Preservation of properties of a map by forcing

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Abstract. Let $f\colon X \to Y$ be a continuous map such as an open map, a closed map or a quotient map. We study under what circumstances f remains an open, closed or quotient map in forcing extensions.

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1. Introduction

All spaces are T_1 and all maps are surjective. Let $\mathbf{V}$ be a ground model and let $\mathbf{V}^{\mathbb{P}}$ denote the extension of $\mathbf{V}$ by a forcing $\mathbb{P}$. For a topological space $(X, \mathcal{T})$ in $\mathbf{V}$, we define a topological space $(X, \mathcal{T}^{\mathbb{P}})$ in $\mathbf{V}^{\mathbb{P}}$ such that $\mathcal{T}^{\mathbb{P}}$ is the topology generated by $\mathcal{T}$ in $\mathbf{V}^{\mathbb{P}}$. By definition $\mathcal{T}$ is a base for $\mathcal{T}^{\mathbb{P}}$.

In [3] R. Grunberg, L. R. Junqueira, and F. D. Tall studied for a normal space $(X, \mathcal{T})$ conditions under which $(X, \mathcal{T}^{\mathbb{P}})$ remains a normal space with the Cohen forcing $\mathbb{P}$. In [4] we studied for a countably compact space or a pseudocompact space $(X, \mathcal{T})$, conditions under which $(X, \mathcal{T}^{\mathbb{P}})$ remains a countably compact space or a pseudocompact space.

Consider a map

$$f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y).$$

Suppose that f has property φ such as being a closed map. We say that a forcing $\mathbb{P}$ preserves property φ if, in $\mathbf{V}^{\mathbb{P}}$, the map

$$f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$$

still has property φ. In this note, we study under what circumstances various properties of a map are preserved by forcing. We are interested in the following maps. Here $\operatorname{int} A$ denotes the interior of a set A.

Definition 1.1. Let $f\colon X \to Y$ be a continuous map. Then f is called an *open map* (or *closed map*) if for every open (or closed) subset S of X, $f(S)$ is open (closed, respectively) in Y.

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We call f a *perfect map* if it is a closed map and for every $y \in Y$, $f^{-1}(y)$ is compact.

We call f a *pseudo-open map* if for every $y \in Y$ and every open set $U \subseteq X$ with $f^{-1}(y) \subseteq U$, it is true that $y \in \operatorname{int} f(U)$.

We call f a *quotient map* if for every $U \subseteq Y$, $f^{-1}(U)$ is open in X if and only if U is open in Y; equivalently, f is a quotient map if for every $F \subseteq Y$, $f^{-1}(F)$ is closed in X if and only if F is closed in Y.

Let us look at some properties of a map that are preserved by forcing.

Theorem 1.2. *Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. For any forcing $\mathbb{P}$, the following holds:*

(1) *If f is a continuous map, then $f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ is a continuous map.*
(2) *If f is an open map, then $f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ is an open map.*
(3) *If f is a homeomorphism, then $f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ is a homeomorphism.*

Proof: (1) To show that f remains continuous in $\mathbf{V}^{\mathbb{P}}$, take $W \in \mathcal{T}_Y^{\mathbb{P}}$. We will show that $f^{-1}(W) \in \mathcal{T}_X^{\mathbb{P}}$. We can write $W = \bigcup\{U_\xi \in \mathcal{T}_Y : \xi \in \Gamma\}$ for some index set Γ. Then $f^{-1}(W) = \bigcup\{f^{-1}(U_\xi) \colon \xi \in \Gamma\}$. Since $f^{-1}(U_\xi) \in \mathcal{T}_X$ for each $\xi \in \Gamma$, we have $f^{-1}(W) \in \mathcal{T}_X^{\mathbb{P}}$.

(2) By (1), f remains continuous in $\mathbf{V}^{\mathbb{P}}$. To show that f is open in $\mathbf{V}^{\mathbb{P}}$, take $W \in \mathcal{T}_X^{\mathbb{P}}$. Write $W = \bigcup\{U_\xi \in \mathcal{T}_X : \xi \in \Gamma\}$. Then $f(W) = \bigcup\{f(U_\xi) \colon \xi \in \Gamma\}$. Since $f(U_\xi) \in \mathcal{T}_Y$ for each $\xi \in \Gamma$, we have $f(W) \in \mathcal{T}_Y^{\mathbb{P}}$.

