Totally Brown subsets of the Golomb space and the Kirch space

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Abstract. A topological space X is totally Brown if for each $n \in \mathbb{N}\setminus\{1\}$ and every nonempty open subsets $U_1, U_2, \dots, U_n$ of X we have $\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) \cap \dots \cap \mathrm{cl}_X(U_n) \neq \emptyset$. Totally Brown spaces are connected. In this paper we consider the Golomb topology τ_G on the set $\mathbb{N}$ of natural numbers, as well as the Kirch topology τ_K on $\mathbb{N}$. Then we examine subsets of these spaces which are totally Brown. Among other results, we characterize the arithmetic progressions which are either totally Brown or totally separated in $(\mathbb{N}, \tau_G)$. We also show that $(\mathbb{N}, \tau_G)$ and $(\mathbb{N}, \tau_K)$ are aposyndetic. Our results generalize properties obtained by A. M. Kirch in 1969 and by P. Szczuka in 2010, 2013 and 2015.

Keywords: arithmetic progression; Golomb topology; Kirch topology; totally Brown space; totally separated space

Classification: 11B25, 54D05, 11A41, 11B05, 54A05, 54D10

1. Introduction

We denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and of natural numbers, respectively. We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_b = \{n \in \mathbb{N} \colon n \geq b\}$ for each $b \in \mathbb{N}$. In this paper we consider arithmetic progressions in both $\mathbb{N}$ and $\mathbb{Z}$. Namely, for each $a, b \in \mathbb{N}$ we define

$$(1) \qquad P(a, b) = \{b + an \colon n \in \mathbb{N}_0\} = b + a\mathbb{N}_0.$$

If $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ we also define

$$(2) \qquad P_F(a, b) = \{b + az \colon z \in \mathbb{Z}\} = b + a\mathbb{Z} \qquad \text{and} \qquad M(a) = \{an \colon n \in \mathbb{N}\}.$$

In [9] H. Furstenberg consider the family

$$(3) \qquad \mathcal{B}_F = \{P_F(a, b) \colon (a, b) \in \mathbb{N} \times \mathbb{Z}\}$$

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which is a base of a topology τ_F in $\mathbb{Z}$. The topological space $(\mathbb{Z}, \tau_F)$, known as the *Furstenberg space*, is metrizable and each member of $\mathcal{B}_F$ is open and closed in $(\mathbb{Z}, \tau_F)$, so this space is zero-dimensional and then hereditarily disconnected.

The purpose of this paper is to consider in $\mathbb{N}$ both the Golomb topology τ_G as well as the Kirch topology τ_K, and present new properties of these spaces. We characterize the arithmetic progressions which are totally Brown in $(\mathbb{N}, \tau_G)$, and show that the basic members of $(\mathbb{N}, \tau_K)$ are totally Brown. We also consider other totally Brown subsets of either $(\mathbb{N}, \tau_G)$ or $(\mathbb{N}, \tau_K)$ as well as properties that involve the closure of a set with respect to these spaces. We characterize the arithmetic progressions which are totally separated in $(\mathbb{N}, \tau_G)$.

The paper is organized in six sections. After this Introduction we consider in Section 2 some notions and results from number theory and from general topology that we use in the paper. In Section 3 we define both Brown and totally Brown spaces. We note that totally Brown spaces are Brown spaces and that Brown spaces are connected. We also give an example of a Brown space which is not totally Brown, and present new properties of Brown and totally Brown spaces. Some of these are similar to known properties of connected spaces. In Theorem 3.6 we show that totally Brown T_2 spaces are aposyndetic.

In Section 4 we present a study of arithmetic progressions in both $\mathbb{N}$ and $\mathbb{Z}$. We focus on the problem of characterizing the intersection of finitely many arithmetic progressions, and on decomposing an arithmetic progression in $\mathbb{N}$ as the union of other arithmetic progressions which are mutually disjoint. Our principal results from this section are Theorems 4.10, 4.12, 4.14 and the ones presented in Subsection 4.3.

In Section 5 we consider the *Golomb space* $(\mathbb{N}, \tau_G)$ and show new properties of it. For example, in [14, Theorem 1, page 169] A. M. Kirch showed that $(\mathbb{N}, \tau_G)$ is not locally connected at 1. In the paper nothing is said about the local connectedness of $(\mathbb{N}, \tau_G)$ at a point distinct to 1. By Corollary 5.3 we infer that $(\mathbb{N}, \tau_G)$ is not locally connected at each of its points. In Subsection 5.2 we present several results that involve the closure in $(\mathbb{N}, \tau_G)$ of arithmetic progressions. By its importance in the rest of the paper, the main result of this subsection is Theorem 5.9.

In Subsection 5.3 we describe totally Brown subsets of $(\mathbb{N}, \tau_G)$. The principal result of this subsection is Theorem 5.12 in which we show that for an arithmetic progression in $\mathbb{N}$, being totally Brown and being connected are equivalent. One consequence of this result is Corollary 5.15 in which we show that an arithmetic progression is either totally separated or totally Brown in $(\mathbb{N}, \tau_G)$ and no matter which is the case, its closure in $(\mathbb{N}, \tau_G)$ is always totally Brown (Theorem 5.16). In Corollary 5.13 we show that $(\mathbb{N}, \tau_G)$ is aposyndetic.

In Section 6 we consider the *Kirch space* $(\mathbb{N}, \tau_K)$ and show new properties of it. For example, this space is totally Brown and aposyndetic. Many results from Section 5 remain valid in $(\mathbb{N}, \tau_K)$. Others differ, like Theorem 6.2. We calculate in Theorem 6.4 the closure in $(\mathbb{N}, \tau_K)$ of an arithmetic progression and in Subsection 6.1 we present subsets of $\mathbb{N}$ that are totally Brown in $(\mathbb{N}, \tau_K)$. The main result of this subsection is Theorem 6.9, in which we show that the members of the base of $(\mathbb{N}, \tau_K)$ are totally Brown.

2. Notions and terminology

For the sets $P(a,b)$, $P_F(a,b)$ and $M(a)$ defined in (1) and (2) we have $P(a,b) \subset \mathbb{N}_b$, $M(a) = P(a,a)$, $M(1) = \mathbb{N}$ and

$$P(a,b) = P_F(a,b) \cap \mathbb{N}_b. \tag{4}$$

Hence $P(a,b) \subset P_F(a,b)$. The symbol $\mathbb{P}$ denotes the set of prime numbers. We consider that $\mathbb{P} \subset \mathbb{N}$. Given nonzero integers a and b, the symbols $\langle a,b\rangle$ and $[a,b]$ denote the greatest common divisor and the least common multiple of a and b, respectively. Note that $\langle a,b\rangle$, $[a,b] \in \mathbb{N}$ and $\langle a,b\rangle[a,b] = |ab|$. If $a,b,c \in \mathbb{N}$ and $\langle b,c\rangle = 1$, then $\langle a,bc\rangle = \langle a,b\rangle\langle a,c\rangle$.

For nonzero integers $a_1, a_2, \ldots, a_k \in \mathbb{N}$, the symbol $[a_1, a_2, \ldots, a_k]$ denotes the least common multiple of $a_1, a_2, \ldots, a_k$, respectively. We say that $a_1, a_2, \ldots, a_k$ are *relatively prime in pairs* if $\langle a_i, a_j\rangle = 1$ for each $i,j \in \{1,2,\ldots,k\}$ with $i \neq j$. For $a,b \in \mathbb{Z}$ the symbol $a|b$ mean that $b = ac$ for some $c \in \mathbb{Z}$. If $a,b \in \mathbb{Z}$ and $m \in \mathbb{N}$, the symbol $a \equiv b \,(\operatorname{mod} m)$ means that $m|(a-b)$. We say that $a \in \mathbb{N}_2$ is *square-free* if it is not divided by the square p^2 of any $p \in \mathbb{P}$. Equivalently a is square-free if its standard prime decomposition is of the form $\prod_{i=1}^k p_i$. It is known that if $a,b \in \mathbb{N}_2$ are square-free and $\langle a,b\rangle \neq 1$, then both $\langle a,b\rangle$ and $[a,b]$ are square-free. Moreover ab is square-free if and only if both a and b are square-free and $\langle a,b\rangle = 1$.

