^2 c_0(k+1)}{k^3}. \tag{4}$$

On the other hand,

$$\mathrm{Im}^2(z_0) = \varrho^2 - \mathrm{Re}^2(z_0) < \varrho^2 - \left(1 - \frac{2\varrho^2 c_0(k+1)}{k^3}\right)^2 < \frac{4\varrho^2 c_0(k+1)}{k^3}. \tag{5}$$

We now write $z_0 = 1 + z_1$. Hence, $z_1 = (\mathrm{Re}(z_0) - 1) + \mathrm{i}\mathrm{Im}(z_0)$. Thus, by (4) and (5), we get

$$\begin{aligned}
|z_1| &= ((\mathrm{Re}(z_0) - 1)^2 + \mathrm{Im}^2(z_0))^{1/2} \\
&< \sqrt{2}\Big(\max\Big\{\Big(\frac{2\varrho^2 c_0(k+1)}{k^3}\Big)^2, \frac{4\varrho^2 c_0(k+1)}{k^3}\Big\}\Big)^{1/2} \\
&< 2\sqrt{2}\varrho c_0^{1/2}\Big(\frac{k+1}{k^3}\Big)^{1/2} \\
&< \frac{4\varrho c_0^{1/2}}{k}.
\end{aligned}$$

Hence, we have proved that

$$z_0 = 1 + z_1, \qquad \text{where } |z_1| < \frac{4\varrho c_0^{1/2}}{k}. \tag{6}$$

We now analyze the polynomial function of complex value $(1+z)^\lambda$ for $\lambda = 1, 2, \ldots, k$ in $z = z_1$, according to the binomial theorem. We put $\eta := ((1 + z_1)^\lambda - 1 - \lambda z_1)/z_1^2$, so

$$\begin{aligned}
|\eta| &= \left|\sum_{j=2}^{\lambda}\binom{\lambda}{j} z_1^{j-2}\right| \le \sum_{j=2}^{\lambda}\binom{\lambda}{j}|z_1|^{j-2} \\
&< \lambda^2\sum_{j=2}^{\lambda}\binom{\lambda}{j-2}|z_1|^{j-2} = \lambda^2(1 + |z_1|)^{\lambda-2} \\
&\le \lambda^2(1 + 2(\lambda - 2)|z_1|).
\end{aligned}$$

Here we have used that $\binom{\lambda}{j} < \lambda^2\binom{\lambda}{j-2}$ for $j = 2, \dots, k$, in addition to

$$(1 + |z_1|)^{\lambda-2} < \mathrm{e}^{(\lambda-2)|z_1|} < 1 + 2(\lambda - 2)|z_1|,$$

which holds because $(\lambda - 2)|z_1| < k|z_1| < 1/2$ by (6). Since $\varrho < 1$, it then follows that

$$|\eta| < \lambda^2(1 + 8c_0^{1/2}) \qquad \text{for} \ \ 1 \le \lambda \le k.$$

Hence, for each $\lambda = 1, 2, \dots, k$

$$z_0^{\lambda} = (1 + z_1)^{\lambda} = 1 + \lambda z_1 + \delta_{\lambda}, \qquad \text{where } |\delta_{\lambda}| < 16c_0(1 + 8c_0^{1/2}). \tag{7}$$

Thus, we obtain

$$\begin{aligned} 0 = f_k(z_0) &= z_0^k - z_0^{k-1} - \cdots - z_0^2 - z_0 - 1 \\ &= (1 + kz_1 + \delta_k) - (1 + (k-1)z_1 + \delta_{k-1}) - \cdots - (1 + z_1 + \delta_1) - 1 \\ &= 1 - k - \frac{k(k-3)}{2} z_1 + \delta_k - \sum_{\lambda=1}^{k-1} \delta_{\lambda}. \end{aligned}$$

Now, by (2) we have that

$$k - 1 < \frac{k(k-3)}{2}|z_1| + \sum_{\lambda=1}^{k} |\delta_{\lambda}| \le (2c_0^{1/2} + 16c_0(1 + 8c_0^{1/2}))k < \frac{k}{3},$$

which is not possible. Thus, our assumption (3) is false. Hence, $\varrho \le 1 - c_0/k^3$. The fact that the inequality is strict follows because ϱ is an algebraic integer, while $1 - 1/(2^8k^3)$ is not. This completes the proof of Theorem 2. □

3. An open problem

In 1950, P. Erdős and P. Turán in [1] investigated the angular distribution of zeros of complex polynomials $f(z) \in \mathbb{C}[z]$. Let I be an arc on the unit circle and let $N(I, f(z))$ be the number of zeros α's with $\alpha/|\alpha| = \mathrm{e}^{\mathrm{i}\theta}$ lying on this arc. A natural way to estimate the equidistribution of the roots of the polynomial $f(z)$ is the discrepancy, defined as

$$D(f(z)) = \max_I \left| N(I, f(z)) - \frac{|I|}{2\pi} N \right|.$$

This measures the maximum difference between the actual count of the number of arguments of roots found in a given arc, and the number that would be expected if all the angles were uniformly distributed.

The version of the Erdős–Turán inequality was recently given by K. Soundararajan in [8]. Assume $f(z)$ is monic and put

$$h(f(z)) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta, \qquad \text{where } \log^+(x) = \max\{\log x, 0\}.$$

Theorem 3. *For any monic polynomial $f(z)$ of degree $k > 1$,*

$$D(f(z)) \le \frac{8}{\pi}\sqrt{kh(f(z))}.$$

Remark. For the case of $f(z) = g_k(z) = z^{k+1} - 2z^k + 1$ it follows that $|f(e^{i\theta})| \le 4$, so $h(g_k(z)) \le \log 4$. In particular, if we take the largest subinterval I of $[0, 2\pi)$ free of any $\theta = \arg(\alpha/|\alpha|)$ for root α of $g_k(z)$ then $|I| < 16(\log 4)^{1/2}k^{-1/2}$. Numerically, since $16(\log 4)^{1/2} = 18.838\ldots$, it follows that by taking short intervals I_h of radius $18.9k^{1/2}$ around each angle $2\pi h/k$, $h = 1, 2, \ldots, k-1$, one will always find at least one $\theta \in I_h$.

Assume that

$$\alpha_1, \quad \widetilde{\alpha}_1, \quad \alpha_2, \quad \overline{\alpha_2}, \quad \ldots \quad \alpha_l, \quad \overline{\alpha_l}, \qquad \text{with } l = \left\lfloor \frac{k-1}{2} \right\rfloor,$$

are the roots of $f_k(z)$, where $\widetilde{\alpha}_1 \in (-1, 0)$ appears only when k is even. For $1 \le j \le l$ we write $\alpha_j = \varrho_j e^{i\theta_j}$, with $\theta_1 = 0$ and $\theta_j \in (0, 2\pi)$. As a consequence of the above remark for each $h \in \{0, \ldots, k-1\}$ there is $j \in \{1, \ldots, k\}$ such that

$$\left|\theta_j - \frac{2\pi h}{k}\right| < \frac{8(\log 4)^{1/2}}{k^{1/2}}.$$

Open problem. Prove that inequality

$$\left|\theta_j - \frac{2j\pi}{k}\right| < \frac{1}{k}$$

holds for all $k \ge 3$.

This problem is motivated by the computational verification made for each $k \in [3, 1000]$.

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