(3) Being one-to-one and being onto are preserved by any forcing. Apply (1) and (2). $\square$

At the end of this section, let us look at a simple example where forcing destroys a closed map. We introduce some notations. Let $\mathbb{C}$ be the Cohen forcing that adds a Cohen real ([8, Definition 5.1], $\mathbb{C} = Fn(\omega, 2)$). Denote by $[0,1]^{\mathbf{V}}$ the closed interval in $\mathbf{V}$ with the usual topology, and $[0,1]^{\mathbf{V}^{\mathbb{C}}}$ the closed interval in $\mathbf{V}^{\mathbb{C}}$ with the usual topology. We have $[0,1]^{\mathbf{V}} \subsetneqq [0,1]^{\mathbf{V}^{\mathbb{C}}}$.

Example 1.3. There is a closed map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ such that $f\colon (X, \mathcal{T}_X^{\mathbb{C}}) \to (Y, \mathcal{T}_Y^{\mathbb{C}})$ is not a closed map.

Proof: Let

$$X = [0,1]^{\mathbf{V}} \times [0,1]^{\mathbf{V}}, \qquad \text{and} \qquad Y = [0,1]^{\mathbf{V}}.$$

Define $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ such that $f((a,b)) = a$ for all $(a,b) \in X$. In $\mathbf{V}$, f is a closed map. In $\mathbf{V}^{\mathbb{C}}$, pick $r \in [0,1]^{\mathbf{V}^{\mathbb{C}}} \setminus [0,1]^{\mathbf{V}}$. Let $\mathbb{N}$ be the set of all natural numbers. Consider the decimal representation of $r = 0.n_1 n_2 n_3 \ldots n_i \ldots$, where $n_i \in \{0, 1, \dots, 9\}$ for each $i \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $q_k = 0.n_1 n_2 \dots n_k$. The set $A := \{(1/k, q_k) \colon k \in \mathbb{N}\}$ has no accumulation point in X because $(0, r)$ is not in X, so A is a closed subset of $(X, \mathcal{T}_X^{\mathbb{C}})$. But $f(A) = \{1/k \colon k \in \mathbb{N}\}$ is not a closed subset of $(Y, \mathcal{T}_Y^{\mathbb{C}})$ because 0 is an accumulation point of $f(A)$. Thus, $f \colon (X, \mathcal{T}_X^{\mathbb{C}}) \to (Y, \mathcal{T}_Y^{\mathbb{C}})$ is not a closed map. $\square$

In this example, f is an open map because it is a projection. By Theorem 1.2 (2), f remains open in $\mathbf{V}^{\mathbb{C}}$. In the last section, we give an example where f is a closed map in $\mathbf{V}$, but in $\mathbf{V}^{\mathbb{C}}$ f is not even a quotient map (Example 3.6).

2. Preserving a closed map

In the diagram below, "$\varphi \to \psi$" means that if a map has property φ, then it has property ψ. Properties preserved by any forcing are in boldface.

$$
\begin{tikzcd}
\textbf{homeomorphic} \arrow[d] \arrow[ddr] & \\
\text{perfect} \arrow[d] & \\
\text{closed} \arrow[d] & \textbf{open} \arrow[dl] \\
\text{pseudo-open} \arrow[d] & \\
\text{quotient} \arrow[d] & \\
\textbf{continuous} &
\end{tikzcd}
$$

By Theorem 1.2, "homeomorphic", "open" and "continuous" are preserved by any forcing, but all other properties can be destroyed by forcing. In this section, we study conditions under which the properties in the diagram are preserved. The following lemma is useful for our purpose. A topological space is said to be *scattered* if every nonempty subspace contains an isolated point.

Lemma 2.1 ([7, Lemma 7]). *If $(X, \mathcal{T})$ is a compact scattered space, then $(X, \mathcal{T}^{\mathbb{P}})$ is a compact scattered space for any forcing $\mathbb{P}$.*

We use the following characterization of a closed map.

Lemma 2.2 ([2, Theorem 1.4.13]). *For a continuous map $f\colon X \to Y$, the following are equivalent:*

(1) *f is a closed map.*
(2) *For every $y \in Y$ and every open set $W \subseteq X$ with $f^{-1}(y) \subseteq W$, there exists an open set $H \subseteq Y$ such that $y \in H$ and $f^{-1}(H) \subseteq W$.*

We show that if the boundary of each fiber is compact and scattered, then a closed map remains closed in any forcing extension. Here ∂A denotes the boundary of a set A.

