Non-normality points and nice spaces

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Abstract. J. Terasawa in "$\beta X - \{p\}$ are non-normal for non-discrete spaces X" (2007) and the author in "On non-normality points and metrizable crowded spaces" (2007), independently showed for any metrizable crowded space X that each point p of its Čech–Stone remainder X^* is a non-normality point of βX. We introduce a new class of spaces, named nice spaces, which contains both of Sorgenfrey line and every metrizable crowded space. We obtain the result above for every nice space.

Keywords: non-normality point; butterfly-point; nice family; nice space; metrizable crowded space; Sorgenfrey line

Classification: 54D15, 54D35, 54D40, 54D80, 54E35, 54G20

1. Introduction

A point p of a normal space X is called a non-normality point, if $X \setminus \{p\}$ is not normal. In a similar way, p is called a butterfly-point (b-point) of X, if $\{p\} = [F] \cap [G]$ for some subsets F and G of $X \setminus \{p\}$, see [7]. We modify this notion for Čech–Stone compactification βX as follows: a point p of remainder $X^* = \beta X \setminus X$ is called a butterfly-point (b-point) of βX, if $\{p\} = [F]_{\beta X} \cap [G]_{\beta X}$ for some subsets F and G of $X^* \setminus \{p\}$, which are closed in $\beta X \setminus \{p\}$. It implies, obviously, that $\beta X \setminus \{p\}$ is not normal.

Every point p of ω^* is a non-normality point of ω^* if [CH] holds, see [9]. But so far despite several efforts not much is known within ZFC (Zermelo–Fraenkel set theory). For example, p is called a Kunen point if there exists a discrete set D in ω^* such that $|D| = \omega_1$ and $D \setminus O$ is countable for each neighbourhood O of p. If p is either an accumulation point of some countable discrete subset of ω^*, see [1], or p is a Kunen point (E. K. van Douwen, unpublished), then p is a non-normality point of ω^*.

As for crowded spaces, J. Terasawa and the author independently obtained the following result.

Theorem 1 ([8], [5]). *Let X be a non-compact metrizable crowded space. Then any point p of X^* is a butterfly-point in βX. Hence $\beta X \setminus \{p\}$ is not normal.*

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Some facts for Tychonoff products were obtained by the author.

Theorem 2 ([6]). *Let τ be an arbitrary cardinal number and for every $k < \tau$ let $\mathcal{F}_k$ be a family of metrizable spaces with the following properties: $\mathcal{F}_k$ contains a crowded space and $\mathcal{F}_k$ contains at most one non-compact space. Let a space S be a free union $\bigcup_{k<\tau} S_k$ of Tychonoff products $S_k = \prod\{X : X \in \mathcal{F}_k\}$. Then every point p of S^* is a butterfly-point in βS. Hence $\beta S \setminus \{p\}$ is not normal.*

For instance, this is true if a space S is a free union of arbitrary powers of closed segments $\bigcup_{k<\tau} I^k$ or, in particular, $S = \omega \times I^c$. Some other relevant facts may be seen in [2], [3] and [4].

Now we define a new class of spaces, nice spaces (see the definitions below) so that Sorgenfrey line and all metrizable crowded spaces belong to this class and prove the following

Theorem 3. *Let X be a non-compact nice space. Then every point p of X^* is a butterfly-point in βX. Hence $\beta X \setminus \{p\}$ is not normal.*

Corollary 1. *Let S be a Sorgenfrey line. Then every point p of S^* is a butterfly-point in βS. Hence $\beta S \setminus \{p\}$ is not normal.*

We obtain also the following more technical result.

Theorem 4. *Let a space X be p-nice for some point p of X^*. Then p is a butterfly-point in βX. Hence $\beta X \setminus \{p\}$ is not normal.*

Theorems 3 and 4 follow from the last result of our paper, Theorem 5.

2. Preliminaries

In our article every space X is normal and crowded, i.e. X has no isolated points. By a neighbourhood of a point or a set we always mean an open neighbourhood. The closure of an open set is called a canonically closed set. By $X^* = \beta X \setminus X$ we denote a remainder of Čech–Stone compactification βX of X, by $[\ ]$ and $[\ ]_{\beta X}$ – the closure operators in X and βX, respectively, $3 = \{0, 1, 2\}$ and $\omega = \{0, 1, 2, \dots\}$. By O^ε we denote the biggest open in βX set, which trace on X equals open set $O \subset X$. A family of nonempty open sets $\mathcal{B}$ is called a π-base of X, if every nonempty open subset of X contains some member of $\mathcal{B}$. A π-base $\mathcal{B}$ is σ-locally finite, if it can be represented as $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$, where every $\mathcal{B}_i$ is locally finite. A base $\mathcal{B}$ is called a regular base of Arhangelskii, if for every neighbourhood O of any point x in X there is another or the same neighbourhood O' of x with the following properties: $O' \subset O$ and at most finitely many members of $\mathcal{B}$ meet both O' and $X \setminus O$ simultaneously.