Theorem 2.1. *If $a \in \mathbb{N}_2$ is square-free and $b \in \mathbb{N}$, then there exists $q \in \mathbb{N}$ so that $[a,b] = bq$, $q|a$ and $\langle q,b\rangle = 1$.*

Proof: Let $A = \{p \in \mathbb{P} \colon p|a \text{ and } p \nmid b\}$. Define $q = 1$ if $A = \emptyset$ and $q = \prod_{p \in A} p$ if $A \neq \emptyset$. □

Note that $x \in P_F(a,b)$ if and only if $a|(x-b)$, i.e., $x \equiv b \,(\operatorname{mod} a)$ and $x \in P(a,b)$ if and only if $a|(x-b)$ and $x \geq b$, i.e., $x \equiv b \,(\operatorname{mod} a)$ and $x \in \mathbb{N}_b$.

Let (X,τ) be a topological space and $A \subset X$. Then the symbols $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ denote the closure and the interior of A in (X,τ), respectively. If $A \subset Y \subset X$, then $\operatorname{cl}_Y(A) = Y \cap \operatorname{cl}_X(A)$. If we like to specify the topology τ

on X we write $\mathrm{cl}_{(X,\tau)}(A)$ and $\mathrm{int}_{(X,\tau)}(A)$, respectively. We say that $x \in X$ is an *indiscrete point* of X if $\{U \in \tau \colon x \in U\} = \{X\}$. The topological space (X, τ) is said to be

1) *indiscrete* if $\tau = \{\emptyset, X\}$;
2) T_2 or *Hausdorff* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$;
3) $T_{2(1/2)}$ or *Urysohn* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$ and $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$;
4) *regular* if for each $x \in X$ and every closed subset C of X with $x \notin C$, there exist $U, V \in \tau$ so that $x \in U$, $C \subset V$ and $U \cap V = \emptyset$; and X is T_3 if X is regular and for each $x \in X$ the one-point-set $\{x\}$ is closed in X;
5) *hereditarily disconnected* if X does not contain any connected subset of cardinality larger than one;
6) *totally separated* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$, $X = U \cup V$ and $U \cap V = \emptyset$;
7) *connected im kleinen at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is a connected subset V of X such that $x \in \mathrm{int}_X(V) \subset V \subset U$;
8) *almost connected im kleinen at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is a closed and connected subset V of X such that $\mathrm{int}_X(V) \neq \emptyset$ and $V \subset U$;
9) *locally connected at* $x \in X$ if for each $U \in \tau$ with $x \in U$, there is $V \in \tau$ connected so that $x \in V \subset U$; and *locally connected* if X is locally connected at each of its points.

Urysohn spaces, also called completely Hausdorff spaces, are Hausdorff spaces. The Bing space (B, τ_B) defined in Section 3, the Golomb space $(\mathbb{N}, \tau_G)$ defined in Section 5 and the Kirch space $(\mathbb{N}, \tau_K)$ defined in Section 6, are all Hausdorff spaces which are not Urysohn. Note that X is hereditarily disconnected if and only if the connected component of any point $x \in X$ is the one-point-set $\{x\}$, while X is totally separated if and only if the quasi-component of any point $x \in X$ is the one-point-set $\{x\}$. By [7, Theorem 6.1.22, page 356] totally separated spaces are hereditarily disconnected. A space which is hereditarily disconnected but not totally separated is presented in [17, Example 72, page 91]. If X is compact and Hausdorff then, by [7, Theorem 6.1.23, page 357], each hereditarily disconnected space is totally separated. Being totally separated is hereditary. Though the notions are not equivalent, in the literature both totally separated spaces as well as hereditarily disconnected spaces have been called totally disconnected.

If X is locally connected at $x \in X$, then X is connected im kleinen at x. A space which is connected im kleinen at some point y but not locally connected at y is shown in [17, Examples 119 and 120, page 139].

For notions and results related with number theory and not defined here, we refer the reader to [8]. For those related with general topology, we refer the reader to [7].

3. Totally Brown spaces

In this section we present some properties of the spaces described in the following definition.

Definition 3.1. Let (X, τ) be a topological space. We say that X is

1) a *Brown space* if for every nonempty open subsets U and V of X we have $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \emptyset$;
2) a *totally Brown space* if for every $n \in \mathbb{N}_2$ and each nonempty open subsets $U_1, U_2, \ldots, U_n$ of X we have $\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) \cap \cdots \cap \mathrm{cl}_X(U_n) \neq \emptyset$.

Brown spaces were introduced in [5, page 77], while totally Brown spaces appear in [2, page 424] under the name of superconnected spaces. We prefer the given name in Definition 3.1 since the notion of a superconnected space has appear in the literature with different meanings. For example, in [6] X is said to be superconnected if it is connected and every subset which contains a nonempty open subset is open, while in [15] a space has this property if it contains no disjoint nonempty open sets.

Note that

$$\text{totally Brown} \implies \text{Brown} \implies \text{connected}.$$

The fact that Brown spaces are connected appear in [5, Proposition 7, page 77]. In the same proposition it is shown that every space with an indiscrete point is a Brown space and that a Brown space X is regular if and only if X is indiscrete. Clearly nondegenerate Brown spaces are not Urysohn, so each nondegenerate connected Urysohn space is neither Brown nor totally Brown.

We now present a Brown space (B, τ_B) which is not totally Brown. Let

$$B = \{(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q} \colon r_2 \geq 0\}.$$

For every $x = (r_1, r_2) \in B$ and each $i \in \mathbb{N}$, let

$$\begin{aligned} U_i(x) = \{x\} &\cup \left\{(r, 0) \in B \colon \left|r - \left(r_1 - \frac{r_2}{\sqrt{3}}\right)\right| < \frac{1}{i}\right\} \\ &\cup \left\{(r, 0) \in B \colon \left|r - \left(r_1 + \frac{r_2}{\sqrt{3}}\right)\right| < \frac{1}{i}\right\}. \end{aligned}$$

If $x = (r_1, r_2) \in B$ and $r_2 > 0$, then the set $U_i(x)$ consists of x together with all members of B on the x-axis whose distance from a vertex of the equilateral triangle, with one vertex at x and the other two on the x-axis, is less than $1/i$. If $r_2 = 0$, then $U_i(x)$ consists of all members of B on the x-axis whose distance from x is less than $1/i$. For each $x \in B$, let

$$\mathcal{B}_x = \{U_i(x) \colon i \in \mathbb{N}\}.$$

It is straightforward to see that the collection $\{\mathcal{B}_x \colon x \in B\}$ generates a topology τ_B on B. The topological space (B, τ_B) is called the *Bing space* and was defined in [4] by R. H. Bing.