Theorem 2.3. *Suppose that $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a closed map such that for every $y \in Y$, $\partial f^{-1}(y)$ is compact and scattered. Then for any forcing $\mathbb{P}$, $f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ is a closed map.*

Proof: The map f remains continuous in $\mathbf{V}^{\mathbb{P}}$ by Theorem 1.2 (1). We use Lemma 2.2 to show that f remains closed in $\mathbf{V}^{\mathbb{P}}$. Let $y \in Y$ and take $W \in \mathcal{T}_X^{\mathbb{P}}$ such that $f^{-1}(y) \subseteq W$. We will find $H \in \mathcal{T}_Y^{\mathbb{P}}$ such that $y \in H$ and $f^{-1}(H) \subseteq W$. We are assuming that all spaces are T_1 in this paper, so $\{y\}$ is a closed set. Since f is continuous, $f^{-1}(y)$ is closed, and we have $\partial f^{-1}(y) \subseteq f^{-1}(y)$. In $\mathbf{V}^{\mathbb{P}}$, for each $x \in \partial f^{-1}(y)$ choose $U_x \in \mathcal{T}_X$ such that $x \in U_x \subseteq W$. Then $\{U_x : x \in \partial f^{-1}(y)\}$ is an open cover of $\partial f^{-1}(y)$ in $\mathbf{V}^{\mathbb{P}}$, and by Lemma 2.1, $\partial f^{-1}(y)$ remains compact in $\mathbf{V}^{\mathbb{P}}$, so we can choose a finite subcover, say $\{U_{x_i} : i < n\}$. We have $f^{-1}(y) \cup \bigcup_{i<n} U_i \in \mathcal{T}_X$. Using Lemma 2.2 and the fact that f is a closed map in $\mathbf{V}$, we can find $H \in \mathcal{T}_Y$ such that $y \in H$ and $f^{-1}(H) \subseteq f^{-1}(y) \cup \bigcup_{i<n} U_i$. Since $f^{-1}(y) \cup \bigcup_{i<n} U_i \subseteq W$, we have $f^{-1}(H) \subseteq W$. $\square$

In a similar way, we can prove that being a pseudo-open map is preserved by any forcing if the boundary of each fiber is compact and scattered.

Proposition 2.4. *Suppose that $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a pseudo-open map such that for each $y \in Y$, $\partial f^{-1}(y)$ is compact and scattered. Then for any forcing $\mathbb{P}$, $f\colon (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ is a pseudo-open map.*

Proof: By Theorem 1.2 (1), f is continuous in $\mathbf{V}^{\mathbb{P}}$. To show that f is pseudo-open in $\mathbf{V}^{\mathbb{P}}$, let $y \in Y$ and take $W \in \mathcal{T}_X^{\mathbb{P}}$ such that $f^{-1}(y) \subseteq W$. We will show that $y \in \mathrm{int} f(W)$. Using the same argument in the proof of Theorem 2.3, we can find a finite open cover $\{U_i : i < n\}$ of $\partial f^{-1}(y)$ such that $U_i \in \mathcal{T}_X$ and $U_i \subseteq W$ for each $i < n$, and so $f^{-1}(y) \cup \bigcup_{i<n} U_i \in \mathcal{T}_X$. By the fact that f is pseudo-open in $\mathbf{V}$, we have $y \in \mathrm{int} f(f^{-1}(y) \cup \bigcup_{i<n} U_i)$. Since $f^{-1}(y) \cup \bigcup_{i<n} U_i \subseteq W$, we have $y \in \mathrm{int} f(W)$. $\square$

According to the example below, we cannot weaken the condition that the boundary of each fiber is compact in Theorem 2.3 to the condition that it is countably compact, even if the map is open.

Example 2.5. There are a closed and open map $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ and a ccc forcing $\mathbb{P}$ such that

(1) for each $y\in Y$, $\partial f^{-1}(y)$ is countably compact and scattered, and
(2) $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is not a closed map.