Let π and σ be arbitrary families of sets. For any set A we put $\pi(A) = \{U \in \pi\colon U \cap A \neq \emptyset\}$. By $\mathrm{Exp}(\pi)$ we denote all subfamilies of π, i.e. $\mathrm{Exp}(\pi) = \{F\colon F \subset \pi\}$. We define a map $f^\pi_\sigma\colon \mathrm{Exp}(\pi) \to \mathrm{Exp}(\sigma)$ in every $F \in \mathrm{Exp}(\pi)$ as follows: $f^\pi_\sigma(F) = \big\{V \in \sigma\colon \bigcup F \cap V \neq \emptyset\big\}$. If members of π are mutually disjoint (with closure), then π is called (strongly) cellular. A set U is a proper subset of a set V, denoted $U \subsetneq V$, if both $U \subset V$ and $U \neq V$. A set U of π is a maximal member of π, if $U \subsetneq V$ for no $V \in \pi$. We say, that π (strongly) refines σ, denoted $(\pi \succ \sigma)$ $\pi \succeq \sigma$, if $U \in \pi$ is a (proper) subset of $V \in \sigma$ whenever they are not disjoint. The family

$$\mathrm{Cell}(\pi) = \left\{U_\varphi = \bigcap \varphi \setminus \left[\bigcup (\pi \setminus \varphi)\right]\colon\ \varphi \subset \pi \text{ is nonempty}\right\}$$

is a cellular refinement of π.

Let π and σ be nice families, i.e. maximal locally finite cellular families of open in X sets and $p \in X^*$. A collection $\mathcal{F} \subset \mathrm{Exp}(\pi)$ is called a p-filter on π, see [5], if $p \in \big[\bigcup \bigcap_{i \le n} F_i\big]_{\beta X}$ for any finite subcollection $\{F_1, \ldots, F_n\} \subset \mathcal{F}$. We write $\pi \succeq_{\mathcal{F}} \sigma$ $(\pi \succ_{\mathcal{F}} \sigma)$, if there is $F \in \mathcal{F}$ with $F \succeq \sigma$ $(F \succ \sigma)$. Obviously, the union of any increasing family of p-filters is also a p-filter. So by Kuratowski–Zorn lemma there are maximal p-filters or p-ultrafilters $\mathcal{F}$ on π, that is $\mathcal{F} = \mathcal{G}$ whenever $\mathcal{G}$ is a p-filter and $\mathcal{F} \subset \mathcal{G}$. Enriching any p-filter with new subfamilies of π, while possible, we can embed it into some p-ultrafilter. It may be not unique one, if a point p is not remote. But every p-ultrafilter contains $\pi(O)$ for any neighborhood O of p. We denote

$$\bigcap \mathcal{F}^* = \bigcap \left\{\left[\bigcup F\right]_{\beta X}\colon\ F \in \mathcal{F}\right\}.$$

3. Nice spaces

Definition 1. A $\pi-$base $\mathcal{B}$ of X is called a *nice* $\pi-$*base*, if $\mathcal{B}$ is σ-locally finite and for every neighbourhood O of any closed set F there is a nice subfamily π of $\mathcal{B}$ such that $\bigcup \pi(F) \subset O$.

Definition 2. A normal crowded space X is called *nice*, if for any point p of X^* there is a nice π-base $\mathcal{B}$ of X with the following property: $p \notin [U]_{\beta X}$ for every $U \in \mathcal{B}$.

Definition 3. Let p be any point of βX. A π-base $\mathcal{B}$ of X is called a *p-nice π-base*, if $\mathcal{B}$ is σ-locally finite and for any neighbourhood O of p in βX there is a neighbourhood O' of p and a nice subfamily π of $\mathcal{B}$ such that $\bigcup \pi(O') \subset O$.

Definition 4. Let $p \in X^*$. A normal crowded space X is called *p-nice*, if there is a p-nice π-base $\mathcal{B}$ of X with the following property: $p \notin [U]_{\beta X}$ for every $U \in \mathcal{B}$.