It is known that (B, τ_B) is a Hausdorff space. Moreover, the closure of the set $U_i(x)$ consists of all points of B whose distance from the line passing through the point x and forming an angle of $60°$ with the x-axis is not larger than $\sqrt{3}/(2i)$, as well as of all points of B whose distance from the line passing through the point x and forming an angle of $120°$ with the x-axis is not larger than $\sqrt{3}/(2i)$, see [17, Example 75, page 93] for a picture of such closure. Hence, for each $x_1, x_2 \in V$ and all $i_1, i_2 \in \mathbb{N}$ we have

$$\mathrm{cl}_B(U_{i_1}(x_1)) \cap \mathrm{cl}_B(U_{i_2}(x_2)) \neq \emptyset.$$

This implies that (B, τ_B) is a Brown space. Hence (B, τ_B) is a Hausdorff space which is not Urysohn. Such space is not totally Brown. For example, if $x_1 = (1, 0)$, $x_2 = (2, 0)$ and $x_3 = (3, 0)$, then

$$\mathrm{cl}_B(U_2(x_1)) \cap \mathrm{cl}_B(U_2(x_2)) \cap \mathrm{cl}_B(U_2(x_3)) = \emptyset.$$

Let X be a topological space and $Y \subset X$. Then Y is totally Brown in X if and only if for every $n \in \mathbb{N}_2$ and each nonempty open subsets $O_1, O_2, \dots, O_n$ of Y we have

$$Y \cap \mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) \cap \cdots \cap \mathrm{cl}_X(O_n) \neq \emptyset.$$

Similarly Y is Brown in X if and only if for every nonempty open subsets U and V of Y we have

$$Y \cap \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \emptyset.$$

Let $f \colon X \to Y$ be a continuous function. In [2, Proposition 2.1, page 424] it is claimed that if X is totally Brown, then $f(X)$ is totally Brown too and that if $C \subset X$ is both dense and totally Brown in X, then X is totally Brown. These results remain valid for Brown spaces.

We now present other properties of Brown and of totally Brown spaces. Some of them are similar to known properties of connected spaces, like the following three.

Theorem 3.2. *Let τ_1 and τ_2 be two topologies on X so that $\tau_1 \subset \tau_2$. If $C \subset X$ is totally Brown (Brown, respectively) in (X, τ_2), then C is totally Brown (Brown, respectively) in (X, τ_1).*

Proof: The result follows from the fact that $\mathrm{cl}_{(X,\tau_2)}(A) \subset \mathrm{cl}_{(X,\tau_1)}(A)$ for each $A \subset X$. □

Under the assumptions of Theorem 3.2, if (X, τ_2) is totally Brown (Brown, respectively), then (X, τ_1) is totally Brown (Brown, respectively).

Theorem 3.3. *Let X be a topological space and $B, Y \subset X$ so that $Y \subset B \subset \mathrm{cl}_X(Y)$. If Y is totally Brown (Brown, respectively) in X, then B is totally Brown (Brown, respectively) in X.*

Proof: Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of B. For each $i \in \{1, 2, \ldots, n\}$ let U_i be an open subset of X so that $O_i = B \cap U_i$. Then $U_i \cap \mathrm{cl}_X(Y) \neq \emptyset$ so $Y \cap U_i \neq \emptyset$ for every $i \in \{1, 2, \ldots, n\}$ and since Y is totally Brown in X, we infer that

$$\emptyset \neq Y \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(Y \cap U_i) \right) \subset B \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(B \cap U_i) \right) = B \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(O_i) \right).$$

This shows that B is totally Brown in X. If Y is Brown in X, proceeding as before we obtain that B is Brown in X. □

Corollary 3.4. *Let X be a topological space and $Y \subset X$. If Y is totally Brown (Brown, respectively) in X, then $\mathrm{cl}_X(Y)$ is totally Brown (Brown, respectively) in X. In particular, if Y is both dense and totally Brown (Brown, respectively) in X, then X is totally Brown (Brown, respectively).*

A subset A of a topological space X is a *closed domain* if $A = \mathrm{cl}_X(\mathrm{int}_X(A))$. Closed domains sets are also called regular closed sets. In Section 5 we show that the Golomb space $(\mathbb{N}, \tau_G)$ is totally Brown and contains subspaces which are not Brown, so being Brown and being totally Brown is not hereditary. By the following result, Brown and totally Brown spaces are hereditary with respect to closed domains.

Theorem 3.5. *If X is totally Brown (Brown, respectively) and U is a nonempty open subset of X, then $\mathrm{cl}_X(U)$ is totally Brown (Brown, respectively) in X. In particular if $A \subset X$ is a nonempty closed domain, then A is totally Brown (Brown, respectively) in X.*

Proof: Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of $\mathrm{cl}_X(U)$. For each $i \in \{1, 2, \ldots, n\}$ let U_i be an open subset of X so that $O_i = \mathrm{cl}_X(U) \cap U_i$.

Then $U \cap U_i \neq \emptyset$ for every $i \in \{1, 2, \ldots, n\}$ and then

$$\emptyset \neq \bigcap_{i=1}^{n} \mathrm{cl}_X(U \cap U_i) = \mathrm{cl}_X(U) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(U \cap U_i) \right)$$
$$\subset \mathrm{cl}_X(U) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(\mathrm{cl}_X(U) \cap U_i) \right) = \mathrm{cl}_X(U) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_X(O_i) \right).$$

This shows that $\mathrm{cl}_X(U)$ is totally Brown in X. If X is Brown, proceeding as before we obtain that $\mathrm{cl}_X(U)$ is Brown in X. □

Let X be a topological space and $a, b \in X$ with $a \neq b$. We say that X is *aposyndetic at a with respect to b* if there exists a closed and connected subset M of X such that $a \in \mathrm{int}_X(M)$ and $b \notin M$. If for each $x \in X \setminus \{a\}$, X is aposyndetic at a with respect to x, we say that X is *aposyndetic at a*. Finally, we say that X is *aposyndetic* if X has this property at each of its points. This notion was defined by F. B. Jones in [12]. If X is T_3 and connected im kleinen at a, then X is aposyndetic at a, so in this sense aposyndesis at a point is a generalization of the notion of connectedness im kleinen at such point. Both Brown and totally Brown T_2 spaces are aposyndetic, according with the following result.

Theorem 3.6. *If X is totally Brown (Brown, respectively) and T_2, then X is aposyndetic.*

Proof: Fix $a \in X$ and let $b \in X \setminus \{a\}$. Since X is T_2 there exist open subsets U and V of X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Using this and Theorem 3.5, the set $M = \mathrm{cl}_X(U)$ is a closed and connected subset of X such that $a \in \mathrm{int}_X(M)$ and $b \notin M$. Hence X is aposyndetic at a. □

In [1] more properties of totally Brown and of Brown spaces are presented.

4. A study of arithmetic progressions

4.1 Elementary properties. In this section we write a systematic study of the arithmetic progressions $P(a, b)$ and $P_F(a, b)$ defined in (1) and (2). We start with the following results, from which it follows that an arithmetic progression in $\mathbb{N}$ is uniquely determined by its initial term and the common difference of successive members.

Proposition 4.1. *Let $a, c \in \mathbb{N}$. If $b \in \mathbb{Z}$ then $P_F(a, b) = P_F(c, b)$ if and only if $a = c$. Moreover if $b \in \mathbb{N}$, then $P(a, b) = P(c, b)$ if and only if $a = c$.*

Proof: Assume that $P_F(a, b) = P_F(c, b)$. Then $a + b \in P_F(a, b) = P_F(c, b)$, so $c|[(a+b)-b]$, i.e., $c|a$. Similarly $c+b \in P_F(c, b) = P_F(a, b)$ and then $a|[(c+b)-b]$, i.e., $a|c$. Hence $a = c$. This shows the first part. The proof of the second part is similar. □

Corollary 4.2. *Let $a, b, c, d \in \mathbb{N}$. Then $P(a, b) = P(c, d)$ if and only if $a = c$ and $b = d$.*

Proof: Assume that $P(a, b) = P(c, d)$. Since $b \in P(a, b) = P(c, d)$, there is $n \in \mathbb{N}_0$ such that $b = cn + d$. Similarly $d \in P(c, d) = P(a, b)$, so there exists $m \in \mathbb{N}_0$ so that $d = am + b$. Hence $b = cn + d = cn + am + b$, so $am + cn = 0$ and then $n = m = 0$ so $b = d$ and, by Proposition 4.1, $a = c$. □

The next result is the core property to show that the family $\mathcal{B}_F$ defined in (3) is a base of the Furstenberg topology τ_F on $\mathbb{Z}$.

