Classification of quasigroups according to directions of translations II

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Abstract. In each quasigroup Q there are defined six types of translations: the left, right and middle translations and their inverses. Two translations may coincide as permutations of Q, and yet be different when considered upon the web of the quasigroup. We shall call each of the translation types a direction and will associate it with one of the elements ι, l, r, s, ls and rs, i.e., the elements of a symmetric group S_3. Properties of the directions are considered in part 1 of "Classification of quasigroups according to directions of translations I" by F. M. Sokhatsky and A. V. Lutsenko.

Let ${}^{\sigma}\mathcal{M}$ denote the set of all translations of a direction $\sigma \in S_3$. The conditions ${}^{\sigma}\mathcal{M} = {}^{\kappa}\mathcal{M}$, where $\sigma, \kappa \in S_3$ and $\sigma \neq \kappa$, define nine quasigroup varieties. Four of them are well known: LIP, RIP, MIP and CIP. The remaining five quasigroup varieties are relatively new because they are left and right inverses of CIP variety and generalization of commutative, left and right symmetric quasigroups.

Keywords: quasigroup; parastrophe; parastrophic symmetry; parastrophic orbit; translation; direction; matrix quasigroup

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1. Introduction

In quasigroup theory many types of inverse properties come under consideration. The most prominent types are known under abbreviations IP, LIP, RIP, CIP, WIP and AAIP, cf. [1], [2], [6], [4] and [13]. Loops with these properties are closely connected to some of the most widely studied varieties of loops, e.g. Moufang loops, Bol loops and Bruck loops. For example, a loop is Moufang if and only if all its loop isotopes are IP loops. Recall also that the class of IP loops itself has a number of strong properties: (1) the left, right and middle nuclei coincide, (2) isotopic IP loops are pseudo-isomorphic, and (3) any two commutative IP loops that are isotopic are also isomorphic [13]. Cf. [6], [7] and [8] for further generalizations and applications.

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Thus, the study of the relationships between classes of quasigroups with different inversion properties are topical. In particular, the description of parastrophy orbits of these quasigroup classes should be studied.

Let ${}^{\sigma}\mathcal{M}$ denote the set of all σ-translations in a quasigroup. It is easy to verify that the class of all LIP quasigroups is exactly the class of quasigroups with ${}^{l}\mathcal{M} = {}^{ls}\mathcal{M}$; the class of all CIP quasigroups is exactly the class of quasigroups with ${}^{l}\mathcal{M} = {}^{r}\mathcal{M}$. The symmetric group S_3 acts on the set of all quasigroup classes defined by the equalities ${}^{\sigma}\mathcal{M} = {}^{\kappa}\mathcal{M}$, $\sigma \neq \kappa$. It is proved that there are three orbits under this action and each orbit has three pairwise parastrophic classes (Theorem 4). The first one is the orbit consisting of the well-known classes: MIP, LIP, RIP. Note that these classes of quasigroups were considered in detail in [15].

The second orbit corresponds to the well-known class of cross inverse quasigroups. The two classes parastrophic to the CIP quasigroups have not been described before. We shall call them the left and right cross inverse quasigroups. The classical CIP quasigroup are called, in this context, middle cross inverse.

The third orbit consists of three new classes whose quasigroups are certain generalizations of commutative, left symmetric and right symmetric quasigroups, respectively. They are called middle, left and right mirror quasigroups.

All nine classes of quasigroups are varieties, the defining identities may be found in Theorem 5, Theorem 6 and Theorem 7. Each variety has some inverse property. The corresponding inversion mappings are presented in the above described theorems.

2. Preliminaries

We shall be using notions and notations defined in [16]. From Theorem 1 in [16] we obtained following corollary.

Corollary 1. *Let S_3 act on a set K. If k is such an element of K that ${}^{s}k = k$ and k does not coincide with another element from* $\mathrm{Po}(k)$, *then k is singly symmetric.*

Proof: Since ${}^{s}k = k$, then the element k is neither asymmetric nor semisymmetric. It is not totally symmetric, because $k \neq {}^{\sigma}k$ for some σ in S_3. □

According to [16, Definition 2], a σ-parastrophe ${}^{\sigma}(\omega = \upsilon)$ of an identity $\omega = \upsilon$ is an identity which is obtained from $\omega = \upsilon$ by replacing the main functional variable with its σ-parastrophe.

For example, sl-parastrophe of the commutativity law $x \cdot y = y \cdot x$ is the identity $x \overset{(sl)^{-1}}{\cdot} y = y \overset{(sl)^{-1}}{\cdot} x$, i.e., $x \overset{sr}{\cdot} y = y \overset{sr}{\cdot} x$. By definition of s-parastrophe, $y \overset{r}{\cdot} x = x \overset{r}{\cdot} y$.

By definition of r-parastrophe, $y\cdot(x\overset{r}{\cdot}y)=x$. If $z:=x\overset{r}{\cdot}y$, then $y=x\cdot z$, therefore, $(x\cdot z)\cdot z=x$. Thus, the left symmetry law is sl-parastrophe of the commutativity law.

Corollary 2 ([14]). *An identity $\omega=\upsilon$ defines a variety of quasigroups $\mathfrak{A}$ if and only if a σ-parastrophe ${}^{\sigma}(\omega=\upsilon)$ of this identity defines the variety ${}^{\sigma}\mathfrak{A}$, where $\sigma\in S_3$.*

Two identities are called, see [14]:

1) *equivalent*, if they define the same variety;
2) *σ-parastrophically equivalent*, if they define σ-parastrophic varieties;
3) *parastrophically equivalent*, if they are σ-parastrophically equivalent for some σ.

Some of inversion mappings have *invertibility properties* of their elements, for example, IP quasigroups. Recall the definitions of quasigroups with an inverse property, see [1], [2], [15], [6], [5].