Definition 5. Let π be any subfamily of a π-base $\mathcal{B}$. Then a *cap* of π in $\mathcal{B}$, denoted $\mathcal{B}'(\pi)$, are all the sets $U \in \mathcal{B}$ with the following property: if U meets some $V \in \pi$, then U is a proper subset of V, i.e.

$$\mathcal{B}'(\pi) = \{U \in \mathcal{B} \colon \forall\, V \in \pi(U \cap V = \emptyset \vee U \subsetneq V)\}.$$

Definition 6. Let π be any subfamily of a π-base $\mathcal{B}$. Then a *little cap* of π in $\mathcal{B}$, denoted $\mathcal{B}(\pi)$, are all maximal sets of a cap $\mathcal{B}'(\pi)$, i.e.

$$\mathcal{B}(\pi) = \{U \in \mathcal{B}'(\pi) \colon \forall\, V \in \mathcal{B}'(\pi)(\neg(U \subsetneq V))\}.$$

Lemma 1. *Let π be any family of open sets, $U_\varphi \in \mathrm{Cell}(\pi)$ and $x \in U_\varphi$. Then for any $V \in \pi$ the following hold: $x \in V$ if and only if $V \in \varphi$.*

Proof: Let $x \in V$ and $V \notin \varphi$. Then $U_\varphi \cap [V] = \emptyset$ implies $x \notin U_\varphi$. Let $x \notin V$ and $V \in \varphi$. Then $U_\varphi \subset V$ implies $x \notin U_\varphi$. □

Lemma 2. *Let π and σ be any families of open sets such that $\pi \subset \sigma$. Then* $\mathrm{Cell}(\pi) \preceq \mathrm{Cell}(\sigma)$.

Proof: Let $U_\varphi \cap U_{\varphi'} \neq \emptyset$ for some $\varphi \subset \pi$ and $\varphi' \subset \sigma$. For any point $x \in U_\varphi \cap U_{\varphi'}$ we have $\varphi = \{V \in \pi \colon x \in V\}$ and $\varphi' = \{V \in \sigma \colon x \in V\}$. Hence $\varphi \subset \varphi'$ implies $\bigcap \varphi' \subset \bigcap \varphi$. Moreover, $\pi \setminus \varphi = \{V \in \pi \colon x \notin V\}$ and $\sigma \setminus \varphi' = \{V \in \sigma \colon x \notin V\}$. Hence $\pi \setminus \varphi \subset \sigma \setminus \varphi'$ and $\left[\bigcup(\pi \setminus \varphi)\right] \subset \left[\bigcup(\sigma \setminus \varphi')\right]$. Finally, $U_{\varphi'} \subset U_\varphi$. □

Lemma 3. *Let a family π be open locally finite and everywhere dense in X. Then* $\mathrm{Cell}(\pi)$ *is a nice family, refining π.*

Proof: If $U_\varphi \neq \emptyset$ for some $\varphi \subset \pi$, then φ is finite and U_φ is open.

If $U \in \varphi \setminus \varphi'$, then $U_\varphi \subset U$ and $U_{\varphi'} \cap U = \emptyset$. So $\mathrm{Cell}(\pi)$ is cellular.

Let an open set O meet only finitely many sets of π, say $U_0, \dots, U_k$. Then $O \cap U_\varphi \neq \emptyset$ implies $\varphi \subset \{U_0, \dots, U_k\}$. So O meets at most 2^{k+1} members of $\mathrm{Cell}(\pi)$, which is locally finite.

Let x not be a boundary point of any $U \in \pi$. Then $x \in U_\varphi$ for $\varphi = \{U \in \pi \colon x \in U\}$ and $\mathrm{Cell}(\pi)$ is everywhere dense.

Let U_φ meet some $V \in \pi$. Then $V \in \varphi$ by our definition. Hence $U_\varphi \subset \bigcap \varphi$ implies $U_\varphi \subset V$, i.e., $\mathrm{Cell}(\pi)$ refines π. □

Lemma 4. *Sorgenfrey line S has a nice π-base.*

Proof: Every $\mathcal{B}_n = \{[z + k/2^n, z + k + 1/2^n) \colon z \in Z \text{ and } k = 0, \dots, 2^n - 1\}$ is a nice family and $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ is a nice π-base. Indeed, let O be any neighbourhood of a closed set F. Define σ to be all maximal sets of the cover $\mathcal{A} = \{U \in \mathcal{B} \colon U \cap F = \emptyset \vee U \subset O\}$ of X. Since $\mathcal{B}$ is embedded, σ is cellular. Any $x \in F$ belongs to some $U \in \mathcal{A}$. Let V be the maximal set of $\mathcal{A}$, containing U. Then $V \in \sigma$ and σ is a cover. Hence σ is nice and $\bigcup \sigma(F) \subset O$. □