Theorem 4.3. *For each $(a, b) \in \mathbb{N} \times \mathbb{Z}$ the following properties are satisfied:*

1) *$P_F(a, b)$ is infinite, $b \in P_F(a, b)$ and $P_F(1, b) = \mathbb{Z}$;*
2) *if $c \in P_F(a, b)$, then $P_F(a, c) = P_F(a, b)$;*
3) *if $c \in \mathbb{N}$, then $P_F(ac, b) \subset P_F(a, b) \cap P_F(c, b)$.*

Similarly, we have the next result.

Theorem 4.4. *For each $(a, b) \in \mathbb{N} \times \mathbb{N}$ the following properties are satisfied:*

1) *$P(a, b)$ is infinite, $b \in P(a, b)$ and $P(1, b) = \mathbb{N}_b$;*
2) *if $c \in P(a, b)$, then $P(a, c) \subset P(a, b)$;*
3) *if $c \in \mathbb{N}$, then $P(ac, b) \subset P(a, b) \cap P(c, b)$.*

4.2 Intersection of arithmetic progressions. Given $k \in \mathbb{N}_2$, it is important to detect when the intersection of k arithmetic progressions is nonempty and, in such case, what we obtain as such intersection. The following result is proved in [13, Theorem 3.12, page 60], and is an extension of the Chinese Remainder theorem.

Theorem 4.5. *Let $k \in \mathbb{N}_2$, $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k \in \mathbb{Z}$. Then the simultaneous congruences*

$$(5) \qquad x \equiv b_1 \pmod{a_1} \qquad x \equiv b_2 \pmod{a_2} \qquad \ldots \qquad x \equiv b_k \pmod{a_k}$$

have a solution if and only if $\langle a_i, a_j \rangle | (b_i - b_j)$ for each $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. When this condition is satisfied, the general solution forms a single congruence class mod $[a_1, a_2, \ldots, a_k]$.

Note that x is a solution of the simultaneous congruences (5) if and only if $x \in \bigcap_{i=1}^{k} P_F(a_i, b_i)$. Using this and Theorem 4.5 we obtain the following result.

Theorem 4.6. *Let $k \in \mathbb{N}_2$, $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k \in \mathbb{Z}$. Then the following conditions are equivalent:*

1) $\bigcap_{i=1}^{k} P_F(a_i, b_i) \neq \emptyset$;
2) $\langle a_i, a_j\rangle|(b_i - b_j)$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$;
3) $P_F(a_i, b_i) \cap P_F(a_j, b_j) \neq \emptyset$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$;
4) *the intersection* $\bigcap_{i=1}^{k} P_F(a_i, b_i)$ *contains an arithmetic progression in* $\mathbb{Z}$.

Take notice that $\bigcap_{i=1}^{k} P_F(a_i, b_i) \neq \emptyset$ if and only if the members of the family $\{P_F(a_i, b_i) \colon i \in \{1, 2, \dots, k\}\}$ have nonempty intersections by pairs. Theorem 4.6 remains valid for arithmetic progressions in $\mathbb{N}$, see [2, Theorem 1.1, page 424] and for $k = 2$, compare with part 2 of [17, Examples 60 and 61, page 82].

Theorem 4.7. *Let* $k \in \mathbb{N}_2$, $a_1, b_1, a_2, b_2, \dots, a_k, b_k \in \mathbb{N}$. *Then the following conditions are equivalent:*

1) $\bigcap_{i=1}^{k} P(a_i, b_i) \neq \emptyset$;
2) $\langle a_i, a_j\rangle|(b_i - b_j)$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$;
3) $P(a_i, b_i) \cap P(a_j, b_j) \neq \emptyset$ *for each* $i, j \in \{1, 2, \dots, k\}$ *with* $i \neq j$;
4) *the intersection* $\bigcap_{i=1}^{k} P(a_i, b_i)$ *contains an arithmetic progression in* $\mathbb{N}$.

If $a_1, a_2, \dots, a_k \in \mathbb{N}$ are relatively prime in pairs, then assertion 2) of Theorems 4.6 and 4.7 is valid, so assertion 1) of such theorems is valid too. Let $a, c \in \mathbb{N}$ and $b, d \in \mathbb{Z}$. By Theorem 4.6,

$$P_F(a, b) \cap P_F(c, d) \neq \emptyset \qquad \text{if and only if } \langle a, c\rangle|(b - d). \tag{6}$$

Moreover if $b, d \in \mathbb{N}$ then by Theorem 4.7,

$$P(a, b) \cap P(c, d) \neq \emptyset \qquad \text{if and only if } \langle a, c\rangle|(b - d). \tag{7}$$

In particular if $\langle a, c\rangle = 1$, then $P_F(a, b) \cap P_F(c, d) \neq \emptyset$ and $P(a, b) \cap P(c, d) \neq \emptyset$.

Corollary 4.8. *If* $a, b, c \in \mathbb{N}$ *and* $\langle a, c\rangle = 1$, *then* $P(a, b) \cap M(c) \neq \emptyset$.

Corollary 4.9. *Let* $a, b, c \in \mathbb{N}$ *be such that* $b \neq c$ *and* $\max\{b, c\} < a$. *Then* $P(a, b) \cap P(a, c) = \emptyset$.

Proof: Let $d = \max\{b, c\}$ and $e = \min\{b, c\}$. If $P(a, b) \cap P(a, c) \neq \emptyset$ then, by (7), $\langle a, a\rangle|(b - c)$, so $a|(d - e)$ and then $a \leq d - e < d$, a contradiction with the fact that $d < a$. Hence $P(a, b) \cap P(a, c) = \emptyset$. □

In the next result we analyze the intersection of finitely many arithmetic progressions. We show that, when such intersection is nonempty, it is another arithmetic progression.

Theorem 4.10. *Let $a_1, a_2, \ldots, a_k \in \mathbb{N}$. If $b_1, b_2, \ldots, b_k \in \mathbb{Z}$ are such that $\bigcap_{i=1}^k P_F(a_i, b_i) \neq \emptyset$ then for each $t \in \bigcap_{i=1}^k P_F(a_i, b_i)$, we have*

$$P_F([a_1, a_2, \ldots, a_k], t) = \bigcap_{i=1}^{k} P_F(a_i, b_i) \tag{8}$$

and, if $a_1, a_2, \ldots, a_k$ are relatively prime in pairs, then

$$P_F(a_1 a_2 \cdots a_k, t) = \bigcap_{i=1}^{k} P_F(a_i, b_i). \tag{9}$$

If $b_1, b_2, \ldots, b_k \in \mathbb{N}$ are such that $\bigcap_{i=1}^k P(a_i, b_i) \neq \emptyset$ then for each $t \in \bigcap_{i=1}^k P(a_i, b_i)$, we have

$$P([a_1, a_2, \ldots, a_k], t) = \left(\bigcap_{i=1}^{k} P(a_i, b_i) \right) \cap \mathbb{N}_t \tag{10}$$

and, if $a_1, a_2, \ldots, a_k$ are relatively prime in pairs, then

$$P(a_1 a_2 \cdots a_k, t) = \left(\bigcap_{i=1}^{k} P(a_i, b_i) \right) \cap \mathbb{N}_t. \tag{11}$$

Proof: Let $b_1, b_2, \ldots, b_k \in \mathbb{Z}$ be such that $\bigcap_{i=1}^k P_F(a_i, b_i) \neq \emptyset$. Fix $t \in \bigcap_{i=1}^k P_F(a_i, b_i)$. Then $a_i | (t - b_i)$ for each $i \in \{1, 2, \ldots, k\}$. Let

$$z \in P_F([a_1, a_2, \ldots, a_k], t) \qquad \text{and} \qquad i \in \{1, 2, \ldots, k\}.$$

Note that $[a_1, a_2, \ldots, a_k] | (z - t)$, so $a_i | (z - t)$. Hence $a_i | [(z - t) + (t - b_i)]$, i.e., $a_i | (z - b_i)$, and then $z \in P_F(a_i, b_i)$. This shows that $z \in \bigcap_{i=1}^k P_F(a_i, b_i)$ so the left side of (8) is contained in its right side. To prove the reverse inclusion, let $z \in \bigcap_{i=1}^k P_F(a_i, b_i)$. Given $i \in \{1, 2, \ldots, k\}$ we know that $a_i | (z - b_i)$ and since $a_i | (t - b_i)$, we have $a_i | [(z - b_i) - (t - b_i)]$, i.e., $a_i | (z - t)$. This implies that $[a_1, a_2, \ldots, a_k] | (z - t)$ and then $z \in P_F([a_1, a_2, \ldots, a_k], t)$. Hence (8) is satisfied. If $a_1, a_2 \ldots, a_k$ are relatively prime in pairs then $[a_1, a_2, \ldots, a_k] = a_1 a_2 \cdots a_k$, so (9) follows from this and (8).