PROOF: In [2, Problem 3.12.17 (d)], $\gamma'\mathbb{N}$ is a compactification of $\mathbb{N}$ whose remainder coincides with an ordinal $\delta+1$. Let $Z=\gamma'\mathbb{N}\setminus\{\delta\}$. Then Z is a separable, scattered, non-compact, countably compact, locally compact space. Let

$$X=(\omega+1)\times Z, \qquad \text{and} \qquad Y=\omega+1,$$

where $\omega+1$ is equipped with the usual order topology. Define a map $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ such that $f((a,b))=a$ for all $(a,b)\in X$. By [2, Theorem 3.10.7], f is a closed map. Moreover, f is an open map because it is a projection. For each $n\in\omega$, $\partial f^{-1}(n)=\partial(\{n\}\times Z)=\emptyset$, and $\partial f^{-1}(\omega)=\partial(\{\omega\}\times Z)=\{\omega\}\times Z$, which is homeomorphic to Z, so $\partial f^{-1}(y)$ is countably compact and scattered for each $y\in Y$. Let $\mathbb{P}=\mathbb{B}(\mathcal{F})$ be the forcing in [4, Lemma 2.7], where $\mathcal{F}$ is a free filter on a countable dense subset of Z. Then $\mathbb{P}$ is ccc and, in $\mathbf{V}^{\mathbb{P}}$, Z contains an infinite closed discrete subset, say $\{a_i\colon i\in\omega\}$. The set $\{(i,a_i)\colon i\in\omega\}$ has no accumulation point in $(X,\mathcal{T}_X^{\mathbb{P}})$, and so it is a closed subset of $(X,\mathcal{T}_X^{\mathbb{P}})$. But $f(\{(i,a_i)\colon i\in\omega\})=\omega$, which is not closed in $(Y,\mathcal{T}_Y^{\mathbb{P}})$. Thus, $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is not a closed map. □

If we assume that a space X is separable and regular, then we can obtain a partial converse of Theorem 2.3.

Theorem 2.6. *Let X be a separable regular space. Suppose that $f\colon X\to Y$ is a closed map. If f remains a closed map in any ccc forcing extension, then for each $y\in Y$, $\partial f^{-1}(y)$ is countably compact and scattered.*

PROOF: We prove by contrapositive. Assume that there is a point $y\in Y$ such that $\partial f^{-1}(y)$ is non-countably compact or non-scattered. Since X is separable, $X\setminus\overline{f^{-1}(y)}$ is separable. Let C be a countable dense subset of $X\setminus\overline{f^{-1}(y)}$. For each $x\in\partial f^{-1}(y)$, we have $x\in\overline{C}$. Using the notation in [5, Definition 2.3], we have $bd_c^*(f^{-1}(y))=\partial f^{-1}(y)$. Since $bd_c^*(f^{-1}(y))$ is non-countably compact or non-scattered, by [5, Theorem 2.4], there is a ccc forcing $\mathbb{P}$ which destroys a neighborhood base of $f^{-1}(y)$; that is, there is $W\in\mathcal{T}_X^{\mathbb{P}}$ with $f^{-1}(y)\subseteq W$ such that for all $U\in\mathcal{T}_X$ with $f^{-1}(y)\subseteq U$, we have $U\nsubseteq W$. We use Lemma 2.2 to show that $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is not a closed map. Take $H\in\mathcal{T}_Y^{\mathbb{P}}$ such that $y\in H$; we will show that $f^{-1}(H)\nsubseteq W$. Choose $H'\in\mathcal{T}_Y$ such that $y\in H'\subseteq H$. Since $f^{-1}(H')\in\mathcal{T}_X$ and $f^{-1}(y)\subseteq f^{-1}(H')$, we have $f^{-1}(H')\nsubseteq W$. Thus, $f^{-1}(H)\nsubseteq W$. □

Perfect maps are preserved by forcing if each fiber is scattered.

Proposition 2.7. *Suppose that $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is a perfect map such that for each $y\in Y$, $f^{-1}(y)$ is scattered. Then for any forcing $\mathbb{P}$, $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a perfect map.*

Proof: For each $y\in Y$, $f^{-1}(y)$ is compact and scattered, and so it remains compact in any forcing extension by Lemma 2.1. Since $\partial f^{-1}(y)$ is a closed subset of $f^{-1}(y)$, it is compact and scattered. By Theorem 2.3, f remains a closed map in any forcing extension. $\square$

Combining previous results, we can show that if X is a compact scattered space, then all maps in Definition 1.1 are preserved.