A quasigroup $(Q;\cdot)$ is called: an *LIP quasigroup*, an *RIP quasigroup*, an *MIP quasigroup*, if there exist transformations λ, ϱ, μ called a *left*, *right*, *middle inversion mapping* such that for all x and y the respective equalities are true:

$$\lambda(x)\cdot xy=y;\qquad yx\cdot\varrho(x)=y;\qquad xy=\mu(yx).$$

A quasigroup $(Q;\cdot)$ will be called: a *middle CIP quasigroup*, a *left CIP quasigroup*, a *right CIP quasigroup*, if there exist transformations α, β, γ called a *middle*, *left*, *right inversion mapping* such that for all x and y the respective equalities are true:

$$xy\cdot\alpha(x)=y;\qquad yx\cdot y=\beta(x);\qquad y\cdot xy=\gamma(x).$$

Note that the middle CIP quasigroup is the well known CIP quasigroup.

It is easy to show, see [2], [5], [6], that the formulas

$$(1)\qquad xy\cdot\gamma(x)=y,\qquad x\cdot y\gamma(x)=y,\qquad \gamma(x)\cdot yx=y,\qquad \gamma(x)y\cdot x=y$$

are equivalent in a quasigroup $(Q;\cdot)$. In other words, each of these formulas describes the class of all middle CIP *quasigroup*. Moreover in [5], it has been proved that the class of all CIP quasigroups is a variety.

A loop $(Q;\cdot)$ is CIP, if the identity $xy\cdot x^{-1}=y$ holds.

A quasigroup $(Q;\cdot)$ will be called: a *middle mirror quasigroup*, a *left mirror quasigroup*, a *right mirror quasigroup*, if there exists a transformation φ, δ, ξ called a *middle*, *left*, *right inversion mapping* such that for all x and y the respective equalities are true:

$$\varphi(x) \cdot y = y \cdot x; \qquad y \cdot yx = \delta(x); \qquad xy \cdot y = \xi(x).$$

3. Sets of translations having the same direction

The *σ-direction set of translations*, i.e., the *set of all translations of the direction σ* of a quasigroup $(Q;\circ)$ is defined by

$$^{\sigma}\mathcal{M}^{\circ} := \{^{\sigma}M_x^{\circ} \colon x \in Q\}, \qquad \sigma \in S_3.$$

We will also write $^{\sigma}\mathcal{M}^{\tau}$ instead of $^{\sigma}\mathcal{M}^{\overset{\tau}{\circ}}$. Let

$$\mathcal{M}^{\circ} := \{^{\iota}\mathcal{M}^{\circ}, {}^{l}\mathcal{M}^{\circ}, {}^{r}\mathcal{M}^{\circ}, {}^{s}\mathcal{M}^{\circ}, {}^{ls}\mathcal{M}^{\circ}, {}^{lr}\mathcal{M}^{\circ}\}.$$

Some quasigroups satisfy the property: *two or more sets of translations of the same directions coincide.* For example, an LIP quasigroup, i.e., a quasigroup with the left inverse property, is defined by

$$(\exists\, \alpha)(\forall\, x, y) \quad \alpha(x) \circ (x \circ y) = y.$$

This condition is equivalent to $(\exists\, \alpha)(\forall\, x)\ L^{\circ}_{\alpha(x)} = (L^{\circ}_x)^{-1}$. Using [16, (14)], it is transformed to

$$(\exists\, \alpha)(\forall\, x) \quad {}^{ls}M^{\circ}_{\alpha(x)} = {}^{l}M^{\circ}_x.$$

Since α is a bijection of the set Q, the condition means that the sets $^{ls}\mathcal{M}^{\circ}$ and $^{l}\mathcal{M}^{\circ}$ are equal.

Therefore, a quasigroup $(Q;\circ)$ has the left inverse property if and only if

$$^{ls}\mathcal{M}^{\circ} = {}^{l}\mathcal{M}^{\circ},$$

that is its translation sets of the directions ls and l coincide. Consequently, the class of all LIP quasigroups is defined by the equality

$$^{ls}\mathcal{M}^{\cdot} = {}^{l}\mathcal{M}^{\cdot}$$

of terms in which $(\cdot)$ is a functional variable taking its values among the main operations in the class of quasigroups. For short, we will omit the sign $(\cdot)$.

Definition 1. A term $^{\sigma}M_y := x \overset{r\sigma^{-1}}{\cdot} y$ is called an *abstract translation* of the direction σ defined by y; $^{\sigma}\mathcal{M}$ is called an *abstract set of translations* of the direction σ; the formula $^{\sigma}\mathcal{M} = {}^{\kappa}\mathcal{M}$ is the brief notation of the formula

$$(\exists\, \alpha)(\forall\, x)(\forall\, y) \quad x \overset{r\sigma^{-1}}{\cdot} y = x \overset{r\kappa^{-1}}{\cdot} \alpha(y).$$

This equality will be called an *abstract equality of two translation sets of the directions σ and κ.*

In Theorem 4, we will determine all classes of quasigroups which are defined by the above considered equalities.

To do this, we need the following property.

Lemma 3. *For all $\sigma, \kappa \in S_3$, the κ-direction set of translations of the σ-parastrophe of a quasigroup has the direction $\nu\kappa$ in the $\nu\sigma$-parastrophe of the quasigroup:*

$$ {}^{\kappa}\mathcal{M}^{\sigma} = {}^{\nu\kappa}\mathcal{M}^{\nu\sigma}. $$

Proof: Let $(Q;\cdot)$ be a quasigroup, then according to [16, Lemma 6] the equality ${}^{\nu\kappa}M_a^{\nu\sigma} = {}^{\kappa}M_a^{\sigma}$ holds for all $\sigma, \kappa \in S_3$ and for all $a \in Q$. That is why the sets ${}^{\kappa}\mathcal{M}^{\sigma}$ and ${}^{\nu\kappa}\mathcal{M}^{\nu\sigma}$ consist of exactly the same elements and therefore they are equal. $\square$

Two conditions are known to be: *equivalent*, if they determine the same class of quasigroups; *parastrophically equivalent*, if they determine parastrophic classes of quasigroups.