Now assume that $b_1, b_2, \ldots, b_k \in \mathbb{N}$ are such that $\bigcap_{i=1}^k P(a_i, b_i) \neq \emptyset$. Fix $t \in \bigcap_{i=1}^k P(a_i, b_i)$. Then $t \geq b_i$ for each $i \in \{1, 2, \ldots, k\}$, so $\mathbb{N}_t \subset \bigcap_{i=1}^k \mathbb{N}_{b_i}$ and by (4) and (8),

$$\begin{aligned} P([a_1,a_2,\dots,a_k],t) &= P_F([a_1,a_2,\dots,a_k],t)\cap\mathbb{N}_t \\ &= \left(\bigcap_{i=1}^{k} P_F(a_i,b_i)\right)\cap\left(\bigcap_{i=1}^{k}\mathbb{N}_{b_i}\right)\cap\mathbb{N}_t = \left(\bigcap_{i=1}^{k} P_F(a_i,b_i)\cap\mathbb{N}_{b_i}\right)\cap\mathbb{N}_t \\ &= \left(\bigcap_{i=1}^{k} P(a_i,b_i)\right)\cap\mathbb{N}_t = \bigcap_{i=1}^{k}[P(a_i,b_i)\cap\mathbb{N}_t]. \end{aligned}$$

Hence (10) is satisfied. The proof of (11) is like the proof of (9). □

Corollary 4.11. *Let* $a_1,a_2\dots,a_k\in\mathbb{N}$*. If* $b\in\mathbb{Z}$*, then*

$$P_F([a_1,a_2,\dots,a_k],b) = \bigcap_{i=1}^{k} P_F(a_i,b)$$

and if $a_1,a_2,\dots,a_k$ *are relatively prime in pairs, then*

$$P_F(a_1a_2\cdots a_k,b) = \bigcap_{i=1}^{k} P_F(a_i,b). \tag{12}$$

Moreover if $b\in\mathbb{N}$*, then*

$$P([a_1,\dots,a_k],b) = \bigcap_{i=1}^{k} P(a_i,b)$$

and if $a_1,a_2,\dots,a_k$ *are relatively prime in pairs, then*

$$P(a_1a_2\cdots a_k,b) = \bigcap_{i=1}^{k} P(a_i,b).$$

Let $a,c\in\mathbb{N}$. By Theorem 4.10 if $b,d\in\mathbb{Z}$ are such that $P_F(a,b)\cap P_F(c,d)\neq\emptyset$ then for each $t\in P_F(a,b)\cap P_F(c,d)$, we have

$$P_F([a,c],t) = P_F(a,b)\cap P_F(c,d).$$

Moreover if $b,d\in\mathbb{N}$ and $P(a,b)\cap P(c,d)\neq\emptyset$ then for every $t\in P(a,b)\cap P(c,d)$, we have

$$P([a,c],t) = P(a,b)\cap P(c,d)\cap\mathbb{N}_t. \tag{13}$$

Hence,

$$P([a,c],t)\subset P(a,b)\cap P(c,d). \tag{14}$$

The inclusion in (14) might be proper. Take, for example, $P(2,3)$ and $P(6,1)$. Clearly $13 \in P(6,1) \cap P(2,3)$ and by (14),

$$P(6,13) = P([6,2],13) \subset P(6,1) \cap P(2,3).$$

Note that $7 \in [P(6,1) \cap P(2,3)] \setminus P(6,13)$. By (13)

$$P(6,13) = P(6,1) \cap P(2,3) \cap \mathbb{N}_{13} = P(6,1) \cap P(2,3) \cap P(1,13).$$

Now we show that the nonempty intersection of finitely many arithmetic progressions in $\mathbb{N}$ is another arithmetic progression in $\mathbb{N}$.

Theorem 4.12. *For each $i \in \{1,2,\ldots,k\}$ let $a_i, b_i \in \mathbb{N}$ be such that*

$$\bigcap_{i=1}^{k} P(a_i,b_i) \neq \emptyset.$$

If $z \in \mathbb{N}$, then

$$P([a_1,a_2,\ldots,a_k],z) = \bigcap_{i=1}^{k} P(a_i,b_i) \tag{15}$$

if and only if $z = \min\Big(\bigcap_{i=1}^{k} P(a_i,b_i)\Big)$.

Proof: Let $a = [a_1,a_2,\ldots,a_k]$. Assume first that $z = \min\Big(\bigcap_{i=1}^{k} P(a_i,b_i)\Big)$. Then $\bigcap_{i=1}^{k} P(a_i,b_i) \subset \mathbb{N}_z$. Using this and (10) we have

$$P(a,z) = \left(\bigcap_{i=1}^{k} P(a_i,b_i)\right) \cap \mathbb{N}_z = \bigcap_{i=1}^{k} P(a_i,b_i).$$

Now assume that $z \in \mathbb{N}$ is such that (15) holds. Since $z \in P(a,z)$ we have $z \in \bigcap_{i=1}^{k} P(a_i,b_i)$ and if $t \in \bigcap_{i=1}^{k} P(a_i,b_i)$, then $t \in P(a,z)$ so $t \geq z$. Hence, $z = \min\Big(\bigcap_{i=1}^{k} P(a_i,b_i)\Big)$. □

Notice that $7 = \min(P(6,1) \cap P(2,3))$ so, by Theorem 4.12,

$$P(6,7) = P(6,1) \cap P(2,3).$$

If $a,b,c,d \in \mathbb{N}$ are such that $P(a,b) \cap P(c,d) \neq \emptyset$ then, by Theorem 4.12,

$$P([a,c],z) = P(a,b) \cap P(c,d) \qquad \text{if and only if } z = \min(P(a,b) \cap P(c,d)).$$

Let $a_1,a_2,\ldots,a_k \in \mathbb{N}$. Since $[a_1,a_2,\ldots,a_k] = \min\Big(\bigcap_{i=1}^{k} M(a_i)\Big)$ and $M(a_i) = P(a_i,a_i)$ for each $i \in \{1,2,\ldots,k\}$, by Corollary 4.11 and Theorem 4.12 we have the following result.

Theorem 4.13. *Let* $a_1, a_2, \ldots, a_k \in \mathbb{N}$. *Then* $\bigcap_{i=1}^{k} M(a_i) \neq \emptyset$ *and*

$$M([a_1, a_2, \ldots, a_k]) = \bigcap_{i=1}^{k} M(a_i). \tag{16}$$

In particular, if $a_1, a_2, \ldots, a_k$ *are relatively prime in pairs, then*

$$M(a_1 a_2 \cdots a_k) = M(a_1) \cap M(a_2) \cap \cdots \cap M(a_k). \tag{17}$$

As an application of Corollary 4.11 we have the following result (compare (19) with (3.1) of [21, page 15]).