Lemma 5. *Every metrizable crowded space X has a nice π-base.*

Proof: For every $i \in \omega$ let $\mathcal{P}_i$ be a locally finite open cover of X, consisting of sets with diameter at most $1/(i+1)$. Obviously, $\mathcal{P} = \bigcup_{i\in\omega} \mathcal{P}_i$ is a regular base of Arhangelskii. Every $\mathcal{B}_i = \mathrm{Cell}\big(\bigcup_{j\le i} \mathcal{P}_i\big)$ is nice and $\mathcal{B}_i \succeq \mathcal{P}_i$ by Lemma 3, $\mathcal{B}_{i+1} \succeq \mathcal{B}_i$ by Lemma 2. Then $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ is a nice π-base. Indeed, let O be any neighbourhood of a closed set F. Assume π to be all maximal sets of the cover $\{U \in \mathcal{P} \colon U \cap F = \emptyset \vee U \subset O\}$. It is easy to see that π is a locally finite cover of X and $\bigcup \pi(F) \subset O$. For any $U \in \pi$ we fix unique $i_0 \in \omega$ so that $U \in \mathcal{P}_{i_0}$. If U meets some $V \in \mathcal{B}_{i_0}$, where the index i_0 is one and the same, then $V \subset U$. Hence $\mathcal{B}_U = \{V \in \mathcal{B}_{i_0} \colon V \subset U\}$ is nice in U. Let $\mathcal{B}_\pi$ be all maximal members of $\bigcup_{U\in\pi} \mathcal{B}_U$. Since $\mathcal{B}$ is embedded, $\mathcal{B}_\pi$ is nice. Let $V \in \mathcal{B}_\pi$ intersect F. Then $V \in \mathcal{B}_U$ for some $U \in \pi$ by our construction. It implies $V \subset U$ and $U \cap F \ne \emptyset$. But then $U \subset O$ implies $V \subset O$ and $\bigcup \mathcal{B}_\pi(F) \subset O$. □

Lemma 6. *Let $\mathcal{B}$ be a σ-locally finite π-base. Then $\mathcal{B}$ is nice if and only if for any two closed disjoint subsets F and G of X there is a nice subfamily σ of $\mathcal{B}$ such that $\bigcup \sigma(F) \cap \big(\bigcup \sigma(G)\big) = \emptyset$.*

Proof: Let $\mathcal{B}$ be nice and let F and G be closed and disjoint. Then there is a nice subfamily σ of $\mathcal{B}$ such that $\bigcup \sigma(F) \subset X \setminus G$. Since σ is cellular, $\sigma(F) \cap \sigma(G) = \emptyset$ implies $\bigcup \sigma(F) \cap \big(\bigcup \sigma(G)\big) = \emptyset$.

Vice versa. Let O and O' be any neighbourhoods of a closed set F such that $[O'] \subset O$. Then every nice subfamily σ of $\mathcal{B}$ is everywhere dense in canonically closed $G = [X \setminus [O']]$. Hence $\bigcup \sigma(F) \cap \big(\bigcup \sigma(G)\big) = \emptyset$ implies $\bigcup \sigma(F) \subset O$. □

Lemma 7. *Let there be a nice π-base $\mathcal{A}$ with the following properties: $\mathcal{A} = \bigcup_{i\in\omega} \mathcal{A}_i$ and every $\mathcal{A}_i$ is locally finite. Then there is a nice π-base $\mathcal{B}$ such that $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ and for every $i \in \omega$ the following hold:*

1) *$\mathcal{B}_i$ is a nice family;*
2) *$\mathcal{A}_i \prec \mathcal{B}_i$;*
3) *$\mathcal{B}_i \prec \mathcal{B}_{i+1}$;*
4) *there is a strongly cellular family $\{U(\nu) \colon U \in \mathcal{B}_i$ and $\nu \in 3\}$ of sets $U(\nu) \in \mathcal{B}_{i+1}$ with $[U(\nu)] \subset U$.*

