

## Analysis of periodic solutions for nonlinear coupled integro-differential systems with variable delays

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*Abstract.* The objective of this work is the application of Krasnosel'skii's fixed point technique to prove the existence of periodic solutions of a system of coupled nonlinear integro-differential equations with variable delays. An example is given to illustrate this work.

*Keywords:* integro-differential equation; periodic solution; Krasnosel'skii's fixed point theorem

*Classification:* 34K20, 45J05, 45D05

### 1. Introduction

There are many papers written on the subject of existence of periodic solutions of nonlinear differential equations and nonlinear integro-differential equations, for such topics we refer the interested reader to [1]–[7], [10], [12] and the references therein. In 2007, in the paper [14] Y. Wang, H. Lian and W. Ge consider the second order nonlinear differential equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t)x'(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$

and by using fixed point theorem technique, the authors obtained existence of periodic solutions. H. Deham in [8] considers the second order nonlinear integro-differential equation

$$x''(t) + p(t)x'(t) + q(t)h(x(t)) = \int_{-\infty}^t Q(t, s)f(s, x(s - g(s))) \, ds,$$

and by Krasnosel'skii–Burton's fixed point theorem shows that the existence of periodic solutions is concluded. In the paper [11] Y. N. Raffoul studies the existence of periodic and asymptotically periodic solutions of the following system of

coupled nonlinear Volterra integro-differential equations with infinite delay

$$\begin{cases} x'(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^t a(t, s)f(x(s), y(s)) \, ds, \\ y'(t) = p_1(t)y(t) + p_2(t)x(t) + \int_{-\infty}^t b(t, s)g(x(s), y(s)) \, ds, \end{cases}$$

the author uses Schauder's fixed point theorem to obtain his results.

Motivated by the papers [8], [11], [14] and the references therein and by using Krasnosel'skii's fixed point theorem, in this paper we study the existence of periodic solutions of the following system of coupled nonlinear integro-differential equations with variable delays

$$\begin{aligned} (1.1) \quad & x_i''(t) + p_i(t)x_i'(t) + q_i(t)x_i(t) \\ & = g_i(t, x_1(t), x_2(t), x_1(t - \tau_1(t)), x_2(t - \tau_2(t))) \\ & + c_i(t)x_i'(t - \tau_i(t)) \\ & + \int_{-\infty}^t C_i(t, s)f_i(x_1(s), x_2(s)) \, ds, \quad i = 1, 2, \end{aligned}$$

where  $p_i$ ,  $q_i$ ,  $i = 1, 2$ , are positive continuous real-valued functions and the functions  $c_i$ ,  $C_i$ ,  $i = 1, 2$ , are assumed to be continuous in their arguments throughout the paper. The functions  $g_i(t, x, y, z, w)$ ,  $i = 1, 2$ , are continuous, periodic in  $t$  and Lipschitz continuous in  $x, y, z$  and  $w$ ,  $f_i(x, y)$ ,  $i = 1, 2$ , are continuous and Lipschitz continuous in  $x$  and  $y$ , and for some positive constants  $\eta_{ji}$ ,  $j = 1, \dots, 4$ , and  $i = 1, 2$ , we have

$$|g_i(t, y_1, y_2, y_3, y_4) - g_i(t, x_1, x_2, x_3, x_4)| \leq \sum_{j=1}^4 \eta_{ji}|y_j - x_j|,$$

and for some positive constants  $\varrho_{ji}$ ,  $j = 1, 2$ , and  $i = 1, 2$ , we have

$$|f_i(y_1, y_2) - f_i(x_1, x_2)| \leq \sum_{j=1}^2 \varrho_{ji}|y_j - x_j|,$$

we also assume that  $g_i(t, 0, 0, 0, 0) = f_i(0, 0) = 0$ .

We assume that there exists a positive real number  $T$ , such that

$$(1.2) \quad \begin{cases} C_i(t + T, s + T) = C_i(t, s), \\ c_i(t + T) = c_i(t), \quad \tau_i(t + T) = \tau_i(t), \end{cases} \quad i = 1, 2,$$

for all  $t \in \mathbb{R}$ , with  $\tau_i$  being scalar functions, continuous and  $\tau_i(t) \geq \tau_i^* > 0$ ,  $\tau_i'(t) \neq 1$ .

To have a well behaved mapping we must assume that

$$(1.3) \quad \begin{cases} p_i(t+T) = p_i(t), & \int_0^T p_i(s) \, ds > 0, \\ q_i(t+T) = q_i(t), & \int_0^T q_i(s) \, ds > 0, \end{cases} \quad i = 1, 2.$$

Define

$$P_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\},$$

$$P_T^2 = P_T \times P_T = \{(x_1, x_2) : x_1 \in P_T, x_2 \in P_T\}.$$

Then  $P_T^2$  is a Banach space when endowed with the maximum norm

$$\|(x_1, x_2)\| = \max \left\{ \max_{t \in [0, T]} |x_1(t)|, \max_{t \in [0, T]} |x_2(t)| \right\}.$$

**Lemma 1.1** ([9]). *Suppose that (1.2) and (1.4) hold and for  $i = 1, 2$ ,*

$$(1.4) \quad \frac{R_i}{Q_i T} (e^{\int_0^T p_i(u) \, du} - 1) \geq 1,$$

where

$$R_i = \max_{t \in [0, T]} \left| \int_t^{t+T} \frac{e^{\int_t^s p_i(u) \, du}}{e^{\int_0^T p_i(u) \, du} - 1} q_i(s) \, ds \right|, \quad Q_i = (1 + e^{\int_0^T p_i(u) \, du})^2 R_i^2.$$

Then there are continuous  $T$ -periodic functions  $a_i$  and  $b_i$  such that

$$b_i(t) > 0, \quad \int_0^T a_i(u) \, du > 0,$$

and

$$a_i(t) + b_i(t) = p_i(t), \quad b_i'(t) + a_i(t)b_i(t) = q_i(t) \quad \text{for all } t \in \mathbb{R}.$$