Theorem 4.14. *Let* $a \in \mathbb{N}_2$ *and assume that* $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ *is the standard prime decomposition of* a. *If* $b \in \mathbb{Z}$, *then*

$$P_F(a, b) = \bigcap_{i=1}^{k} P_F(p_i^{\alpha_i}, b) \qquad \text{and} \qquad M(a) = \bigcap_{i=1}^{k} M(p_i^{\alpha_i}). \tag{18}$$

Moreover, if $b \in \mathbb{N}$, *then*

$$P(a, b) = \bigcap_{i=1}^{k} P(p_i^{\alpha_i}, b). \tag{19}$$

Proof: Assume first that $b \in \mathbb{Z}$. Given $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ the equality $\langle p_i, p_j \rangle = 1$ implies that $\langle p_i^{\alpha_i}, p_j^{\alpha_j} \rangle = 1$, so $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ are relatively prime in pairs and then $[p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Since $b \in \bigcap_{i=1}^{k} P_F(p_i^{\alpha_i}, b)$, by (12) and (17),

$$P_F(a, b) = P_F(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, b) = \bigcap_{i=1}^{k} P_F(p_i^{\alpha_i}, b)$$

and

$$M(a) = M(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \bigcap_{i=1}^{k} M(p_i^{\alpha_i}).$$

Hence (18) holds. If $b \in \mathbb{N}$ then, proceeding as before we obtain (19). □

The following result generalizes [18, Lemma 4.1, page 904].

Theorem 4.15. *Let* $a, c \in \mathbb{N}$ *be such that* $a|c$. *If* $b, d \in \mathbb{Z}$ *are such that* $P_F(a, b) \cap P_F(c, d) \neq \emptyset$, *then* $P_F(c, d) \subset P_F(a, b)$. *Moreover if* $b, d \in \mathbb{N}$, $P(a, b) \cap P(c, d) \neq \emptyset$ *and* $b < a$, *then* $P(c, d) \subset P(a, b)$.

Proof: To show the first part let $b, d \in \mathbb{Z}$ be such that $P_F(a, b) \cap P_F(c, d) \neq \emptyset$. By (6), $\langle a, c \rangle | (b - d)$. Since $a|c$ we infer that $\langle a, c \rangle = a$, so $a|(b - d)$. Let $y \in P_F(c, d)$. Then $c|(y - d)$ and since $a|c$ we have $a|(y - d)$. This implies that $a|[(y - d) - (b - d)]$, i.e., $a|(y - b)$, so $y \in P_F(a, b)$. Hence $P_F(c, d) \subset P_F(a, b)$.

To show the second part let $b, d \in \mathbb{N}$ be such that $P(a,b) \cap P(c,d) \neq \emptyset$ and $b < a$. Let $y \in P(c,d)$. Proceeding as in the first part, we infer that $a|(y-b)$. Let $x \in \mathbb{Z}$ be such that $y - b = ax$. If $x \leq -1$, then $ax \leq -a$. Hence $y - b \leq -a$ and then $0 \leq y \leq b - a$, so $a \leq b$, a contradiction. Hence $x \geq 0$ and then $y \in P(a,b)$. □

Corollary 4.16. *Let $a, c \in \mathbb{N}$ be such that $a|c$. If $b \in \mathbb{Z}$, then $P_F(c,b) \subset P_F(a,b)$. Moreover, if $b \in \mathbb{N}$ and $b < a$, then $P(c,b) \subset P(a,b)$.*

Corollary 4.17. *Let $a, c \in \mathbb{N}$. If $b \in \mathbb{Z}$, then $P_F(ac,b) \subset P_F([a,c],b)$. Moreover $P_F(ac,b) = P_F([a,c],b)$ if and only if $\langle a,c \rangle = 1$. If $b \in \mathbb{N}$ then $P(ac,b) \subset P([a,c],b)$ and $P(ac,b) = P([a,c],b)$ if and only if $\langle a,b \rangle = 1$.*

Theorem 4.18. *Let $a, c \in \mathbb{N}$ and $b, d \in \mathbb{Z}$. If $P_F(c,d) \subset P_F(a,b)$, then $a|c$. On the other hand, if $b, d \in \mathbb{N}$ and $P(c,d) \subset P(a,b)$, then $a|c$.*

Proof: Since $d, c + d \in P_F(c,d)$ we have $d, c + d \in P_F(a,b)$, so $a|(d-b)$ and $a|[(c+d)-b)]$. Hence $a|[(c+d-b)-(d-b)]$, i.e., $a|c$. □

4.3 Decompositions of arithmetic progressions. Without proof, in several papers of P. Szczuka (also known as P. Szyszkowska), like [18, page 902], [19, page 878], [22, page 1010] and [24, page 93], an arithmetic progression in $\mathbb{N}$ is decomposed as the union of pairwise disjoint arithmetic progressions in $\mathbb{N}$. In the next result, we present the precise meaning of such decomposition and give a proof.

Theorem 4.19. *Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$. Then for each $t \in \mathbb{N}_2$,*

$$P(a,b) = \bigcup_{k=0}^{a^{t-1}-1} P(a^t, ak+b) \tag{20}$$

and the members of the family $\mathcal{F} = \{P(a^t, ak+b) \colon k \in \{0,1,\ldots,a^{t-1}-1\}\}$ are pairwise disjoint.

Proof: Let $A = \{0,1,\ldots,a^{t-1}-1\}$. Fix $t \in \mathbb{N}_2$ and let x in the right side of (20). Then there exist $m \in \mathbb{N}_0$ and $k_0 \in A$ so that $x = a^t m + (ak_0 + b)$. Since $t > 1$ we have $t - 1 \geq 1$ so $a^{t-1}m + k_0 \in \mathbb{N}_0$ satisfies

$$x = a(a^{t-1}m) + (ak_0 + b) = a(a^{t-1}m + k_0) + b \in P(a,b).$$

This shows that the right side of (20) is a subset of its left side. Now assume that x is in the left side of (20). Let $l, m \in \mathbb{N}_0$ and $k \in A$ be such that $x = al + b$ and $l = a^{t-1}m + k$. Then

$$x = al + b = a(a^{t-1}m + k) + b = a^t m + (ak+b) \in P(a^t, ak+b).$$

Hence x is in the right side of (20) and such equality is satisfied.

To show the second part, let $n_1, n_2 \in \mathbb{N}_0$ and $k_1, k_2 \in A$ be such that $k_1 \neq k_2$ and $a^t n_1 + (ak_1 + b) = a^t n_2 + (ak_2 + b)$. Then $a^{t-1}(n_1 - n_2) = k_2 - k_1$. Hence $k_2 \equiv k_1 \pmod{a^{t-1}}$. Now, since k_1 and k_2 are distinct members of the set A, which is a complete residue system modulo a^{t-1}, the integers k_1 and k_2 are not congruent modulo a^{t-1}. From this contradiction, we infer that the members of $\mathcal{F}$ are pairwise disjoint. □

Now, given an arithmetic progression $P(a, b)$ with $a \in \mathbb{N}_2$, and $x, y \in P(a, b)$ with $x \neq y$, we decompose $P(a, b)$ as the union of two disjoint sets U and V so that $x \in U$ and $y \in V$.

Theorem 4.20. *Let $a \in \mathbb{N}_2$, $b \in \mathbb{N}$ and $x, y \in P(a, b)$ with $x < y$. Write $x = am + b$, $y = an + b$ with $0 \leq m < n$. Then $P(a, b) = U \cup V$, where*

$$(21) \qquad U = \bigcup_{k=0}^{m} P(a^{n+1}, ak + b) \qquad \text{and} \qquad V = \bigcup_{k=m+1}^{a^n - 1} P(a^{n+1}, ak + b).$$

Moreover, $x \in U$, $y \in V$ and the members of the family

$$\mathcal{F} = \{P(a^{n+1}, ak + b) \colon k \in \{0, 1, \ldots, a^n - 1\}\}$$

are pairwise disjoint. In particular, $U \cap V = \emptyset$.