Proof: Every

$$\mathcal{D}_i = \mathrm{Cell}\bigg(\bigcup_{j\le i} \mathcal{A}_j \cup \{X\}\bigg)$$

is nice and $\mathcal{A}_i \preceq \mathcal{D}_i$ by Lemma 3, $\mathcal{D}_i \preceq \mathcal{D}_{i+1}$ by Lemma 2. To provide (4) we put $\mathcal{B}_0 = \mathcal{D}_0$ and assume $\mathcal{B}_i$ to be constructed for some $i \in \omega$. There is a strongly cellular family of nonempty open sets

$$\mathcal{W}_i = \{U(\nu)\colon U \in \mathcal{B}_i \text{ and } \nu \in 3\}$$

with $[U(\nu)] \subset U$. If $\mathcal{B}_{i+1} = \mathrm{Cell}(\mathcal{B}_i \cup \mathcal{W}_i \cup \mathcal{D}_{i+1})$, then $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ is as required.

Indeed, leaving the conditions 1)–4) to the reader we will show only that $\mathcal{B}$ is nice. Let O be any neighbourhood of a closed set F in X. There is nice $\sigma \subset \mathcal{A}$ such that $\bigcup \sigma(F) \subset O$. For any $U \in \sigma$ we choose unique $i_0 \in \omega$ so that $U \in \mathcal{A}_{i_0}$. By our construction, $\mathcal{A}_{i_0} \preceq \mathcal{D}_{i_0} \preceq \mathcal{B}_{i_0}$, where the index i_0 is one and the same. So $V \cap U \neq \emptyset$ implies $V \subset U$ for every $V \in \mathcal{B}_{i_0}$. Hence $\mathcal{B}_U = \{V \in \mathcal{B}_{i_0}\colon V \subset U\}$ is nice in U. Since σ is nice, $\mathcal{B}_\sigma = \bigcup_{U\in\sigma} \mathcal{B}_U$ is also nice. Let $V \cap F \neq \emptyset$ for some $V \in \mathcal{B}_\sigma$. Then $V \in \mathcal{B}_U$ implies $V \subset U$ for unique $U \in \sigma$ and $U \cap F \neq \emptyset$ implies $U \subset O$. Hence $V \subset O$ implies $\bigcup \mathcal{B}_\sigma(F) \subset O$ and our proof is complete. □

From now on we may assume that every nice π-base $\mathcal{B}$ satisfies the conditions 1)–4). Then $\mathcal{B}$ is embedded and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ if $i \neq j$. So for each $U \in \mathcal{B}$ we can put $n(U) = i$ if $U \in \mathcal{B}_i$.

Lemma 8. *If $\mathcal{A} \subset \mathcal{B}$ is locally finite, then "little cap" $\mathcal{B}(\mathcal{A})$ is nice.*

Proof: Since $\mathcal{B}(\mathcal{A}) \subset \mathcal{B}$, it is a family of open sets.

Since $\mathcal{B}(\mathcal{A})$ is the family of maximal sets of $\mathcal{B}'(\mathcal{A})$, which is embedded, then $\mathcal{B}(\mathcal{A})$ is cellular.

Let an open O intersect at most finitely many sets of $\mathcal{A}$ and let $x \in O$ not be in the boundary of any of them. There is a neighbourhood O_0 of x such that $O_0 \subset O$ and for any $U \in \mathcal{A}$ the following hold: either $O_0 \cap U = \emptyset$ or $O_0 \subsetneq U$. If $V \in \mathcal{B}$ and $V \subset O_0$, then $V \in \mathcal{B}'(\mathcal{A})$. Let W be the maximal set of $\mathcal{B}'(\mathcal{A})$, containing V. Then $W \cap O \neq \emptyset$ and $W \in \mathcal{B}(\mathcal{A})$, which is maximal.

Now we have to show only that $\mathcal{B}(\mathcal{A})$ is locally finite. Let a neighbourhood O of a point x intersect at most finitely many sets of $\mathcal{A}$. We put either $k_0 = \max\{n(U)\colon O \text{ meets } U \in \mathcal{A}\}$, if the last set is not empty, or $k_0 = 1$ otherwise. For any neighbourhood O_0 of x with $[O_0] \subset O$ there is a nice subfamily σ of $\mathcal{B}$ such that $\bigcup \sigma(O_0) \subset O$. Let a neighbourhood O_1 of x satisfy both $O_1 \subset O_0$ and O_1 meets at most finitely many members of σ. We set $k_1 = \max\{n(U)\colon O_1 \text{ meets } U \in \sigma\}$ and $k = k_0 + k_1$.