**Lemma 1.2** ([14]). *Suppose the conditions of Lemma 1.1 hold and  $\varphi_i \in P_T$ ,  $i = 1, 2$ . Then the equation*

$$x_i''(t) + p_i(t)x_i'(t) + q_i(t)x_i(t) = \varphi_i(t),$$

has a  $T$ -periodic solution. Moreover, the periodic solution can be expressed as

$$x_i(t) = \int_t^{t+T} G_i(t, s) \varphi_i(s) \, ds,$$

where

$$G_i(t, s) = \frac{\int_t^s e^{\int_t^u b_i(v) \, dv} + \int_u^s a_i(v) \, dv \, du + \int_s^{t+T} e^{\int_t^u b_i(v) \, dv} + \int_u^{s+T} a_i(v) \, dv \, du}{(e^{\int_0^T a_i(u) \, du} - 1)(e^{\int_0^T b_i(u) \, du} - 1)}.$$

**Corollary 1.3** ([14]). *Green's functions  $G_i$ ,  $i = 1, 2$ , satisfies the following properties*

$$\begin{aligned} G_i(t, t+T) &= G_i(t, t), & G_i(t+T, s+T) &= G_i(t, s), \\ \frac{\partial}{\partial s} G_i(t, s) &= a_i(s)G_i(t, s) - H_i(t, s), \\ \frac{\partial}{\partial t} G_i(t, s) &= -b_i(t)G_i(t, s) + H_i^*(t, s), \end{aligned}$$

where

$$H_i(t, s) = \frac{e^{\int_t^s b_i(v) dv}}{e^{\int_0^T b_i(v) dv} - 1}, \quad H_i^*(t, s) = \frac{e^{\int_t^s a_i(v) dv}}{e^{\int_0^T a_i(v) dv} - 1}.$$

**Lemma 1.4.** *Assume (1.2)–(1.4). If  $(x_1, x_2) \in P_T^2$ , then  $x_i$  is a solution of (1.1) if and only if*

$$\begin{aligned} (1.5) \quad x_i(t) &= \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) du \\ &+ \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) ds du \\ &+ \int_t^{t+T} [h_i(u)H_i(t, u) - r_i(u)G_i(t, u)]x_i(u - \tau_i(u)) du, \quad i = 1, 2, \end{aligned}$$

where

$$(1.6) \quad h_i(u) = \frac{c_i(u)}{1 - \tau_i'(u)}, \quad i = 1, 2,$$

$$(1.7) \quad r_i(u) = \frac{(a_i(u)c_i(u) + c_i'(u))(1 - \tau_i'(u)) + \tau_i''(u)c_i(u)}{(1 - \tau_i'(u))^2}, \quad i = 1, 2.$$

PROOF: Let  $(x_1, x_2) \in P_T^2$  be a solution of (1.1). From Lemma 1.2 we have

$$\begin{aligned} (1.8) \quad x_i(t) &= \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) du \\ &+ \int_t^{t+T} G_i(t, u) c_i(u) x_i'(u - \tau_i(u)) du \\ &+ \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) ds du, \quad i = 1, 2. \end{aligned}$$

Letting

$$\int_t^{t+T} G_i(t, u) c_i(u) x_i'(u - \tau_i(u)) du = \int_t^{t+T} \frac{G_i(t, u) c_i(u)}{1 - \tau_i'(u)} (1 - \tau_i'(u)) x_i'(u - \tau_i(u)) du,$$

performing an integration by parts, we get

$$\begin{aligned} \int_t^{t+T} G_i(t, u) c_i(u) x'_i(u - \tau_i(u)) du &= \left[ \frac{G_i(t, u) c_i(u)}{1 - \tau'_i(u)} x_i(u - \tau_i(u)) \right]_t^{t+T} \\ &\quad - \int_t^{t+T} [r_i(u) G_i(t, u) - h_i(u) H_i(t, u)] x_i(u - \tau_i(u)) du. \end{aligned}$$

Since

$$\left[ \frac{G_i(t, u) c_i(u)}{1 - \tau'_i(u)} x_i(u - \tau_i(u)) \right]_t^{t+T} = 0,$$

we obtain

$$\begin{aligned} (1.9) \quad &\int_t^{t+T} G_i(t, u) c_i(u) x'_i(u - \tau_i(u)) du \\ &= \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u)) du, \end{aligned}$$

where  $h_i$ ,  $r_i$  are given by (1.6) and (1.7). Substituting (1.9) into (1.8), we obtain

$$\begin{aligned} x_i(t) &= \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) du \\ &\quad + \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) ds du \\ &\quad + \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u)) du, \quad i = 1, 2. \end{aligned}$$

□

**Lemma 1.5** ([14]). *Let  $\Gamma_i = \int_0^T p_i(u) du$ ,  $\Lambda_i = T^2 e^{(1/T) \int_0^T \ln(q_i(u)) du}$ ,  $i = 1, 2$ . If  $\Gamma_i^2 \geq 4\Lambda_i$ , then we have*

$$\min \left\{ \int_0^T a_i(u) du, \int_0^T b_i(u) du \right\} \geq \frac{1}{2} \left( \Gamma_i - \sqrt{\Gamma_i^2 - 4\Lambda_i} \right) := l_i,$$

and

$$\max \left\{ \int_0^T a_i(u) du, \int_0^T b_i(u) du \right\} \leq \frac{1}{2} \left( \Gamma_i + \sqrt{\Gamma_i^2 - 4\Lambda_i} \right) := m_i.$$

**Corollary 1.6** ([14]). *Functions  $G_i$  and  $H_i$ ,  $i = 1, 2$ , satisfy*

$$\frac{T}{(e^{m_i} - 1)^2} \leq G_i(t, s) \leq \frac{T e^{\int_0^T p_i(v) dv}}{(e^{l_i} - 1)^2}, \quad |H_i(t, s)| \leq \frac{e^{m_i}}{e^{l_i} - 1}.$$