Proposition 2.8. *Let $(X,\mathcal{T}_X)$ be a compact scattered space and let $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a map. For any forcing $\mathbb{P}$, the following hold:*

1. *If f is a perfect map, then $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a perfect map.*
2. *If f is a closed map, then $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a closed map.*
3. *If f is a pseudo-open map, then $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a pseudo-open map.*
4. *If f is a quotient map, then $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a quotient map, assuming that $(Y,\mathcal{T}_Y)$ is Hausdorff.*

Proof: (1) is by Proposition 2.7. (2) is by Theorem 2.3. (3) is by Proposition 2.4. To prove (4), let $E\subseteq Y$ in $\mathbf{V}^{\mathbb{P}}$ and suppose that $f^{-1}(E)$ is a closed subset of $(X,\mathcal{T}_X^{\mathbb{P}})$. We want to show that E is closed in $(Y,\mathcal{T}_Y^{\mathbb{P}})$. Since $(X,\mathcal{T}_X^{\mathbb{P}})$ is compact by Lemma 2.1, $f^{-1}(E)$ is compact in $(X,\mathcal{T}_X^{\mathbb{P}})$. Since $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is continuous by Theorem 1.2(1), $f(f^{-1}(E))=E$ is compact in $(Y,\mathcal{T}_Y^{\mathbb{P}})$. Forcing preserves Hausdorffness, so $(Y,\mathcal{T}_Y^{\mathbb{P}})$ is Hausdorff. Thus, E is a closed subset of $(Y,\mathcal{T}_Y^{\mathbb{P}})$. $\square$

3. Preserving a quotient map

In order for a quotient map to be preserved by forcing, in addition to $\partial f^{-1}(y)$ being compact and scattered, $\{f^{-1}(y)\colon\ y\in Y,\ |f^{-1}(y)|>1\}$ needs to be a discrete family.

Theorem 3.1. *Let $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a quotient map. Suppose that $\partial f^{-1}(y)$ is compact and scattered for each $y\in Y$. Further, suppose that $\{f^{-1}(y)\colon y\in Y$ and $|f^{-1}(y)|>1\}$ is a discrete family. Then for any forcing $\mathbb{P}$, $f\colon (X,\mathcal{T}_X^{\mathbb{P}})\to(Y,\mathcal{T}_Y^{\mathbb{P}})$ is a quotient map.*

Proof: Let $\mathbb{P}$ be a forcing. By Theorem 1.2 (1), f is continuous in $\mathbf{V}^{\mathbb{P}}$. Suppose that $f^{-1}(E)$ is an open subset of $(X, \mathcal{T}_X^{\mathbb{P}})$ for some $E \subseteq Y$. We will show that E is an open subset of $(Y, \mathcal{T}_Y^{\mathbb{P}})$. Fix $y \in E$. We will find $H \in \mathcal{T}_Y$ such that $y \in H \subseteq E$. If $\partial f^{-1}(y) = \emptyset$, then $f^{-1}(y)$ is open. Since f is a quotient map, y is an isolated point, and $y \in \{y\} \subseteq E$. So suppose that $\partial f^{-1}(y) \neq \emptyset$. For each $x \in \partial f^{-1}(y)$, we can pick $U \in \mathcal{T}_X$ such that $x \in U \subseteq f^{-1}(E)$ and $U \cap f^{-1}(w) \neq \emptyset$ for at most one $w \in Y$ with $|f^{-1}(w)| > 1$. If $w = y$, then let $U_x = U$. If $w \neq y$, then let $U_x = U \setminus f^{-1}(w)$. Since Y is T_1, $\{w\}$ is closed so $f^{-1}(w)$ is closed, and so U_x is open. In either case, we have

$$(*) \qquad (\forall\, x \in \partial f^{-1}(y))\, [f^{-1} f(U_x) \subseteq U_x \cup f^{-1}(y)].$$

We have that $\{U_x \colon x \in \partial f^{-1}(y)\}$ is an open cover of $\partial f^{-1}(y)$ in $\mathbf{V}^{\mathbb{P}}$ and $\partial f^{-1}(y)$ remains compact in $\mathbf{V}^{\mathbb{P}}$ by Lemma 2.1, so there is a finite subcover, say $\{U_{x_i} \colon i < n\}$. Let $V = f^{-1}(y) \cup \bigcup_{i<n} U_{x_i}$. Then $V \in \mathcal{T}_X$ and $V \subseteq f^{-1}(E)$.