Theorem 4. *Each equality of two translation sets of different directions determines exactly one class of quasigroups. Namely,*

- *the parastrophy orbit of (middle, left, right) inverse property quasigroups:*

${}^{\iota}\mathcal{M} = {}^{s}\mathcal{M}$	$M_x^{-1} = M_{\mu(x)}$	$yz = \mu(zy)$	MIP quas.	$\mathfrak{I} = {}^{s}\mathfrak{I}$
${}^{l}\mathcal{M} = {}^{ls}\mathcal{M}$	$L_x^{-1} = L_{\lambda(x)}$	$\lambda(x) \cdot xy = y$	LIP quas.	${}^{l}\mathfrak{I} = {}^{ls}\mathfrak{I}$
${}^{r}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x^{-1} = R_{\varrho(x)}$	$yx \cdot \varrho(x) = y$	RIP quas.	${}^{r}\mathfrak{I} = {}^{rs}\mathfrak{I}$

- *the parastrophy orbit of cross inverse quasigroups:*

${}^{l}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{rs}\mathcal{M} = {}^{ls}\mathcal{M}$	$L_x^{-1} = R_{\alpha(x)}$ $R_x^{-1} = L_{\alpha(x)}$	$\alpha(x) \cdot yx = y$	CIP quas. $MCIP$ quas.	$\mathfrak{C} = {}^{s}\mathfrak{C}$
${}^{\iota}\mathcal{M} = {}^{rs}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{r}\mathcal{M}$	$R_x^{-1} = M_{\beta(x)}$ $M_x^{-1} = R_{\beta(x)}$	$xy \cdot x = \beta(y)$	$LCIP$ quas.	${}^{l}\mathfrak{C} = {}^{ls}\mathfrak{C}$
${}^{ls}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{l}\mathcal{M}$	$L_x = M_{\gamma(x)}$ $M_x^{-1} = L_{\gamma(x)}^{-1}$	$x \cdot yx = \gamma(y)$	$RCIP$ quas.	${}^{r}\mathfrak{C} = {}^{rs}\mathfrak{C}$

- *the parastrophy orbit of mirror quasigroups:*

${}^{sr}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{l}\mathcal{M} = {}^{rs}\mathcal{M}$	$L_x = R_{\varphi(x)}$ $R_x^{-1} = L_{\varphi(x)}^{-1}$	$\varphi(x) \cdot y = y \cdot x$	MM quas.	$\mathfrak{M} = {}^{s}\mathfrak{M}$
${}^{r}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x = M_{\delta(x)}$ $M_x^{-1} = R_{\delta(x)}^{-1}$	$x \cdot xy = \delta(y)$	LM quas.	${}^{l}\mathfrak{M} = {}^{ls}\mathfrak{M}$
${}^{l}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{ls}\mathcal{M}$	$L_x^{-1} = M_{\xi(x)}$ $M_x^{-1} = L_{\xi(x)}$	$xy \cdot y = \xi(x)$	RM quas.	${}^{r}\mathfrak{M} = {}^{rs}\mathfrak{M}$

Proof: Let ${}^{\sigma}\mathcal{M} = {}^{\kappa}\mathcal{M}$ be an equality of abstract translation sets. Lemma 3 implies that the equality is parastrophically equivalent to ${}^{\iota}\mathcal{M} = {}^{\sigma^{-1}\kappa}\mathcal{M}$. That is why it is enough to consider the equalities

$$ {}^{\iota}\mathcal{M} = {}^{l}\mathcal{M}, \qquad {}^{\iota}\mathcal{M} = {}^{r}\mathcal{M}, \qquad {}^{\iota}\mathcal{M} = {}^{s}\mathcal{M}, \qquad \mathcal{M} = {}^{sl}\mathcal{M}, \qquad \mathcal{M} = {}^{sr}\mathcal{M}. $$

By Lemma 3, the equality ${}^{\iota}\mathcal{M} = {}^{rs}\mathcal{M}$ is parastrophically equivalent to ${}^{ls}\mathcal{M} = {}^{(ls)(rs)}\mathcal{M}$, i.e., ${}^{\iota}\mathcal{M} = {}^{ls}\mathcal{M}$ is parastrophically equivalent to ${}^{r}\mathcal{M} = {}^{rls}\mathcal{M}$, in other words, ${}^{r}\mathcal{M} = {}^{l}\mathcal{M}$.

The equality ${}^{\iota}\mathcal{M} = {}^{l}\mathcal{M}$ is parastrophically equivalent to ${}^{s}\mathcal{M} = {}^{sl}\mathcal{M}$ and this equality is equivalent to $({}^{s}\mathcal{M})^{-1} = ({}^{sl}\mathcal{M})^{-1}$. According to [16, (12)], it is ${}^{\iota}\mathcal{M} = {}^{r}\mathcal{M}$ which is parastrophically equivalent to ${}^{s}\mathcal{M} = {}^{sr}\mathcal{M}$.

Thus, each abstract translation set is parastrophically equivalent to at least one of the equalities

$$ {}^{\iota}\mathcal{M} = {}^{s}\mathcal{M}, \qquad {}^{r}\mathcal{M} = {}^{l}\mathcal{M}, \qquad {}^{sr}\mathcal{M} = {}^{r}\mathcal{M}. \tag{2} $$

We denote the quasigroup classes defined by these equalities with $\mathfrak{I}, \mathfrak{C}, \mathfrak{M}$, respectively.