To simplify notation, we introduce for  $i = 1, 2$ , the constants

$$\begin{aligned}\alpha_i &= \frac{T e^{\int_0^T p_i(v) dv}}{(e^{t_i} - 1)^2}, & \gamma_i &= \frac{e^{m_i}}{e^{t_i} - 1}, & \theta_i &= \max_{t \in [0, T]} |h_i(t)|, \\ \beta_i &= \max_{t \in [0, T]} |r_i(t)|, & \lambda_i &= \max_{t \in [0, T]} |a_i(t)|, & \delta_i &= \max_{t \in [0, T]} |b_i(t)|.\end{aligned}$$

## 2. Periodic solutions

**Lemma 2.1** ([13]). *Let  $\mathbb{M}$  be a closed convex nonempty subset of a Banach space  $(S, \|\cdot\|)$ . Suppose that  $A$  and  $B$  map  $\mathbb{M}$  into  $S$  such that*

- (i)  $x, y \in \mathbb{M}$  implies  $Ax + By \in \mathbb{M}$ ;
- (ii)  $A$  is compact and continuous;
- (iii)  $B$  is a contraction mapping.

*Then there exists  $z \in \mathbb{M}$  with  $z = Az + Bz$ .*

We assume the existence of positive constants  $M_i$ ,  $K_i$  and  $L_i$ ,  $i = 1, 2$ , such that

$$(2.1) \quad |f_i(x, y)| \leq M_i,$$

$$(2.2) \quad |g_i(t, x, y, z, w)| \leq K_i,$$

and

$$(2.3) \quad \int_t^{t+T} \int_{-\infty}^u |C_i(u, s)| ds du \leq L_i.$$

Set

$$(2.4) \quad M = \max \left\{ \frac{(TK_i + L_i M_i) \alpha_i}{1 - T(\theta_i \gamma_i + \beta_i \alpha_i)} : i = 1, 2 \right\},$$

with  $0 < T(\theta_i \gamma_i + \beta_i \alpha_i) < 1$ ,  $i = 1, 2$ .

We define subset  $\Omega_M$  of  $P_T^2$  as follows

$$\Omega_M = \{(x_1, x_2) \in P_T^2 : \|(x_1, x_2)\| \leq M\}.$$

Then  $\Omega_M$  is a bounded, closed and convex subset of  $P_T^2$ .

Now for  $(x_1, x_2) \in \Omega_M$  we can define an operator  $E: \Omega_M \rightarrow P_T^2$  by

$$E(x_1, x_2)(t) = (E_1(x_1, x_2)(t), E_2(x_1, x_2)(t)),$$

where

$$\begin{aligned}
 E_i(x_1, x_2)(t) = & \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), \\
 & x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) \, du \\
 (2.5) \quad & + \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) \, ds \, du \\
 & + \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u)) \, du, \\
 & i = 1, 2.
 \end{aligned}$$

To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is compact. Therefore, we state (2.5) as

$$E_i(x_1, x_2)(t) = B_i(x_1, x_2)(t) + A_i(x_1, x_2)(t), \quad i = 1, 2,$$

where  $B_i, A_i: \Omega_M \rightarrow P_T$  are given by

$$B_i(x_1, x_2)(t) = \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) \, du,$$

and

$$\begin{aligned}
 A_i(x_1, x_2)(t) = & \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) \, ds \, du \\
 & + \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u)) \, du.
 \end{aligned}$$

Now for  $(x_1, x_2) \in \Omega_M$  we can define the operators  $B, A: \Omega_M \rightarrow P_T^2$  by

$$\begin{aligned}
 B(x_1, x_2)(t) &= (B_1(x_1, x_2)(t), B_2(x_1, x_2)(t)), \\
 A(x_1, x_2)(t) &= (A_1(x_1, x_2)(t), A_2(x_1, x_2)(t)).
 \end{aligned}$$

Observe that, since the functions  $g_i(t, x_1, x_2, x_3, x_4)$ ,  $i = 1, 2$ , is Lipschitz continuous in  $x_1, x_2, x_3, x_4$  and  $f_i(x_1, x_2)$ ,  $i = 1, 2$ , is Lipschitz continuous in  $x_1, x_2$  we have

$$\begin{aligned}
 |g_i(t, x_1, x_2, x_3, x_4)| &= |g_i(t, x_1, x_2, x_3, x_4) - g_i(t, 0, 0, 0, 0) + g_i(t, 0, 0, 0, 0)| \\
 &\leq |g_i(t, x_1, x_2, x_3, x_4) - g_i(t, 0, 0, 0, 0)| + |g_i(t, 0, 0, 0, 0)| \\
 &\leq \sum_{j=1}^4 \eta_{ji} |x_j|,
 \end{aligned}$$

and

$$\begin{aligned}
|f_i(x_1, x_2)| &= |f_i(x_1, x_2) - f_i(0, 0) + f_i(0, 0)| \\
&\leq |f_i(x_1, x_2) - f_i(0, 0)| + |f_i(0, 0)| \leq \sum_{j=1}^2 \varrho_{ji} |x_j|.
\end{aligned}$$

**Theorem 2.2.** *Suppose (1.2)–(1.4) and (2.1)–(2.3) hold. Suppose that*

$$L_i \alpha_i \sum_{j=1}^2 \varrho_{ji} + T(\theta_i \gamma_i + \beta_i \alpha_i) \leq 1, \quad i = 1, 2,$$

and  $2T\alpha_i V_i < 1$ ,  $i = 1, 2$ , where  $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$ . Then (1.1) has a  $T$ -periodic solution.