Proof: Since $0 \leq m < n < 2^n \leq a^n$ we have $0 \leq m < m + 1 \leq n \leq a^n - 1$, so

$$x = am + b = a^{n+1}(0) + am + b \in P(a^{n+1}, am + b) \subset U$$

and

$$y = an + b = a^{n+1}(0) + an + b \in P(a^{n+1}, an + b) \subset V.$$

Applying (20) with $t = n + 1$, which belongs to $\mathbb{N}_2$, we infer that

$$\begin{aligned} P(a, b) &= \bigcup_{k=0}^{a^{t-1}-1} P(a^t, ak + b) = \bigcup_{k=0}^{a^n - 1} P(a^{n+1}, ak + b) \\ &= \left(\bigcup_{k=0}^{m} P(a^{n+1}, ak + b) \right) \cup \left(\bigcup_{k=m+1}^{a^n - 1} P(a^{n+1}, ak + b) \right) = U \cup V. \end{aligned}$$

Moreover, by Theorem 4.19, the members of the family

$$\begin{aligned} \mathcal{F} &= \{P(a^t, ak + b) \colon k \in \{0, 1, \ldots, a^{t-1} - 1\}\} \\ &= \{P(a^{n+1}, ak + b) \colon k \in \{0, 1, \ldots, a^n - 1\}\} \end{aligned}$$

are pairwise disjoint. In particular, $U \cap V = \emptyset$. □

Corollary 4.21. *Let $a \in \mathbb{N}_2$, $b \in \mathbb{N}$ and $x, y \in P(a,b)$ with $x \neq y$. Then there exist U and V so that $P(a,b) = U \cup V$, $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

5. The Golomb space

In both [10, page 663] and [11, page 179], S. W. Golomb showed that the family

$$\mathcal{B}_G = \{P(a,b)\colon (a,b) \in \mathbb{N} \times \mathbb{N} \text{ and } \langle a,b \rangle = 1\}$$

is a base for a topology τ_G in $\mathbb{N}$. In [17, Examples 60 and 61, page 82] this topology is called the relatively prime integer topology. However as it is now more popular in general topology, we call τ_G the *Golomb topology* and refer to the topological space $(\mathbb{N}, \tau_G)$ as the *Golomb space*. Clearly

$$\tau_G = \{\emptyset\} \cup \{U \subset \mathbb{N}\colon \text{for each } b \in U \text{ there is } a \in \mathbb{N} \text{ so that } \langle a,b \rangle = 1 \text{ and } P(a,b) \subset U\}.$$

In this section we present new properties of $(\mathbb{N}, \tau_G)$. Let $b, c \in \mathbb{N}$ be such that $b \neq c$. Take $a \in \mathbb{P}$ with $\max\{b,c\} < a$. Then $\langle a,b \rangle = \langle a,c \rangle = 1$, so $P(a,b)$ and $P(a,c)$ are open subsets in $(\mathbb{N}, \tau_G)$ that contain b and c, respectively. By Corollary 4.9, $P(a,b) \cap P(a,c) = \emptyset$. This implies that $(\mathbb{N}, \tau_G)$ is Hausdorff, so it is not superconnected in the sense of [15].

If $p \in \mathbb{P}$, then $\langle p,i \rangle = 1$ for each $i \in \{1, 2, \dots, p-1\}$ so each $P(p,i)$ is open in $(\mathbb{N}, \tau_G)$. Moreover

$$\mathbb{N} \setminus M(p) = \bigcup_{i=1}^{p-1} P(p,i),$$

so $M(p)$ is closed in $(\mathbb{N}, \tau_G)$.

5.1 Totally separated subsets of the Golomb space. In this subsection we show that the members of the base $\mathcal{B}_G$ whose common difference of successive members is greater than one are totally separated. We then derive some consequences of this fact. The following result was proved differently in [2, Proposition 3.2, page 427].

Theorem 5.1. *Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ be such that $\langle a,b \rangle = 1$. Then $P(a,b)$ is totally separated. In particular, $P(a,b)$ is hereditarily disconnected.*

Proof: Let $x, y \in P(a,b)$ with $x \neq y$. Assume, without loss of generality, that $x < y$. Write $x = am + b$, $y = an + b$ with $0 \leq m < n$ and consider the sets U and V defined in (21). By Theorem 4.20 we have $P(a,b) = U \cup V$, $U \cap V = \emptyset$, $x \in U$ and $y \in V$. Fix $k \in \mathbb{N}_0$. If $\langle a^{n+1}, ak+b \rangle \neq 1$, there is $p \in \mathbb{P}$ so that $p | a^{n+1}$

and $p|(ak+b)$. Then $p|a$ and $p|b$. Since this contradicts the fact that $\langle a,b\rangle = 1$ we infer that $\langle a^{n+1}, ak+b\rangle = 1$, and then both U and V are open in $(\mathbb{N}, \tau_G)$. □

If $A \subset Y \subset \mathbb{N}$, we denote by $\mathrm{int}_{\mathbb{N}}(A)$ the interior of A in $(\mathbb{N}, \tau_G)$ and by $\mathrm{int}_Y(A)$ the interior of A in the subspace Y of $(\mathbb{N}, \tau_G)$.

Theorem 5.2. *If $a, b \in \mathbb{N}$ and $\langle a,b\rangle = 1$, then $P(a,b)$ is neither connected im kleinen nor almost connected im kleinen at each of its points.*

Proof: Let us assume that $P(a,b)$ is either connected im kleinen or almost connected im kleinen at $c \in P(a,b)$. We divide the proof in two cases. Assume first that $a \in \mathbb{N}_2$. Since $\langle a,c\rangle = 1$ and $P(a,c)$ is an open subset of $P(a,b)$ that contains c, there is a connected subset C of $P(a,c)$ so that $\mathrm{int}_{P(a,b)}(C) \neq \emptyset$. Hence $\mathrm{int}_{\mathbb{N}}(C) \neq \emptyset$ and since nonempty open subsets of $(\mathbb{N}, \tau_G)$ are infinite, the set C is infinite. This contradicts the fact that, by Theorem 5.1, $P(a,c)$ is hereditarily disconnected.

Now assume that $a = 1$. Let $p \in \mathbb{P}$ be so that $c < p$. Then $\langle p,c\rangle = 1$ and since $P(p,c)$ is an open subset of $P(a,b)$ that contains c, there is a connected subset D of $P(p,c)$ so that $\mathrm{int}_{P(a,b)}(D) \neq \emptyset$. Hence $\mathrm{int}_{\mathbb{N}}(D) \neq \emptyset$ and since nonempty open subsets of $(\mathbb{N}, \tau_G)$ are infinite, the set D is infinite. This contradicts the fact that, by Theorem 5.1, $P(p,c)$ is hereditarily disconnected. □

Corollary 5.3. *The Golomb space $(\mathbb{N}, \tau_G)$ is neither connected im kleinen nor almost connected im kleinen at each of its points. In particular, $(\mathbb{N}, \tau_G)$ is not locally connected.*

Proof: Since $\mathbb{N} = P(1,1)$ the result follows from Theorem 5.2. □

As we mention in Section 1, by Corollary 5.3 we infer that $(\mathbb{N}, \tau_G)$ is not locally connected at each of its points.

5.2 The closure in the Golomb topology. We present in this subsection several results that involve the closure of an arithmetic progression with respect to the Golomb space. If $A \subset \mathbb{N}$ we denote by $\mathrm{cl}_{\mathbb{N}}(A)$ the closure of A in $(\mathbb{N}, \tau_G)$. The next result shows that the closure in $(\mathbb{N}, \tau_G)$ of $P(a,b)$ might contain members of $P(a,b)$ which are natural numbers less than b.