Let $U \in \mathcal{B}$ intersect O_1 and $n(U) > k$. Since σ is nice, $U \cap O_1$ meets some $V \in \sigma$. Then $k_1 \geq n(V)$ implies $U \subset V \subset O_0$. Let U intersect some $V \in \mathcal{A}$. Then $k_0 \geq n(V)$ implies $U \subsetneq V$ and $U \in \mathcal{B}'(\mathcal{A})$.

Let $U \in \mathcal{B}$ intersect O_1 and $n(U) > k + 1$. By our construction, U is a proper subset of unique $V \in \mathcal{B}_{k+1}$. Since $V \in B'(\mathcal{A})$, then $U \notin \mathcal{B}(\mathcal{A})$.

Finally, let a neighbourhood O_2 satisfy both $O_2 \subset O_1$ and O_2 intersects at most finitely many members of $\bigcup_{i\leq k+1} \mathcal{B}_i$. Then O_2 intersects at most finitely many members of $\mathcal{B}(\mathcal{A})$. $\square$

Corollary 2. *For any locally finite subfamily π of $\mathcal{B}$ there is a nice subfamily σ of $\mathcal{B}$ such that $\sigma \succ \pi$.*

Lemma 9. *Let $\mathcal{B}$ be a σ-locally finite π-base. Then $\mathcal{B}$ is nice if and only if $\mathcal{B}$ is p-nice for any point p of βX.*

PROOF: Let $\mathcal{B}$ be nice and assume O and O' to be any neighbourhoods of p in βX with $[O']_{\beta X} \subset O$. Then $U = O \cap X$ is an open neighbourhood of $F = [O']_{\beta X} \cap X$. There is a nice subfamily σ of $\mathcal{B}$ such that $\bigcup \sigma(F) \subset U$. But then O contains $\bigcup \sigma(O') = \bigcup \sigma(F)$.

Vice versa. Let O be any neighbourhood of a closed set F in X. Then O^ε is an open neighbourhood of $G = [F]_{\beta X}$ in βX. For any point x of G there is a neighbourhood Ox in βX and a nice subfamily σ_x of $\mathcal{B}$ such that $\sigma_x(Ox) \subset O^\varepsilon$. The open cover $\{Ox\colon x \in G\}$ of G contains a finite subcover $\{Ox_1, \dots, Ox_n\}$. The family $\mathcal{A} = \bigcup_{i\leq n} \sigma_i$, where $\sigma_i = \sigma_{x_i}$, is locally finite in X. Hence $\sigma = \mathcal{B}(\mathcal{A})$ is nice by Lemma 8 and $\bigcup \sigma(F) \subset O$. Indeed, every $U \in \sigma(F)$ intersects some Ox_i. Since σ_i is nice, U meets some $V \in \sigma_i$. Then $U \subset V$ by the definition of σ and $V \cap Ox_i \neq \emptyset$. Hence $V \subset O^\varepsilon$ and our proof is complete. $\square$

4. Butterfly-point

From now on a space X has a nice π-base $\mathcal{B}$, satisfying the conditions 1)–4) of Lemma 7. By $\Sigma = \Sigma(\mathcal{B})$ we denote all nice subfamilies of $\mathcal{B}$, i.e. $\Sigma = \{\sigma \subset \mathcal{B}\colon \sigma \text{ is nice}\}$. For any $\sigma \in \Sigma$ and $\nu \in 3$ we put $\sigma(\nu) = \{U(\nu)\colon U \in \sigma\}$.

Lemma 10. *Let a paracompact space X has a nice π-base. Then X is nice.*

PROOF: For any point p of X^* there is an open locally finite cover $\mathcal{P}$ of X with the following property: $p \notin [U]_{\beta X}$ for every $U \in \mathcal{P}$. Let $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ be a nice π-base, where every $\mathcal{B}_i$ is locally finite. Then each

$$\mathcal{B}'_i = \{U \cap V\colon\ U \in \mathcal{B}_i \ \text{ and } \ V \in \mathrm{Cell}(\mathcal{P})\}$$

is locally finite and $\mathcal{B}' = \bigcup_{i\in\omega} \mathcal{B}'_i$ is as required. Indeed, for any open neighbourhood O of a closed set F there is a nice subfamily σ of $\mathcal{B}$ such that $\bigcup \sigma(F) \subset O$. But then $\sigma' = \{U \cap V : U \in \sigma \text{ and } V \in \mathrm{Cell}(\mathcal{P})\}$ is a nice subfamily of $\mathcal{B}'$, having the same property. $\square$