Claim 3.2. *It holds that $V = f^{-1}(f(V))$.*

Proof: We have $f(V) = \{y\} \cup \bigcup_{i<n} f(U_{x_i})$. By $(*)$, $f^{-1}(f(V)) = f^{-1}(y) \cup \bigcup_{i<n} f^{-1}(f(U_{x_i})) \subseteq f^{-1}(y) \cup \bigcup_{i<n} U_{x_i} = V$. □

By the claim and the fact that f is a quotient map in $\mathbf{V}$, we have $f(V) \in \mathcal{T}_Y$. We also have $y \in f(V) \subseteq E$. □

We give an example of a quotient map such that each fiber is compact and scattered, but after adding a Cohen real, it becomes a non-quotient map.

Example 3.3. There is a quotient map $f \colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ such that

(1) for each $y \in Y$, $\partial f^{-1}(y)$ is compact and scattered, and
(2) $f \colon (X, \mathcal{T}_X^{\mathbb{C}}) \to (Y, \mathcal{T}_Y^{\mathbb{C}})$ is not a quotient map.

Proof: For each $n \in \omega$, let $\langle a_n(i) \colon i < \omega \rangle$ be a sequence of distinct points converging to a_n. Let $\langle b_n \colon n < \omega \rangle$ be a sequence of distinct points converging to p. Let $(X, \mathcal{T}_X)$ be the topological sum of these convergent sequences, each of which is homeomorphic to $\omega+1$ with the usual order topology. Let $(Y, \mathcal{T}_Y)$ be the quotient space of $(X, \mathcal{T}_X)$, where a_n and b_n are identified for each $n \in \omega$. ($(Y, \mathcal{T}_Y)$ is called Arens' space, see [1].) Let $\tilde{a}_n \in Y$ denote the point where a_n and b_n are identified. We have

$$X = \{p\} \cup \{b_n \colon n \in \omega\} \cup \{a_n \colon n \in \omega\} \cup \bigcup_{n \in \omega} \{a_n(i) \colon i \in \omega\};$$

$$Y = \{p\} \cup \{\tilde{a}_n \colon n \in \omega\} \cup \bigcup_{n \in \omega} \{a_n(i) \colon i \in \omega\}.$$

Let $f\colon (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ be the induced quotient map. For each $y \in Y$, we have $|f^{-1}(y)| \le 2$, so $\partial f^{-1}(y)$ is compact and scattered. A neighborhood of p in $(Y,\mathcal{T}_Y)$ has the form

$$(**) \qquad \{p\} \cup \{\tilde{a}_n \colon n > k\} \cup \{a_n(i)\colon n > k \text{ and } i > h(n)\}$$

for some $k \in \omega$ and $h \in {}^{\omega}\omega \cap \mathbf{V}$.

Now we work in $\mathbf{V}^{\mathbb{C}}$. Cohen reals are unbounded in ${}^{\omega}\omega \cap \mathbf{V}$; that is, there is a function $g \in {}^{\omega}\omega \cap \mathbf{V}^{\mathbb{C}}$ such that for every $h \in {}^{\omega}\omega \cap \mathbf{V}$, $\{n \in \omega\colon h(n) < g(n)\}$ is infinite ([6, Lemma 15.30 (ii)]). Let $E = \{a_n(g(n))\colon n \in \omega\}$. Then E is a closed subset of $(X,\mathcal{T}_X^{\mathbb{C}})$, and $f(E) = E$ meets every set of the form $(**)$, so $p \in \overline{E}$, and thus $f(E) = E$ is not closed in $(Y,\mathcal{T}_Y^{\mathbb{C}})$. Since $E = f^{-1}(f(E))$, $f\colon (X,\mathcal{T}_X^{\mathbb{C}}) \to (Y,\mathcal{T}_Y^{\mathbb{C}})$ is not a quotient map. $\square$

Remark 3.4. The quotient map f in the proof of Example 3.3 cannot satisfy the hypothesis of Theorem 3.1. Indeed, for each $n \in \omega$, we have $f^{-1}(\tilde{a}_n) = \{a_n, b_n\}$, and p is an accumulation point of the family $\{\{a_n, b_n\}\colon n \in \omega\}$.

Remark 3.5. By Proposition 2.4, the quotient map f in the proof of Example 3.3 cannot be a pseudo-open map. Indeed, $U := \{p\} \cup \{b_n\colon n \in \omega\}$ is an open subset of X which contains $\{p\} = f^{-1}(p)$, but $p \notin \operatorname{int} f(U)$ because $\operatorname{int} f(U) = \emptyset$.