By Lemma 3, s-parastrophes of these classes are respectively determined by

$$ {}^{s}\mathcal{M} = {}^{ss}\mathcal{M}, \qquad {}^{sr}\mathcal{M} = {}^{sl}\mathcal{M}, \quad {}^{ssr}\mathcal{M} = {}^{sr}\mathcal{M}. $$

Since $sr = ls$ and $sl = rs$, the second equality is ${}^{ls}\mathcal{M} = {}^{rs}\mathcal{M}$. By [16, (12)], it is $({}^{l}\mathcal{M})^{-1} = ({}^{r}\mathcal{M})^{-1}$ which is equivalent to the second equality in (2). Thence, s-parastrophe of each of these classes coincides with itself:

$$ {}^{s}\mathfrak{I} = \mathfrak{I}, \qquad {}^{s}\mathfrak{C} = \mathfrak{C}, \qquad {}^{s}\mathfrak{M} = \mathfrak{M}. \tag{3} $$

According to Corollary 1 to prove that each of the classes is singly symmetric, it is enough to prove that l-parastrophes of these classes does not coincide with themselves. For this aim, we will find conditions which determine these classes of quasigroups.

By Definition 1, the equalities (2) are respectively equivalent to the formulas

$$ (\exists\, \alpha)(\forall\, x, y) \quad x \overset{r}{\cdot} y = x \overset{rs}{\cdot} \mu(y), \qquad (\exists\, \beta)(\forall\, x, y) \quad x \cdot y = x \overset{rl}{\cdot} \alpha(y), $$
$$ (\exists\, \gamma)(\forall\, x, y) \quad x \overset{s}{\cdot} y = x \cdot \varphi(y). $$

As $rs = sl$, then

$$ (\exists\, \alpha)(\forall\, x, y) \quad x \overset{r}{\cdot} y = \mu(y) \overset{l}{\cdot} x, \qquad (\exists\, \alpha)(\forall\, x, y) \quad \alpha(y) = x \overset{l}{\cdot} xy, $$
$$ (\exists\, \alpha)(\forall\, x, y) \quad xy = y \cdot \varphi(x). $$

The first equality can be written as $\mu(y) = \big(x \overset{r}{\cdot} y\big) \cdot x$. Replace $x \overset{r}{\cdot} y$ with z, i.e., $xz = y$, consequenly $\mu(xz) = zx$. Therefore, the classes $\mathfrak{I}, \mathfrak{C}, \mathfrak{M}$ are respectively determined by:

$$(4)\qquad (\exists\,\alpha)(\forall\,x,y) \quad \mu(xy) = yx, \qquad (\exists\,\alpha)(\forall\,x,y) \quad \alpha(y)\cdot xy = x, \qquad (\exists\,\alpha)(\forall\,x,y) \quad xy = y\cdot\varphi(x).$$

There are at least two ways of finding the conditions to describe the σ-parastrophes ${}^{\sigma}\mathfrak{I}, {}^{\sigma}\mathfrak{C}, {}^{\sigma}\mathfrak{M}$ of these classes of quasigroups: 1) find σ-parastrophes of (4) replacing the main functional variable $(\cdot)$ with its σ^{-1}-parastrophe $\big(\overset{\sigma^{-1}}{\cdot}\big)$; 2) find σ-parastrophes of the equalities (2) using Lemma 3 and then find the conditions using Definition 1. We will illustrate both of them if $\sigma = l$ and $\sigma = r$.

Let $\sigma = l$. To find l-parastrophes of the conditions (4), we replace $(\cdot)$ with $\big(\overset{l}{\cdot}\big)$ because $l^{-1} = l$:

$$(\exists\,\alpha)(\forall\,x,y) \quad \lambda\big(x \overset{l}{\cdot} y\big) = y \overset{l}{\cdot} x, \qquad (\exists\,\alpha)(\forall\,x,y) \quad \beta(y) \overset{l}{\cdot} \big(x \overset{l}{\cdot} y\big) = x,$$
$$(\exists\,\alpha)(\forall\,x,y) \quad x \overset{l}{\cdot} y = y \overset{l}{\cdot} \delta(x).$$

Let $z := x \overset{l}{\cdot} y$, i.e., $zy = x$, then

$$(\exists\,\alpha)(\forall\,x,z) \quad \lambda(z) = y \overset{l}{\cdot} zy, \qquad (\exists\,\alpha)(\forall\,x,z) \quad \beta(y) \overset{l}{\cdot} z = zy,$$
$$(\exists\,\alpha)(\forall\,x,z) \quad z = y \overset{l}{\cdot} \delta(zy)$$

that is,

$$(\exists\,\alpha)(\forall\,x,z) \quad \lambda(z)\cdot zy = y, \qquad (\exists\,\alpha)(\forall\,x,z) \quad zy\cdot z = \beta(y),$$
$$(\exists\,\alpha)(\forall\,x,z) \quad z\cdot\delta(zy) = y.$$

We multiply the third equality by z from the left and replace zy with y in this way $z(z\cdot\lambda(y)) = y$. Thus, the quasigroup classes ${}^{l}\mathfrak{I}, {}^{l}\mathfrak{C}, {}^{l}\mathfrak{M}$ are described by the conditions

$$(5)\qquad (\exists\,\alpha)(\forall\,x,z) \quad \lambda(x)\cdot xy = y, \qquad (\exists\,\alpha)(\forall\,x,z) \quad xy\cdot x = \beta(y), \qquad (\exists\,\alpha)(\forall\,x,z) \quad x\cdot xy = \delta(y).$$

Let $\sigma = r$. To find r-parastrophes of the equalities (2) we use Lemma 3:

$${}^{r}\mathcal{M} = {}^{rs}\mathcal{M}, \qquad \mathcal{M} = {}^{rl}\mathcal{M}, \qquad {}^{rsr}\mathcal{M} = \mathcal{M},$$

that is,

$${}^{r}\mathcal{M} = {}^{sl}\mathcal{M}, \qquad \mathcal{M} = {}^{sr}\mathcal{M}, \qquad {}^{l}\mathcal{M} = \mathcal{M}.$$