Lemma 11. *Let $\mathcal{B}$ be a nice π-base of X and $p \in X^*$. If there is a zero-set Z in βX with $p \in Z \subset X^*$, then there is $\sigma \in \Sigma$ with the following property: $p \notin [U]_{\beta X}$ for any $U \in \sigma$.*

Proof: Let $Z = \bigcap_{i\in\omega} O_i$, where O_i is open in βX and $[O_{i+1}]_{\beta X} \subset O_i$ for each $i \in N$. We put $F_0 = [X \setminus [O_2]]$ and $F_i = [O_i \setminus [O_{i+2}]]$. We set $W_0 = X \setminus [O_3]$ and $W_i = O_{i-1} \setminus [O_{i+3}]$. Then every F_i is a canonically closed subset of open W_i and $\bigcup_{i\in\omega} F_i = X$. If $\sigma_i \subset \mathcal{B}$ is nice and $\bigcup \sigma_i(F_i) \subset W_i$, then $\mathcal{A} = \bigcup_{i\in\omega} \sigma_i(F_i)$ is locally finite. Hence "little cap" $\sigma = \mathcal{B}(\mathcal{A})$ is nice by Lemma 8 and $\sigma \succ \mathcal{A}$. If $U \in \sigma$ meets any F_i, then U meets some $V \in \sigma_i(F_i)$. It implies $U \subset V \subset W_i$ and our proof is complete. □

We omit the proofs of Lemmas 12–15, since they coincide with the proofs of Lemmas 2–5 in [5].

Lemma 12. *Let for a point p of X^* there be $\sigma_p \in \Sigma$ such that $p \notin [U]_{\beta X}$ for any $U \in \sigma$. Then there is a well-ordered chain $\{\sigma_\alpha \colon \alpha < \lambda\} \subset \Sigma$ and a p-ultrafilter $\mathcal{F}_\alpha$ on every σ_α, with the following properties for all $\alpha < \beta < \lambda$ and $f^\alpha_\beta = f^{\sigma_\alpha}_{\sigma\beta}$:*

1) *$p \notin [U]_{\beta X}$ for every $U \in \sigma_0$;*
2) *$f^\alpha_\beta(\mathcal{F}_\alpha) \subset \mathcal{F}_\beta$;*
3) *$\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma_\beta$;*
4) *for any $\sigma \in \Sigma \setminus \{\sigma_\alpha \colon \alpha < \lambda\}$ there is $\alpha < \lambda$ with $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$.*

Lemma 13. *We have $\bigcap \mathcal{F}^*_0 \subset X^*$.*

Lemma 14. *If $\alpha < \beta < \lambda$, then $\bigcap \mathcal{F}^*_\beta \subset \bigcap \mathcal{F}^*_\alpha$.*

Lemma 15. *For any neighbourhood O of p in βX there is $\alpha < \lambda$ with $\bigcap \mathcal{F}^*_\alpha \subset O$.*

Lemma 16 coincides with Proposition 6 in [5]. Now we present a new proof, probably clearer and easier to understand.

Lemma 16. *The set*

$$B_\alpha(\nu) = \bigcap \mathcal{F}^*_\alpha \cap \left(\bigcap_{\beta\in\lambda\setminus\alpha} \left[\bigcup \sigma_\beta(\nu) \right]_{\beta X} \right)$$

is not empty for any $\alpha < \lambda$ and $\nu \in 3$.

Proof: Let $F \in \mathcal{F}_\alpha$ and let $\alpha < \beta_0 < \cdots < \beta_i < \cdots < \beta_n < \lambda$ be any finite sequence of indexes. Our goal is to find by induction $U \in \mathcal{B}$ so that $U \subset \bigcup F$ and $U \subset V(\nu)$ for any $i \le n$ and some $V \in \sigma(\beta_i)$. We can assume $F \prec \sigma_{\beta_0}$ and choose $G_i \in \mathcal{F}_{\beta_i}$ so that $G_i \prec \sigma_{\beta_{i+1}}$ for any $i < n$ and $G_n = \sigma_{\beta_n}$. For $F_0 = f^\alpha_{\beta_0} F \cap G_0$ and $F_{i+1} = f^{\beta_i}_{\beta_{i+1}} F_i \cap G_{i+1}$ we get $F_i \in \mathcal{F}_{\beta_i}$, $F_i \prec F_{i+1}$ and $\bigcup F_{i+1} \subset \bigcup F_i$. For any $U_n \in F_n$ and $U_i \in F_i$ with $U_n \subset U_i$ we obtain

$$U_n \subsetneq \cdots \subsetneq U_i \subsetneq \cdots \subsetneq U_1 \subsetneq U_0 \subsetneq \bigcup F. \tag{1}$$

Only in order to simplify the notation assume, that the order of the embedding does not change.