PROOF: In order to prove that (1.1) has a  $T$ -periodic solution, we shall make sure that  $A$  and  $B$  satisfy the conditions of Lemma 2.1. For all  $(x_1, x_2) \in \Omega_M$ , we have  $(x_1, x_2)(t + T) = (x_1, x_2)(t)$  and  $\|(x_1, x_2)\| \leq M$ . Now let us discuss  $B(x_1, x_2) + A(x_1, x_2)$ . We have

$$\begin{aligned}
&B_i(x_1, x_2)(t + T) \\
&= \int_{t+T}^{t+2T} G_i(t + T, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) \, du \\
&= \int_t^{t+T} G_i(t, u) g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u))) \, du \\
&= B_i(x_1, x_2)(t), \quad i = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
&A_i(x_1, x_2)(t + T) \\
&= \int_{t+T}^{t+2T} G_i(t + T, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) \, ds \, du \\
&\quad + \int_{t+T}^{t+2T} [h_i(u) H_i(t + T, u) - r_i(u) G_i(t + T, u)] x_i(u - \tau_i(u)) \, du \\
&= \int_t^{t+T} G_i(t, u) \int_{-\infty}^u C_i(u, s) f_i(x_1(s), x_2(s)) \, ds \, du \\
&\quad + \int_t^{t+T} [h_i(u) H_i(t, u) - r_i(u) G_i(t, u)] x_i(u - \tau_i(u)) \, du \\
&= A_i(x_1, x_2)(t), \quad i = 1, 2.
\end{aligned}$$

Then  $E_i(x_1, x_2)(t + T) = E_i(x_1, x_2)(t)$ ,  $i = 1, 2$ . Therefore,  $E(x_1, x_2)(t + T) = E(x_1, x_2)(t)$ .

For any  $(x_1, x_2) \in \Omega_M$ , we will show that  $|E(x_1, x_2)(t)| \leq M$ . In view of the above estimates, we have for  $i = 1, 2$ ,

$$\begin{aligned} & |B_i(x_1, x_2)(t)| \\ & \leq \int_t^{t+T} G_i(t, u) |g_i(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u)))| du \\ & \leq TK_i\alpha_i, \end{aligned}$$

and

$$\begin{aligned} |A_i(x_1, x_2)(t)| & \leq \int_t^{t+T} G_i(t, u) \int_{-\infty}^u |C_i(u, s)| |f_i(x_1(s), x_2(s))| ds du \\ & \quad + \int_t^{t+T} [|h_i(u)| |H_i(t, u)| + |r_i(u)| |G_i(t, u)| |x_i(u - \tau_i(u))|] du \\ & \leq L_i M_i \alpha_i + T(\theta_i \gamma_i + \beta_i \alpha_i) M. \end{aligned}$$

As a consequence of (2.4), we have

$$\frac{(TK_i + L_i M_i) \alpha_i}{1 - T(\theta_i \gamma_i + \beta_i \alpha_i)} \leq M,$$

so,

$$(TK_i + L_i M_i) \alpha_i \leq (1 - T(\theta_i \gamma_i + \beta_i \alpha_i)) M.$$

This implies that

$$\begin{aligned} |E_i(x_1, x_2)(t)| & \leq TK_i \alpha_i + L_i M_i \alpha_i + T(\theta_i \gamma_i + \beta_i \alpha_i) M \\ & \leq (1 - T(\theta_i \gamma_i + \beta_i \alpha_i)) M + T(\theta_i \gamma_i + \beta_i \alpha_i) M = M. \end{aligned}$$

Thus,  $E$  maps  $\Omega_M$  into itself, i.e.  $E(\Omega_M) \subseteq \Omega_M$ .

We will now show that  $A$  is continuous. For  $n \in \mathbb{N}$ , let  $\{(x_{1n}, x_{2n})\}$  be a sequence in  $\Omega_M$  such that

$$\lim_{n \rightarrow \infty} \|(x_{1n}, x_{2n}) - (x_1, x_2)\| = 0.$$

Since  $\Omega_M$  is closed, we have  $(x_1, x_2) \in \Omega_M$ . Then by the definition of  $A$  we have

$$\begin{aligned} \|A(x_{1n}, x_{2n}) - A(x_1, x_2)\| & = \max \left\{ \max_{t \in [0, T]} |A_1(x_{1n}, x_{2n})(t) - A_1(x_1, x_2)(t)|, \right. \\ & \quad \left. \max_{t \in [0, T]} |A_2(x_{1n}, x_{2n})(t) - A_2(x_1, x_2)(t)| \right\}, \end{aligned}$$

in which for  $i = 1, 2$ ,

$$\begin{aligned} & |A_i(x_{1n}, x_{2n})(t) - A_i(x_1, x_2)(t)| \\ & \leq \int_t^{t+T} G_i(t, u) \int_{-\infty}^u |C_i(u, s)| |f_i(x_{1n}(s), x_{2n}(s)) - f_i(x_1(s), x_2(s))| ds du \\ & \quad + \int_t^{t+T} [|h_i(u)| |H_i(t, u)| + |r_i(u)| |G_i(t, u)|] |x_{in}(u - \tau_i(u)) - x_i(u - \tau_i(u))| du. \end{aligned}$$

The continuity of  $f_i$  along with the Lebesgue dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \max_{t \in [0, T]} |A_i(x_{1n}, x_{2n})(t) - A_i(x_1, x_2)(t)| = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \|A(x_{1n}, x_{2n}) - A(x_1, x_2)\| = 0.$$

This result proves that  $A$  is continuous.