By Definition 1 (for brief, we omit the quantifiers),

$$xy = x \overset{rsr}{\cdot} \varrho(y), \qquad x \overset{r}{\cdot} y = x \overset{s}{\cdot} \gamma(y), \qquad x \overset{rl}{\cdot} y = x \overset{r}{\cdot} \xi(y),$$

i.e.,

$$xy = x \overset{l}{\cdot} \varrho(y), \qquad x \overset{r}{\cdot} y = \gamma(y) \cdot x, \qquad y \overset{r}{\cdot} x = x \overset{r}{\cdot} \xi(y).$$

Using the properties of the left and right inverses, we have

$$xy \cdot \varrho(y) = x, \qquad x(\gamma(y) \cdot x) = y, \qquad x\big(y \overset{r}{\cdot} x\big) = \xi(y).$$

We replace y with $\gamma^{-1}(y)$ in the second equality and transform the third equality replacing $y \overset{r}{\cdot} x =: z$ and $yz = x$: $yz \cdot z = \xi(y)$.

Thus, the quasigroup classes ${}^l\mathfrak{I}, {}^l\mathfrak{C}, {}^l\mathfrak{M}$ are described by the conditions

$$yx \cdot \varrho(x) = y, \qquad x \cdot yx = \gamma(y), \qquad xy \cdot y = \xi(x).$$

Since S_3 acts on each of the sets $\mathrm{Po}(\mathfrak{I}), \mathrm{Po}(\mathfrak{C}), \mathrm{Po}(\mathfrak{M})$ and (3) holds, then according to Corollary 1 to show that each of the classes $\mathfrak{I}, \mathfrak{C}, \mathfrak{M}$ is singly symmetric, it is enough to prove the following inequalities:

$$\mathfrak{I} \neq {}^l\mathfrak{I}, \qquad \mathfrak{C} \neq {}^l\mathfrak{C}, \qquad \mathfrak{M} \neq {}^l\mathfrak{M}.$$

The first inequality is proved in [15].

Consider a quasigroup $(\mathbb{Z}_5; \circ)$, where $\mathbb{Z}_5$ is a ring of modulo 5 and

$$x \circ y := 2x + 2 + 3y.$$

Let $\alpha(x) := 3x + 1$, then $(\mathbb{Z}_5; \circ)$ has the cross invertible property:

$$\alpha(x) \circ (y \circ x) = 2\alpha(x) + 2 + 3(2y + 2 + 3x) = 2(3x + 1) + y + 3 - x = y,$$

that is, $(\mathbb{Z}_5; \circ)$ belongs to $\mathfrak{C}$ and therefore $\big(\mathbb{Z}_5; \overset{l}{\circ}\big)$ belongs to ${}^l\mathfrak{C}$. It is easy to verify that

$$x \overset{l}{\circ} y = 3x + 4 + y.$$

Suppose, the quasigroup $\big(\mathbb{Z}_5; \overset{l}{\circ}\big)$ belongs to $\mathfrak{C}$. It means that there is a bijection β of $\mathbb{Z}_5$ such that

$$\beta(x) \overset{l}{\circ} \big(y \overset{l}{\circ} x\big) = y$$

holds. This equality is equivalent to $3\beta(x) + 4 + (3y + 4 + x) = y$, i.e., $3\beta(x) = 4x + 2y + 3$ and so $\beta(x) = 2x + y + 4$. A contradiction, because $\beta(x)$ depends on y. Consequently, the quasigroup $\big(\mathbb{Z}_5; \overset{l}{\circ}\big)$ does not belong to $\mathfrak{C}$ and that is why $\mathfrak{C} \neq {}^l\mathfrak{C}$.

Let a quasigroup $(\mathbb{Z}_7;*)$ be defined by $x*y := 4x+2+4y$. The quasigroup is commutative, therefore it belongs to $\mathfrak{M}$ if $\alpha=\iota$. It is easy to verify that

$$x \overset{l}{*} y = 2x+3+6y.$$

The quasigroup $\big(\mathbb{Z}_7;\overset{l}{*}\big)$ belongs to ${}^{l}\mathfrak{M}$ by definition. If it belongs to $\mathfrak{M}$, then there exists a bijection γ of $\mathbb{Z}_7$ such that $\gamma(x)\overset{l}{*}y = y\overset{l}{*}x$ for all x and y in $\mathbb{Z}_7$. Consequently, $2\gamma(x)+3+6y = 2y+3+6x$ and suchwise $\gamma(x)=5y+3x$. But γ is a unary operation (bijection). Thus, the quasigroup $\big(\mathbb{Z}_7;\overset{l}{\circ}\big)$ does not belong to the class $\mathfrak{M}$ and in this way $\mathfrak{M}\neq{}^{l}\mathfrak{M}$.

Thereby, each of the classes $\mathfrak{I},\mathfrak{C},\mathfrak{M}$ is singly symmetric. It remains to prove, that all the orbits $\mathrm{Po}(\mathfrak{I}),\mathrm{Po}(\mathfrak{C}),\mathrm{Po}(\mathfrak{M})$ are pairwise different.

All groups belong to ${}^{l}\mathfrak{I}$. A group belonging to a class from $\mathrm{Po}(\mathfrak{C})$ or $\mathrm{Po}(\mathfrak{M})$ is commutative. Therefore, $\mathrm{Po}(\mathfrak{I})\neq\mathrm{Po}(\mathfrak{C})$ and $\mathrm{Po}(\mathfrak{I})\neq\mathrm{Po}(\mathfrak{M})$.