To insert the set $U_0(\nu)$ into the sequence (1), we note the following points.

1) Since every σ_{β_i} is nice and unique, U_i of σ_{β_i} can be replaced with another or the same set U_i' of σ_{β_i} so that

$$\bigcap_{i=1}^{n} U_i' \cap U_0(\nu) \neq \emptyset.$$

2) Since $U_i \subsetneq U_0$, then $U_i' \subset U_0(\nu)$ by the definition of $\mathcal{B}$.

3) Perhaps $U_1' \neq U_0(\nu)$:

$$U_n' \subset \cdots \subset U_i' \subset \cdots \subset U_1' \subsetneq U_0(\nu) \subset U_0 \subset \bigcup F. \tag{2}$$

4) Perhaps some sets of U_i' are equal to $U_0(\nu)$:

$$U_n' \subset \cdots \subset U_i' \subset \cdots \subset U_{i_0}' \subsetneq U_{i_0-1}' = \cdots = U_1' = U_0(\nu) \subset U_0 \subset \bigcup F. \tag{3}$$

To insert the set $U_1'(\nu)$ in sequence (2) we can repeat points 1)–4) to get either

$$U_n'' \subset \cdots \subset U_i'' \subset \cdots \subset U_2'' \subsetneq U_1'(\nu) \subset U_1' \subsetneq U_0(\nu) \subset U_0 \subset \bigcup F \tag{4}$$

or

$$\begin{aligned} U_n'' \subset \cdots \subset U_i'' \subset \cdots \subset U_{i_1}'' \subsetneq U_{i_1-1}'' = \cdots = U_2'' \\ = U_1'(\nu) \subset U_1' \subsetneq U_0(\nu) \subset U_0 \subset \bigcup F. \end{aligned} \tag{5}$$

To insert the set $U_1'(\nu)$ in sequence (3) we can repeat points 1)–4) to get either

$$\begin{aligned} U_n'' \subset \cdots \subset U_i'' \subset \cdots \subset U_{i_0}'' \subsetneq U_1'(\nu) \\ \subset U_{i_0-1}' = \cdots = U_1' = U_0(\nu) \subset U_0 \subset \bigcup F \end{aligned} \tag{6}$$

or

$$\begin{aligned} U_n'' \subset \cdots \subset U_i'' \subset \cdots \subset U_{i_1}'' \subsetneq U_{i_1-1}'' = \cdots = U_{i_0}'' = U_1'(\nu) \\ \subset U_{i_0-1}' = \cdots = U_1' = U_0(\nu) \subset U_0 \subset \bigcup F. \end{aligned} \tag{7}$$

Now we can insert $U_2''(\nu)$ into the sequences (4) and (5). We can insert $U_{i_0}''(\nu)$ into the sequences (6) and (7) and so on. After each "stroke by the tail in front of the set $U_i(\nu)$" it becomes shorter by at least one set. So, after a finite number $k \leq n$ of "strokes" it will be empty. Then $U_k(\nu)$ is as required. □

Theorem 5. *Let for a point p of X^* there be $\sigma_p \in \Sigma$ such that $p \notin [U]_{\beta X}$ for any $U \in \sigma$. Then p is a butterfly-point of βX.*

Proof: For any $\nu \in 3$ denote $F_\nu = \{p_\alpha(\nu)\colon \alpha < \lambda\}$, where $p_\alpha(\nu)$ is any point of the set $B_\alpha(\nu)$ in the previous lemma. By Lemmas 13–15, $F_\nu \subset B_0 \subset X^*$ and for any neighbourhood O of p there is $\alpha < \lambda$ with $\{p_\beta(\nu)\colon \beta \in \lambda \setminus \alpha\} \subset B_\alpha \subset O$. Then the condition $\{p_\beta(\nu)\colon \beta < \alpha\} \subset \left[\bigcup \sigma_\alpha(\nu)\right]_{\beta X}$ implies both that the sets $[F_\nu]_{\beta X} \setminus \{p\}$ are pairwise disjoint and $p \in F_\nu$ for no more than one unique F_ν. The other two ensure that p is a b-point in βX. Our proof is complete. □

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