We now have to show that  $A$  is compact. For  $n \in \mathbb{N}$ , let  $\{(x_{1n}, x_{2n})\}$  be a sequence in  $\Omega_M$ , then we have for  $i = 1, 2$ ,

$$\begin{aligned} |A_i(x_{1n}, x_{2n})(t)| & \leq \int_t^{t+T} G_i(t, u) \int_{-\infty}^u |C_i(u, s)| |f_i(x_{1n}(s), x_{2n}(s))| ds du \\ & \quad + \int_t^{t+T} [|h_i(u)| |H_i(t, u)| + |r_i(u)| |G_i(t, u)|] |x_{in}(u - \tau_i(u))| du \\ & \leq \left( L_i \alpha_i \sum_{j=1}^2 \varrho_{ji} + T(\theta_i \gamma_i + \beta_i \alpha_i) \right) M \leq M. \end{aligned}$$

Thus

$$\|A(x_{1n}, x_{2n})\| \leq M.$$

If we calculate  $(A(x_{1n}, x_{2n}))'(t)$ , then for  $i = 1, 2$ ,

$$\begin{aligned} & (A_i(x_{1n}, x_{2n}))'(t) \\ & = \int_t^{t+T} [-b_i(t)G_i(t, u) + H_i^*(t, u)] \int_{-\infty}^u C_i(u, s) f_i(x_{1n}(s), x_{2n}(s)) ds du \\ & \quad + h_i(t)x_{in}(t - \tau_i(t)) - \int_t^{t+T} [b_i(t)(h_i(u)H_i(t, u) - r_i(u)G_i(t, u)) \\ & \quad + r_i(u)H_i^*(t, u)] x_{in}(u - \tau_i(u)) du. \end{aligned}$$

Hence, for some positive constant  $D_i$ , we obtain

$$\begin{aligned} & |(A_i(x_{1n}, x_{2n}))'(t)| \\ & \leq \int_t^{t+T} (|b_i(t)| |G_i(t, u)| + |H_i^*(t, u)|) \int_{-\infty}^u |C_i(u, s)| |f_i(x_{1n}(s), x_{2n}(s))| ds du \end{aligned}$$

$$\begin{aligned}
& + |h_i(t)| |x_{in}(t - \tau_i(t))| + \int_t^{t+T} [|b_i(t)| (|h_i(u)| |H_i(t, u)| + |r_i(u)| |G_i(t, u)| \\
& + |r_i(u)| |H_i^*(t, u)|) |x_{in}(u - \tau_i(u))| du \\
& \leq (\delta_i \alpha_i + \gamma_i) L_i M \sum_{j=1}^2 \varrho_{ji} + \theta_i M + T[\delta_i(\theta_i \gamma_i + \beta_i \alpha_i) + \beta_i \gamma_i] M \leq D_i.
\end{aligned}$$

Thus

$$\|(A(x_{1n}, x_{2n}))'\| \leq D,$$

where  $D = \max(D_1, D_2)$ . Thus, the sequence  $(A(x_{1n}, x_{2n}))$  is uniformly bounded and equi-continuous. The Arzelà–Ascoli theorem implies that there exists a subsequence  $(A(x_{1n_k}, x_{2n_k}))$  of  $(A(x_{1n}, x_{2n}))$  converging uniformly to a continuous  $T$ -periodic function. Thus,  $A$  is compact.

For all  $(x_{11}, x_{21}), (x_{12}, x_{22}) \in \Omega_M$ , and for  $i = 1, 2$ ,

$$\begin{aligned}
& |B_i(x_{11}, x_{21})(t) - B_i(x_{12}, x_{22})(t)| \\
& \leq \int_t^{t+T} G_i(t, u) |g_i(u, x_{11}(u), x_{21}(u), x_{11}(u - \tau_1(u)), x_{21}(u - \tau_2(u))) \\
& \quad - g_i(u, x_{21}(u), x_{22}(u), x_{21}(u - \tau_1(u)), x_{22}(u - \tau_2(u)))| du \\
& \leq T \alpha_i (\eta_{1i} |x_{11}(t) - x_{21}(t)| + \eta_{3i} |x_{11}(t - \tau_1(t)) - x_{21}(t - \tau_1(t))| \\
& \quad + \eta_{2i} |x_{21}(t) - x_{22}(t)| + \eta_{4i} |x_{21}(t - \tau_2(t)) - x_{22}(t - \tau_2(t))|) \\
& \leq T \alpha_i \left( \eta_{1i} \max_{t \in [0, T]} |x_{11}(t) - x_{21}(t)| + \eta_{3i} \max_{t \in [0, T]} |x_{11}(t - \tau_1(t)) - x_{21}(t - \tau_1(t))| \right. \\
& \quad \left. + \eta_{2i} \max_{t \in [0, T]} |x_{21}(t) - x_{22}(t)| + \eta_{4i} \max_{t \in [0, T]} |x_{21}(t - \tau_2(t)) - x_{22}(t - \tau_2(t))| \right) \\
& \leq T \alpha_i \left( (\eta_{1i} + \eta_{3i}) \max_{t \in [0, T]} |x_{11}(t) - x_{21}(t)| + (\eta_{2i} + \eta_{4i}) \max_{t \in [0, T]} |x_{21}(t) - x_{22}(t)| \right) \\
& \leq 2T \alpha_i V_i \max \left( \max_{t \in [0, T]} |x_{11}(t) - x_{21}(t)|, \max_{t \in [0, T]} |x_{21}(t) - x_{22}(t)| \right),
\end{aligned}$$

hence  $B_i$  is a contraction because  $2T \alpha_i V_i < 1$ . Then

$$\begin{aligned}
& |B(x_{11}, x_{21})(t) - B(x_{12}, x_{22})(t)| \\
& = \max\{|B_1(x_{11}, x_{21})(t) - B_1(x_{12}, x_{22})(t)|, \\
& \quad |B_2(x_{11}, x_{21})(t) - B_2(x_{12}, x_{22})(t)|\},
\end{aligned}$$

this implies that

$$\begin{aligned}
& \|B(x_{11}, x_{21}) - B(x_{12}, x_{22})\| \\
& \leq 2T \alpha V \max \left( \max_{t \in [0, T]} |x_{11}(t) - x_{21}(t)|, \max_{t \in [0, T]} |x_{21}(t) - x_{22}(t)| \right),
\end{aligned}$$

where  $\alpha V = \max(\alpha_1 V_1, \alpha_2 V_2)$ . Hence  $B$  is a contraction.