A loop belonging to a class from $\mathrm{Po}(\mathfrak{C})$ satisfies $xy\cdot y=x$ and so $yy=e$. Every commutative group belongs to the class $\mathfrak{M}$ and so a commutative group of degree greater than 2 belongs to no class from $\mathrm{Po}(\mathfrak{C})$. Therefore, $\mathrm{Po}(\mathfrak{C})\neq\mathrm{Po}(\mathfrak{M})$. □

For example when $\alpha=\beta=\gamma=\iota$, the subvariety is the variety of all semi-symmetric quasigroup, which are investigated in [9], [11], [12].

In the next theorems we find identities of varieties of the parastrophy orbits.

Theorem 5. *Each variety of the parastrophy orbit of inverse property quasigroups can be described by the following identities:*

Variety	*Defining formula*	*Inversion mapping*	*Defining identity*
$\mathfrak{I}={}^{s}\mathfrak{I}$	$xy=\mu(yx)$	$(\forall\, z)\ \mu=L_zR_z^{-1}={}^{ls}M_z{}^{rs}M_z$	$yx=z\cdot\big(xy\overset{l}{\cdot}z\big)$
${}^{l}\mathfrak{I}={}^{sr}\mathfrak{I}$	$\lambda(x)\cdot xy=y$	$(\forall\, z)\ \lambda=M_z^{-1}R_z={}^{s}M_z{}^{r}M_z$	$\big(z\overset{l}{\cdot}xz\big)\cdot xy=y$
${}^{r}\mathfrak{I}={}^{sl}\mathfrak{I}$	$yx\cdot\varrho(x)=y$	$(\forall\, z)\ \varrho=M_zL_z={}^{\iota}M_z{}^{ls}M_z$	$yx\cdot\big(zx\overset{r}{\cdot}z\big)=y$

Proof: In [15], it was proved that the middle, left and right IP quasigroup varieties belong to the same parastrophy orbit. We show these varieties as being defined by the identities presented in the table.

Let $\mathfrak{I}$ denote the class of all middle IP quasigroups, that is, the quasigroups $(Q;\cdot)$ with the equality $xy=\mu(yx)$ for some transformation μ of the set Q. The equality can be written as $L_y=\mu R_y$. Thus, $\mu=L_yR_y^{-1}$ for all $y\in Q$. Substituting $L_zR_z^{-1}$ for μ in $yx=\mu(xy)$, we have $yx=L_zR_z^{-1}(xy)$. Applying [16, (6)] and [16, (7)],

$$yx = z \cdot \big(xy \overset{l}{\cdot} z\big) \tag{6}$$

is obtained.

Vice versa, let $\big(Q; \cdot, \overset{l}{\cdot}, \overset{r}{\cdot}\big)$ be a quasigroup satisfying (6). The identity can be written as

$$z \cdot \big(x \overset{l}{\cdot} z\big) = y \cdot \big(x \overset{l}{\cdot} y\big), \qquad \text{i.e.,} \quad L_z R_z^{-1} = L_y R_y^{-1} := \mu \qquad \text{for all } y, z \in Q.$$

The bijection μ does not depend on y and z, therefore it is a functional constant. Replacing $z \cdot \big(x \overset{l}{\cdot} z\big)$ with $\mu(x)$ in the identity (6), we get $yx = \mu(xy)$. It means that the quasigroup has the middle invertible property.

Consequently, the class of quasigroups with the middle invertible property is a variety $\mathfrak{I}$ which is defined by the identity (6).

By Theorem 4, ${}^l\mathfrak{I}$ and ${}^r\mathfrak{I}$ are the classes of all left and right IP quasigroups, i.e., quasigroups $(Q; \cdot)$ and $(D; \circ)$, respectively, with the defining formulas

$$\lambda(x) \cdot xy = y \qquad \text{and} \qquad (y \circ x) \circ \varrho(x) = y.$$

According to the definitions of translations, these equalities can be written as

$$M_y \lambda(x) = R_y(x) \qquad \text{and} \qquad L_{y,\circ} x \circ \varrho(x) = y. \tag{7}$$

That is, we have

$$\lambda = M_y^{-1} R_y, \qquad \qquad \varrho = M_{y,\circ} L_{y,\circ}$$

for all $y \in Q$. Replacing λ with $M_z^{-1} R_z$ and ϱ with $M_{z,\circ} L_{z,\circ}$ in the equalities $\lambda(x) \cdot xy = y$ and $(y \circ x) \circ \varrho(x) = y$, respectively, we obtain:

$$M_z^{-1} R_z(x) \cdot xy = y \qquad \text{and} \qquad (y \circ x) \circ M_{z,\circ} L_{z,\circ}(x) = y.$$

Taking into account [16, (6)] and [16, (7)], we have

$$\big(z \overset{l}{\cdot} xz\big) \cdot xy = y \qquad \text{and} \qquad (y \circ x) \circ \big((z \circ x) \overset{r}{\circ} z\big) = y. \tag{8}$$

Vice versa, let $\big(Q; \cdot, \overset{l}{\cdot}, \overset{r}{\cdot}\big)$ and $\big(Q; \circ, \overset{l}{\circ}, \overset{r}{\circ}\big)$ be quasigroups satisfying (8). The identities can be written as

$$z \overset{l}{\cdot} xz = y \overset{l}{\cdot} xy, \qquad (y \circ x) \circ \big((z \circ x) \overset{r}{\circ} z\big) = y,$$

i.e.,

$$M_z^{-1} R_z = M_y^{-1} R_y := \lambda, \qquad M_{z,\circ} L_{z,\circ} = M_{y,\circ} L_{y,\circ} := \varrho$$

for all $y, z \in Q$. Hence, the bijections λ and ϱ do not depend on y and z and are functional constants. Replacing $z \overset{l}{\cdot} xz$ with $\lambda(x)$ and $(z \circ x) \overset{r}{\circ} z$ with $\varrho(x)$ in (8), the equalities $\lambda(x) \cdot xy = y$ and $(y \circ x) \circ \varrho(x) = y$ are obtained.