Before concluding this note, we give an example where a perfect map becomes a non-quotient map by adding a Cohen real. For the definition of $[0,1]^{\mathbf{V}}$ and $[0,1]^{\mathbf{V}^{\mathbb{C}}}$, see the paragraph before Example 1.3.

Example 3.6. There is a perfect map $f\colon (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ such that $f\colon (X,\mathcal{T}_X^{\mathbb{C}}) \to (Y,\mathcal{T}_Y^{\mathbb{C}})$ is not a quotient map.

Proof: Let

$$X = [0,1]^{\mathbf{V}} \times [0,1]^{\mathbf{V}}, \qquad \text{and} \qquad Y = \{0^*\} \cup ([0,1]^{\mathbf{V}} \times (0,1]^{\mathbf{V}}),$$

where X is equipped with the usual Euclidean topology and Y is the quotient space of X where points in $[0,1]^{\mathbf{V}} \times \{0\}$ are identified as 0^*. Let $f\colon (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ be the induced quotient map, so $f((x,0)) = 0^*$ for all $x \in [0,1]^{\mathbf{V}}$. Clearly, f is a perfect map. Let $\mathbb{N}$ be the set of all natural numbers. For each $n \in \mathbb{N}$, let $U_n = \{0^*\} \cup ([0,1]^{\mathbf{V}} \times (0,1/n)^{\mathbf{V}})$. Then $\{U_n\colon n \in \mathbb{N}\}$ is a neighborhood base of 0^* in $(Y,\mathcal{T}_Y)$.

Now we work in $\mathbf{V}^{\mathbb{C}}$ and prove that $f\colon (X,\mathcal{T}_X^{\mathbb{C}}) \to (Y,\mathcal{T}_Y^{\mathbb{C}})$ is not a quotient map. Pick $r \in [0,1]^{\mathbf{V}^{\mathbb{C}}} \setminus [0,1]^{\mathbf{V}}$ and define T_1 and T_2 such that

$$T_1 = \Big\{(x,y) \in X\colon\ 0 \le x < r \text{ and } y < -\frac{x}{r} + 1\Big\};$$
$$T_2 = \Big\{(x,y) \in X\colon\ r < x \le 1 \text{ and } y < \frac{x}{1-r} - \frac{r}{1-r}\Big\}.$$

The sets T_1 and T_2 are open subsets of $(X, \mathcal{T}_X^{\mathbb{C}})$ which form right triangles with vertices $(r,0)$, $(0,0)$ and $(0,1)$ for T_1 and $(r,0)$, $(1,0)$ and $(1,1)$ for T_2. Let $T = T_1 \cup T_2$. The set T is an open subset of $(X, \mathcal{T}_X^{\mathbb{C}})$ which contains $[0,1]^{\mathbf{V}} \times \{0\}$.

Claim 3.7. *The set $f(T)$ is not an open subset of $(Y, \mathcal{T}_Y^{\mathbb{C}})$; that is, $f(T) \notin \mathcal{T}_Y^{\mathbb{C}}$.*

Proof: In the space $[0,1]^{\mathbf{V}^{\mathbb{C}}} \times [0,1]^{\mathbf{V}^{\mathbb{C}}}$, we can choose rationals q_n such that $(q_n, 1/n) \notin T$ and $\langle (q_n, 1/n) \colon n < \omega \rangle$ converges to $(r,0)$. In $(Y, \mathcal{T}_Y^{\mathbb{C}})$, $\{U_n \colon n \in \mathbb{N}\}$ is still a neighborhood base of 0^* because forcing preserves a neighborhood base of a point, see [5, Proposition 1.5]. So the sequence $\langle (q_n, 1/n) \colon n < \omega \rangle$ converges to 0^* in $(Y, \mathcal{T}_Y^{\mathbb{C}})$. Since $(q_n, 1/n) \notin f(T)$ for all $n \in \mathbb{N}$ and $0^* \in f(T)$, we conclude that $f(T)$ is not an open subset of $(Y, \mathcal{T}_Y^{\mathbb{C}})$. □

Since $T = f^{-1}(f(T))$ and T is open in $(X, \mathcal{T}_X^{\mathbb{C}})$, $f \colon (X, \mathcal{T}_X^{\mathbb{C}}) \to (Y, \mathcal{T}_Y^{\mathbb{C}})$ is not a quotient map. □

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