Thus, the conditions of Lemma 2.1 are satisfied and there is a  $(x_1, x_2) \in \Omega_M$ , such that  $(x_1, x_2) = A(x_1, x_2) + B(x_1, x_2)$ .  $\square$

In the next theorem for  $i = 2$  we relax condition (2.1).

**Theorem 2.3.** *Suppose (1.2)–(1.4), (2.1) for  $i = 1$ , (2.2) and (2.3) hold. Suppose that*

$$L_i \alpha_i \sum_{j=1}^2 \varrho_{ji} + T(\theta_i \gamma_i + \beta_i \alpha_i) \leq 1, \quad i = 1, 2,$$

and  $2T\alpha_i V_i < 1$ ,  $i = 1, 2$ , where  $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$ . In addition, we assume the existence of continuous nondecreasing function  $W_2$  such that

$$|f_2(x_1, x_2)| \leq f_2(|x_1|, x_2) \leq N_2 W_2(|x_1|)$$

for some positive constant  $N_2$ , and for  $u > 0$  we ask that

$$(2.6) \quad \frac{W_2(u)}{u} + \frac{TK_2}{L_2 N_2 u} \leq \frac{1 - T(\theta_2 \gamma_2 + \beta_2 \alpha_2)}{L_2 N_2 \alpha_2}.$$

Then (1.1) has a  $T$ -periodic solution.

PROOF: Set

$$(2.7) \quad \sigma = \frac{(TK_1 + L_1 M_1) \alpha_1}{1 - T(\theta_1 \gamma_1 + \beta_1 \alpha_1)}.$$

For any  $(x_1, x_2) \in \Omega_\sigma$ , we have by the proof of Theorem 2.2 that

$$|E_1(x_1, x_2)(t)| \leq \sigma.$$

Thus

$$\begin{aligned} & |B_2(x_1, x_2)(t)| \\ & \leq \int_t^{t+T} G_2(t, u) |g_2(u, x_1(u), x_2(u), x_1(u - \tau_1(u)), x_2(u - \tau_2(u)))| du \\ & \leq TK_2 \alpha_2, \end{aligned}$$

and

$$\begin{aligned} |A_2(x_1, x_2)(t)| & \leq \int_t^{t+T} G_2(t, u) \int_{-\infty}^u |C_2(u, s) f_2(|x_1(s)|, x_2(s))| ds du \\ & \quad + \int_t^{t+T} [|h_2(u)| |H_2(t, u)| + |r_2(u)| |G_2(t, u)|] |x_2(u - \tau_2(u))| du \end{aligned}$$

$$\begin{aligned}
&\leq N_2 W_2(\sigma) \int_t^{t+T} G_2(t, u) \int_{-\infty}^u |C_2(u, s)| ds du \\
&\quad + \sigma \int_t^{t+T} [|h_2(u)| |H_2(t, u)| + |r_2(u)| G_2(t, u)] du \\
&\leq L_2 N_2 \alpha_2 W_2(\sigma) + T(\theta_2 \gamma_2 + \beta_2 \alpha_2) \sigma.
\end{aligned}$$

As a consequence of (2.6), we get

$$\frac{(TK_2 + L_2 N_2 W_2(\sigma)) \alpha_2}{1 - T(\theta_2 \gamma_2 + \beta_2 \alpha_2)} \leq \sigma,$$

so, we have

$$(TK_2 + L_2 N_2 W_2(\sigma)) \alpha_2 \leq (1 - T(\theta_2 \gamma_2 + \beta_2 \alpha_2)) \sigma.$$

This implies that

$$\begin{aligned}
|E_2(x_1, x_2)(t)| &\leq TK_2 \alpha_2 + L_2 \alpha_2 N_2 W_2(\sigma) + T(\theta_2 \gamma_2 + \beta_2 \alpha_2) \sigma \\
&\leq (1 - T(\theta_2 \gamma_2 + \beta_2 \alpha_2)) \sigma + T(\theta_2 \gamma_2 + \beta_2 \alpha_2) \sigma = \sigma.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2.  $\square$

In the next theorem for  $i = 1$  we relax condition (2.1).

**Theorem 2.4.** Suppose (1.2)–(1.4), (2.1) for  $i = 2$ , (2.2) and (2.3) hold. Suppose that

$$L_i \alpha_i \sum_{j=1}^2 \varrho_{ji} + T(\theta_i \gamma_i + \beta_i \alpha_i) \leq 1, \quad i = 1, 2,$$

and  $2T\alpha_i V_i < 1$ ,  $i = 1, 2$ , where  $V_i = \max(\eta_{1i} + \eta_{3i}, \eta_{2i} + \eta_{4i})$ . In addition, we assume the existence of continuous nondecreasing function  $W_1$  such that

$$|f_1(x_1, x_2)| \leq f_1(x_1, |x_2|) \leq N_1 W_1(|x_2|)$$

for some positive constant  $N_1$ , and for  $u > 0$  we ask that

$$(2.8) \quad \frac{W_1(u)}{u} + \frac{TK_1}{L_1 N_1 u} \leq \frac{1 - T(\theta_1 \gamma_1 + \beta_1 \alpha_1)}{L_1 N_1 \alpha_1}.$$

Then (1.1) has a  $T$ -periodic solution.

The proof follows along the lines of the proof of Theorem 2.3, and hence we omit it here.