Thus, $\mathfrak{I}$, ${}^l\mathfrak{I}$ and ${}^r\mathfrak{I}$ are quasigroup varieties defined by the identities (6) and (8), respectively.

The theorem has been proved. □

Theorem 6. *Each variety of the parastrophy orbit of cross inverse property quasigroups can be described by the following identities:*

Variety	*Defining formula*	*Inversion mapping*	*Defining identity*
$\mathfrak{C} = {}^s\mathfrak{C}$	$xy \cdot \alpha(x) = y$	$(\forall z)\ \alpha = M_zR_z = {}^lM_z{}^rM_z$	$xy \cdot \big(xz \overset{r}{\cdot} z\big) = y$
${}^l\mathfrak{C} = {}^{sr}\mathfrak{C}$	$yx \cdot y = \beta(x)$	$(\forall z)\ \beta = R_zL_z = {}^rM_z{}^{ls}M_z$	$yx \cdot y = zx \cdot z$
${}^r\mathfrak{C} = {}^{sl}\mathfrak{C}$	$y \cdot xy = \gamma(x)$	$(\forall z)\ \gamma = L_zR_z = {}^{ls}M_z{}^rM_z$	$y \cdot xy = z \cdot xz$

Proof: Let $\mathfrak{C}$ denote the class of all *CIP* (*cross inverse property*) quasigroups, i.e., the quasigroups $(Q;\cdot)$ by the defining formula $xy \cdot \alpha(x) = y$ for some transformation α of the set Q.

By Corollary 2, the variety ${}^s\mathfrak{C}$ is defined by

$$\big(x \overset{s}{\cdot} y\big) \overset{s}{\cdot} \alpha(x) = y, \qquad \text{i.e.,} \qquad \alpha(x) \cdot yx = y.$$

Replacing x with $\alpha^{-1}(x)$, we get $x \cdot (y \cdot \alpha^{-1}(x)) = y$, therefore

$$x(y \cdot \alpha^{-1}(x)) \cdot \alpha^{-1}(x) = y \cdot \alpha^{-1}(x).$$

Changing y with $y \cdot \alpha^{-1}(x)$, we obtain $(x \cdot y) \cdot \alpha^{-1}(x) = y$. Since the formulas

$$(\exists\, \alpha)(\forall\, x, y) \quad xy \cdot \alpha(x) = y \qquad \text{and} \qquad (\exists\, \alpha)(\forall\, x, y) \quad xy \cdot \alpha^{-1}(x) = y$$

are equivalent, then $\mathfrak{C} = {}^s\mathfrak{C}$.

Pursuant to the definition of the right translation, the equality $xy \cdot \alpha(x) = y$ can be written as $R_yx \cdot \alpha(x) = y$. By the definition of a middle translation, we have $\alpha = M_yR_y$ for all $y \in Q$. Substituting M_zR_z for α in the equality $xy \cdot \alpha(x) = y$, the equality

$$xy \cdot M_zR_z(x) = y$$

is obtained.

Taking into account the equality $M_z(x) = x \overset{r}{\cdot} z$, we have

$$xy \cdot \big(xz \overset{r}{\cdot} z\big) = y. \tag{9}$$

Conversely, let $(Q;\cdot,\overset{l}{\cdot},\overset{r}{\cdot})$ be a quasigroup satisfying the identity (9). The identity can be written as

$$xy\overset{r}{\cdot}y=xz\overset{r}{\cdot}z,\qquad \text{i.e.,}\qquad M_yR_y=M_zR_z:=\alpha\qquad \text{for all } y,z\in Q.$$

Thus, the bijection α does not depend on y and z and is a functional constant. Replacing $xz\overset{r}{\cdot}z$ with $\alpha(x)$ in (9), $xy\cdot\alpha(x)=y$ is obtained. It means that the quasigroup has a cross invertible property.

Therefore, the class of quasigroups with the cross invertible property is a variety $\mathfrak{C}$ defined by the identity (9).

It remains to find the identities of the varieties and their inversion mappings. According to Corollary 2, the varieties ${}^{l}\mathfrak{C}$ and ${}^{r}\mathfrak{C}$ are respectively described by the formulas

$$\big(x\overset{l}{\circ}y\big)\overset{l}{\circ}\beta(x)=y\qquad\text{and}\qquad \big(x\overset{r}{*}y\big)\overset{r}{*}\gamma(x)=y$$

for some tranformations β and γ. In other words, the affiliation of the quasigroups $(A;\circ)$ and $(B;*)$ to ${}^{l}\mathfrak{C}$ and ${}^{r}\mathfrak{C}$ means that these identities hold. It is easy to verify that the transformations β and γ are invertible and the formulas are equivalent to

$$(10)\qquad (y\circ x)\circ y=\beta^{-1}(x)\qquad\text{and}\qquad x*(y*x)=\gamma^{-1}(y).$$

Thus, $\beta^{-1}=R_{y,\circ}L_{y,\circ}$ and $\gamma^{-1}=L_{x,*}R_{x,*}$ for all x,y. Since both β and γ are constant functions, then neither β nor γ depends on y and x, respectively. Hence, replacing them with a new variable z, we obtain the identities

$$(11)\qquad (y\circ x)\circ y=(z\circ x)\circ z,\qquad\qquad x*(y*x)=z*(y*z).$$

Vice versa, let the quasigroups $(A;\circ)$ and $(B;*)$ satisfy the identities (11). They can be written as $R_{y,\circ}L_{y,\circ}=R_{z,\circ}L_{z,\circ}=:\beta^{-1}$ and $L_{x,*}R_{x,*}=L_{z,*}R_{z,*}:=\gamma^{-1}$. It follows that both relationships do not depend on any variables and define some constant functions β and γ. That is why (11) implies (10). Thus, the identities (11) define the varieties ${}^{l}\mathfrak{C}$ and ${}^{r}\mathfrak{C}$.