### 3. An example

**Example 3.1.** Consider the following coupled integro-differential system

$$\begin{aligned}
 (3.1) \quad & x_1''(t) + \frac{1}{\pi}x_1'(t) + \frac{1}{10^3}x_1(t) = \frac{1}{10^9}\sin(x_1(t)) + \frac{2}{10^9}\sin(x_2(t)) \\
 & + \frac{1}{10^8}\sin(x_1(t-2\pi)) + \frac{3}{10^8}\sin(x_2(t-4\pi)) \\
 & + \frac{2}{10^8}\sin(t)x_1'(t-2\pi) \\
 & + \int_{-\infty}^t \frac{1-e^{-2\pi}}{\pi 10^8}e^{-2t+2s} \left( \frac{1}{10}\sin(x_1(s)) + \frac{1}{10^3}\sin(x_2(s)) \right) ds, \\
 & x_2''(t) + \frac{1}{\pi}x_2'(t) + \frac{1}{10^3}x_2(t) = \frac{2}{10^8}\sin(x_1(t)) + \frac{1}{10^8}\sin(x_2(t)) \\
 & + \frac{3}{10^9}\sin(x_1(t-2\pi)) + \frac{1}{10^9}\sin(x_2(t-4\pi)) \\
 & + \frac{3}{10^8}\sin(t)x_2'(t-4\pi) \\
 & + \int_{-\infty}^t \frac{1}{10^9}e^{-t+s} \left( \frac{1}{10^2}\sin(x_1(s)) + \frac{1}{10^4}\sin(x_2(s)) \right) ds.
 \end{aligned}$$

Then

$$\begin{aligned}
 p_1(t) &= p_2(t) = \frac{1}{\pi}, & q_1(t) &= q_2(t) = \frac{1}{10^3}, & T &= 2\pi, \\
 \tau_1(t) &= 2\pi, & \tau_2(t) &= 4\pi, & c_1(t) &= \frac{2}{10^8}\sin(t), \\
 c_2(t) &= \frac{3}{10^8}\sin(t), & C_1(t, s) &= \frac{1-e^{-2\pi}}{\pi 10^8}e^{-2t+2s}, & C_2(t, s) &= \frac{1}{10^9}e^{-t+s}, \\
 g_1(t, x_1(t), x_2(t), x_1(t-2\pi), x_2(t-4\pi)) \\
 &= \frac{1}{10^9}\sin(x_1(t)) + \frac{2}{10^9}\sin(x_2(t)) + \frac{1}{10^8}\sin(x_1(t-2\pi)) + \frac{3}{10^8}\sin(x_2(t-4\pi)), \\
 g_2(t, x_1(t), x_2(t), x_1(t-2\pi), x_2(t-4\pi)) \\
 &= \frac{2}{10^8}\sin(x_1(t)) + \frac{1}{10^8}\sin(x_2(t)) + \frac{3}{10^9}\sin(x_1(t-2\pi)) + \frac{1}{10^9}\sin(x_2(t-4\pi)),
 \end{aligned}$$

and

$$\begin{aligned}
 f_1(x_1(t), x_2(t)) &= \frac{1}{10}\sin(x_1(t)) + \frac{1}{10^3}\sin(x_2(t)), \\
 f_2(x_1(t), x_2(t)) &= \frac{1}{10^2}\sin(x_1(t)) + \frac{1}{10^4}\sin(x_2(t)).
 \end{aligned}$$

Hence

$$\begin{aligned}
& |g_1(t, x_1(t), x_2(t), x_1(t - 2\pi), x_2(t - 4\pi))| \\
& \leq \frac{1}{10^9}|x_1(t)| + \frac{2}{10^9}|x_2(t)| + \frac{1}{10^8}|x_1(t - 2\pi)| + \frac{3}{10^8}|x_2(t - 4\pi)|, \\
& |g_2(t, x_1(t), x_2(t), x_1(t - 2\pi), x_2(t - 4\pi))| \\
& \leq \frac{2}{10^8}|x_1(t)| + \frac{1}{10^8}|x_2(t)| + \frac{3}{10^9}|x_1(t - 2\pi)| + \frac{1}{10^9}|x_2(t - 4\pi)|,
\end{aligned}$$

$$|f_1(x_1, x_2)| \leq \frac{1}{10}|x_1(t)| + \frac{1}{10^3}|x_2(t)|,$$

and

$$|f_1(x_1, x_2)| \leq \frac{1}{10^2}|x_1(t)| + \frac{1}{10^4}|x_2(t)|.$$

So,

$$\begin{array}{llll}
\eta_{11} = \frac{1}{10^9}, & \eta_{21} = \frac{2}{10^9}, & \eta_{31} = \frac{1}{10^8}, & \eta_{41} = \frac{3}{10^8}, \\
\eta_{12} = \frac{2}{10^8}, & \eta_{22} = \frac{1}{10^8}, & \eta_{32} = \frac{3}{10^9}, & \eta_{42} = \frac{1}{10^9}, \\
\varrho_{11} = \frac{1}{10}, & \varrho_{21} = \frac{1}{10^3}, & \varrho_{12} = \frac{1}{10^2}, & \varrho_{22} = \frac{1}{10^4}.
\end{array}$$

We check the conditions of Lemma 1.1 for  $i = 1, 2$ ,

$$\begin{aligned}
R_i &= \max_{t \in [0, 2/\pi]} \left| \int_t^{t+2\pi} \frac{e^{\int_t^s (1/\pi) du}}{e^{\int_0^{2\pi} (1/\pi) du} - 1} \frac{1}{10^3} ds \right| \simeq 0.003, \\
Q_i &= \left( 1 + e^{\int_0^{2\pi} (1/\pi) du} \right)^2 R_i^2 \simeq 0.0006,
\end{aligned}$$

and

$$\frac{R_i}{2\pi Q_i} \left( e^{\int_0^{2\pi} (1/\pi) du} - 1 \right) \simeq 5.0842 \geq 1,$$

this implies

$$a_i(t) = 0.0032, \quad b_i(t) = 0.3152, \quad i = 1, 2.$$

We check the conditions of Lemma 1.5

$$\Gamma_i = \int_0^{2\pi} \frac{1}{\pi} du = 2, \quad \Lambda_i = (2\pi)^2 e^{(1/2\pi) \int_0^{2\pi} \ln(1/10^3) du} \simeq 0.0395,$$