The theorem has been proved. □

Theorem 7. *Each variety of the parastrophy orbit of mirror quasigroups can be described by the following identities:*

Variety	*Defining formula*	*Inversion mapping*	*Defining identity*
$\mathfrak{M}={}^{s}\mathfrak{M}$	$\varphi(x)\cdot y=y\cdot x$	$(\forall\, z)\ \ \varphi=R_z^{-1}L_z={}^{rs}M_z{}^{ls}M_z$	$\big(zx\overset{l}{\cdot}z\big)\cdot y=yx$
${}^{l}\mathfrak{M}={}^{sr}\mathfrak{M}$	$y\cdot yx=\delta(x)$	$(\forall\, z)\ \ \delta=L_z^2=({}^{ls}M_z)^2$	$y\cdot yx=z\cdot zx$
${}^{r}\mathfrak{M}={}^{sl}\mathfrak{M}$	$xy\cdot y=\xi(x)$	$(\forall\, z)\ \ \xi=R_z^2=({}^{r}M_z)^2$	$xy\cdot y=xz\cdot z$

Proof: Let $\mathfrak{M}$ denote the class of quasigroups defined by the following condition: there exists a transformation φ such that the equality $\varphi(x)\cdot y = y\cdot x$ is true for all x, y. According to Corollary 2, the variety ${}^{s}\mathfrak{M}$ is defined by $\varphi(x)\overset{s}{\cdot} y = y\overset{s}{\cdot} x$, i.e., $y\cdot\varphi(x) = x\cdot y$. As a result of fixing y, φ is bijective. Accordingly, x can be replaced with $\varphi^{-1}(x)$, and we get $y\cdot x = \varphi^{-1}(x)\cdot y$, suchwise $\varphi^{-1}(x)\cdot y = y\cdot x$. Since the formulas

$$(\exists\,\alpha)(\forall\,x,y)\quad \varphi(x)\cdot y = y\cdot x \qquad\text{and}\qquad (\exists\,\varphi)(\forall\,x,y)\quad \varphi^{-1}(x)\cdot y = y\cdot x$$

are equivalent, then $\mathfrak{M} = {}^{s}\mathfrak{M}$. The formula $\varphi(x)\cdot y = y\cdot x$ can be written as $R_y\varphi = L_y$. Thence, $\varphi = R_y^{-1}L_y$ for all $y\in Q$. Substituting $R_z^{-1}L_z$ for φ in the equality $\varphi(x)\cdot y = y\cdot x$, we obtain $R_z^{-1}L_z(x)\cdot y = y\cdot x$. Taking into account the equality $R_z^{-1}(x) = x\overset{l}{\cdot} z$, we have

$$\left(zx\overset{l}{\cdot} z\right)\cdot y = yx. \tag{12}$$

Vice versa, let $\left(Q;\cdot,\overset{l}{\cdot},\overset{r}{\cdot}\right)$ be a quasigroup satisfying the identity (12) that is equivalent to $zx\overset{l}{\cdot} z = yx\overset{l}{\cdot} y$, i.e., $R_z^{-1}L_z = R_y^{-1}L_y := \varphi$ for all $y,z\in Q$. Hence, the bijection φ is a functional constant. Replacing $zx\overset{l}{\cdot} z$ with $\varphi(x)$ in the identity (12), we get $\varphi(x)\cdot y = yx$. Thus, the class of quasigroups which is defined by the identity (12) is a variety $\mathfrak{M}$.

The identities of the varieties and the inversion mapping still need to be found. According to Corollary 2, the varieties ${}^{l}\mathfrak{M}$ and ${}^{r}\mathfrak{M}$ are respectively described by the formulas

$$\delta(x)\overset{l}{*} y = y\overset{l}{*} x, \qquad\qquad \xi(x)\overset{r}{\cdot} y = y\overset{r}{\cdot} x$$

for some tranformations δ and ξ. Namely, the affiliation of quasigroups $(A;*)$ and $(B;\cdot)$ to ${}^{l}\mathfrak{M}$ and ${}^{r}\mathfrak{M}$, respectively, means that these identities hold. As a result of fixing y, both β and ξ are bijective. Using the definitions of the left inverse $\left(\overset{l}{\circ}\right)$ and the right inverse $\left(\overset{r}{\circ}\right)$, it is easy to verify that the formulas are equivalent to

$$y*(y*x) = \delta(x) \qquad\text{and}\qquad (y\cdot x)\cdot x = \xi(y). \tag{13}$$

Therefore, $\delta = L^2_{y,*}$ and $\xi = R^2_{x,\cdot}$ for all x, y. Since both δ and ξ are constant functions, then neither δ nor ξ depend on y and x. Hence, replacing them with a new variable z, we obtain

$$y*(y*x) = L^2_{y,*}(x) \qquad\text{and}\qquad (y\cdot x)\cdot x = R^2_{x,\cdot}(y)$$

which are respectively equivalent to the identities

$$y*(y*x) = z*(z*x) \qquad\text{and}\qquad (y\cdot x)\cdot x = (y\cdot z)\cdot z. \tag{14}$$

Conversely, let quasigroups $(A;*)$ and $(B;\cdot)$ simultaneously satisfy the identities (14). These identities can be written as $L^2_{y,*} = L^2_{z,*} =: \delta$ and $R^2_{x,\cdot} = R^2_{z,\cdot} := \xi$. Thereby, both relationships do not depend on variables and define some constant functions δ and ξ. That is why (14) implies (13).

Thus, the identities (14) define the varieties ${}^{l}\mathfrak{M}$ and ${}^{r}\mathfrak{M}$, respectively. The theorem has been proved. $\square$

The identities $yx \cdot y = zx \cdot z$, $y \cdot xy = z \cdot xz$, $y \cdot yx = z \cdot zx$, $xy \cdot y = xz \cdot z$ from Theorem 6 and Theorem 7 were found in [10]. See also [3].

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