and

$$2^2 \geq 4 \cdot 0.0395 \Rightarrow \Gamma_i^2 \geq 4\Lambda_i, \quad i = 1, 2,$$

then we have for  $i = 1, 2$ ,

$$\begin{aligned} \min \left\{ \int_0^{2\pi} 0.0032 \, du, \int_0^{2\pi} 0.3152 \, du \right\} &\geq \frac{1}{2} \left( \Gamma_i - \sqrt{\Gamma_i^2 - 4\Lambda_i} \right) \\ &\simeq \frac{1}{2} \left( 2 - \sqrt{2^2 - 4 \cdot 0.0395} \right) = l_i \simeq 0.0199, \\ \max \left\{ \int_0^{2\pi} 0.0032 \, du, \int_0^{2\pi} 0.3152 \, du \right\} &\leq \frac{1}{2} \left( \Gamma_i + \sqrt{\Gamma_i^2 - 4\Lambda_i} \right) \\ &\simeq \frac{1}{2} \left( 2 + \sqrt{2^2 - 4 \times 0.0395} \right) = m_i \simeq 1.9801. \end{aligned}$$

By Corollary 1.6, we get

$$\begin{aligned} \frac{2\pi}{(e^{1.9801} - 1)^2} &\simeq 0.1611 \leq G_i(t, s) \leq \frac{2\pi e^{\int_0^{2\pi} (1/\pi) \, dv}}{(e^{0.0199} - 1)^2} \simeq 1.15 \cdot 10^5, \quad i = 1, 2, \\ |H_i(t, s)| &\leq \frac{e^{1.9801}}{e^{0.0199} - 1} \simeq 3.61 \cdot 10^2, \quad i = 1, 2. \end{aligned}$$

We obtain

$$\begin{aligned} \alpha_i &= 1.15 \cdot 10^5, \quad \gamma_i = 3.61 \cdot 10^2, \quad \lambda_i = 0.0032, \quad \delta_i = 0.3152, \quad i = 1, 2, \\ \theta_1 &= \max_{t \in [0, 2\pi]} |h_1(t)| = \max_{t \in [0, 2\pi]} |c_1(t)| = \max_{t \in [0, 2\pi]} \left| \frac{2}{10^8} \sin(t) \right| = \frac{2}{10^8}, \\ \theta_2 &= \max_{t \in [0, 2\pi]} |h_2(t)| = \max_{t \in [0, 2\pi]} |c_2(t)| = \max_{t \in [0, 2\pi]} \left| \frac{3}{10^8} \sin(t) \right| = \frac{3}{10^8}, \\ \beta_1 &= \max_{t \in [0, 2\pi]} |r_1(t)| = \max_{t \in [0, 2\pi]} |a_1(t)c_1(t) + c'_1(t)| \\ &= \max_{t \in [0, 2\pi]} \left| 0.0032 \cdot \frac{2}{10^8} \sin(t) + \frac{2}{10^8} \cos(t) \right| = \frac{2.0064}{10^8}, \\ \beta_2 &= \max_{t \in [0, 2\pi]} |r_2(t)| = \max_{t \in [0, 2\pi]} |a_2(t)c_2(t) + c'_2(t)| \\ &= \max_{t \in [0, 2\pi]} \left| 0.0032 \cdot \frac{3}{10^8} \sin(t) + \frac{3}{10^8} \cos(t) \right| = \frac{3.0096}{10^8}, \end{aligned}$$

$$|f_1(x_1(t), x_2(t))| \leq M_1 = \frac{101}{10^3}, \quad |f_2(x_1(t), x_2(t))| \leq M_2 = \frac{101}{10^4},$$

$$\begin{aligned} |g_1(t, x_1(t), x_2(t), x_1(t - 2\pi), x_2(t - 4\pi))| &\leq K_1 = \frac{43}{10^9}, \\ |g_2(t, x_1(t), x_2(t), x_1(t - 2\pi), x_2(t - 4\pi))| &\leq K_2 = \frac{34}{10^9}, \end{aligned}$$

$$\int_t^{t+2\pi} \int_{-\infty}^u \left| \frac{1 - e^{-2\pi}}{\pi 10^8} e^{-2u+2s} \right| ds du \leq L_1 = \frac{0.9982}{10^8},$$

$$\int_t^{t+2\pi} \int_{-\infty}^u \left| \frac{1}{10^9} e^{-u+s} \right| ds du \leq L_2 = \frac{2\pi}{10^9}.$$

Therefore,

$$L_1 \alpha_1 \sum_{j=1}^2 \varrho_{j1} + T(\theta_1 \gamma_1 + \beta_1 \alpha_1) = 0.0145 \leq 1,$$

$$L_2 \alpha_2 \sum_{j=1}^2 \varrho_{j2} + T(\theta_2 \gamma_2 + \beta_2 \alpha_2) = 0.0218 \leq 1,$$

and

$$V_1 = \max(\eta_{11} + \eta_{31}, \eta_{21} + \eta_{41}) = \max\left(\frac{1}{10^9} + \frac{1}{10^8}, \frac{2}{10^9} + \frac{3}{10^8}\right) = \frac{32}{10^9},$$

$$V_2 = \max(\eta_{12} + \eta_{32}, \eta_{22} + \eta_{42}) = \max\left(\frac{2}{10^8} + \frac{3}{10^9}, \frac{1}{10^8} + \frac{1}{10^9}\right) = \frac{23}{10^9},$$

so,

$$2T\alpha_1 V_1 = 4\pi \cdot 1.15 \cdot 10^5 \cdot \frac{32}{10^9} = 0.0462 < 1,$$

$$2T\alpha_2 V_2 = 4\pi \cdot 1.15 \cdot 10^5 \cdot \frac{23}{10^9} = 0.0332 < 1.$$

The conditions of Theorem 2.2 are satisfied, then (3.1) has a  $2\pi$ -periodic solution.

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