Theorem 5.4. *If $a, b \in \mathbb{N}$, then $P_F(a,b) \cap \mathbb{N} \subset \mathrm{cl}_{\mathbb{N}}(P(a,b))$.*

Proof: Let $x \in P_F(a,b) \cap \mathbb{N}$ and $z \in \mathbb{Z}$ be so that $x = az+b$. Let W be an open subset of $(\mathbb{N}, \tau_G)$ with $x \in W$. Take $c \in \mathbb{N}$ so that $\langle c,x\rangle = 1$ and $P(c,x) \subset W$. Since $\langle c,a\rangle | az$ we have $\langle c,a\rangle | (x-b)$ so, by (7), $P(c,x) \cap P(a,b) \neq \emptyset$ and then $x \in \mathrm{cl}_{\mathbb{N}}(P(a,b))$. □

Let $a, b \in \mathbb{N}$ and $n \in \mathbb{N}$. Clearly $P(a^n, b) \subset \mathrm{cl}_{\mathbb{N}}(P(a^n, b))$. In the next result we show that $M(a) \subset \mathrm{cl}_{\mathbb{N}}(P(a^n, b))$. Note that if $\langle a, b\rangle = 1$, then $\langle a^n, b\rangle = 1$, so $P(a^n, b) \in \mathcal{B}_G$.

Theorem 5.5. *If $a, b \in \mathbb{N}$, then*

$$M(a) \subset \mathrm{cl}_{\mathbb{N}}(P(a^n, b)) \qquad \text{for every } n \in \mathbb{N}. \tag{22}$$

Moreover, for each nonempty open subset U of $(\mathbb{N}, \tau_G)$, there is $c \in \mathbb{N}$ so that $M(c) \subset \mathrm{cl}_{\mathbb{N}}(U)$.

Proof: Fix $n \in \mathbb{N}$ and let $c \in M(a)$ and W be an open subset of $(\mathbb{N}, \tau_G)$ with $c \in W$. Take $d \in \mathbb{N}$ so that $\langle d, c\rangle = 1$ and $P(d, c) \subset W$. Assume that $\langle d, a^n\rangle \neq 1$ and let $p \in \mathbb{P}$ be so that $p|d$ and $p|a^n$. Hence $p|a$, so $p|d$ and $p|c$ and then $\langle d, c\rangle \neq 1$, a contradiction. Therefore $\langle d, a^n\rangle = 1$. This implies, by (7), that $P(d, c) \cap P(a^n, b) \neq \emptyset$ and then $W \cap P(a^n, b) \neq \emptyset$ so $c \in \mathrm{cl}_{\mathbb{N}}(P(a^n, b))$. This shows (22).

Now assume that U is a nonempty open subset of $(\mathbb{N}, \tau_G)$. Let $b \in U$ and $c \in \mathbb{N}$ so that $\langle c, b\rangle = 1$ and $P(c, b) \subset U$. Applying (22) we obtain that $M(c) \subset \mathrm{cl}_{\mathbb{N}}(P(c, b)) \subset \mathrm{cl}_{\mathbb{N}}(U)$. □

Corollary 5.6. *For each $b \in \mathbb{N}$ the arithmetic progression $P(1, b)$ is dense in $(\mathbb{N}, \tau_G)$, i.e., $\mathrm{cl}_{\mathbb{N}}(P(1, b)) = \mathbb{N}$.*

Corollary 5.7. *For any finite collection $\{P(a_i, b_i)\colon i \in \{1, 2, \dots, k\}\}$ of arithmetic progressions in $\mathbb{N}$, we have*

$$M([a_1, a_2, \dots, a_k]) \subset \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)). \tag{23}$$

In particular, $\bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)) \neq \emptyset$. Moreover, for each $c \in \mathbb{N}$ we have

$$M(c) \cap \left(\bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))\right) \neq \emptyset. \tag{24}$$

Proof: Let us assume that $a = [a_1, a_2, \dots, a_k]$ and $c \in \mathbb{N}$. By Theorem 5.5, $M(a_i) \subset \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))$ for each $i \in \{1, 2, \dots, k\}$. Using this and (16) we have

$$\emptyset \neq M(a) = \bigcap_{i=1}^{k} M(a_i) \subset \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)).$$

Moreover

$$\emptyset \neq M([c, a]) = M(c) \cap M(a) \subset M(c) \cap \left(\bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))\right).$$

□

The next result is part 10 of [17, Examples 60 and 61, page 83]. The proof that we present is different.

Theorem 5.8. *If* $b \in \mathbb{N}$ *and* $p \in \mathbb{P}$*, then*

$$\mathrm{cl}_{\mathbb{N}}(P(p^n, b)) = M(p) \cup [P_F(p^n, b) \cap \mathbb{N}] \qquad \text{for each } n \in \mathbb{N}. \tag{25}$$

PROOF: Fix $n \in \mathbb{N}$. By Theorems 5.4 and 5.5 the right side of (25) is contained in its left side. To show the other inclusion take $x \in \mathrm{cl}_{\mathbb{N}}(P(p^n, b))$ and assume that $x \notin M(p)$. If $\langle p^n, x\rangle \neq 1$, then there is $q \in \mathbb{P}$ so that $q|p^n$ and $q|x$. Then $q|p$ so $q = p$ and then $p|x$ contradicting the fact that $x \notin M(p)$. Then $\langle p^n, x\rangle = 1$, so $P(p^n, x) \in \mathcal{B}_G$. Hence $P(p^n, x) \cap P(p^n, b) \neq \emptyset$ and by (7), we have $p^n|(x - b)$ implying that $x \in P_F(p^n, b) \cap \mathbb{N}$. □

If in Theorem 5.8 we assume that $\langle p, b\rangle = 1$, then $\langle p^n, b\rangle = 1$ for each $n \in \mathbb{N}$, so $P(p^n, b) \in \mathcal{B}_G$.

By (23) the intersection of the closures in $(\mathbb{N}, \tau_G)$ of finitely many arithmetic progressions is always nonempty. Note that the intersection of such arithmetic progressions might be empty, but when this is not the case, the next result calculates the intersection of such closures just as the closure of the intersection of the progressions.

Theorem 5.9. *Let* $a_1, b_1, a_2, b_2, \ldots, a_k, b_k \in \mathbb{N}$ *be such that* $\bigcap_{i=1}^k P(a_i, b_i) \neq \emptyset$*. Then*

$$\mathrm{cl}_{\mathbb{N}}\left(\bigcap_{i=1}^{k} P(a_i, b_i)\right) = \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)). \tag{26}$$

PROOF: Clearly the left side of (26) is contained in its right side, so to show the reverse inclusion, let b be a member of the right side of (26) and W be an open subset of $(\mathbb{N}, \tau_G)$ with $b \in W$. Take $a \in \mathbb{N}$ such that $\langle a, b\rangle = 1$ and $P(a, b) \subset W$. Then

$$P(a, b) \cap P(a_i, b_i) \neq \emptyset \qquad \text{for every } i \in \{1, 2, \ldots, k\}. \tag{27}$$

Since $\bigcap_{i=1}^k P(a_i, b_i) \neq \emptyset$, by Theorem 4.7, we have

$$P(a_i, b_i) \cap P(a_j, b_j) \neq \emptyset \qquad \text{for each } i, j \in \{1, 2, \ldots, k\} \text{ with } i \neq j. \tag{28}$$

Combining (27) and (28) we infer, applying again Theorem 4.7, that

$$P(a, b) \cap \left(\bigcap_{i=1}^{k} P(a_i, b_i)\right) \neq \emptyset,$$

so $W \cap \left(\bigcap_{i=1}^k P(a_i, b_i)\right) \neq \emptyset$ and then b is in the left side of (26). □

The following result appears with a different proof in [2, Lemma 2.2, page 425].

Theorem 5.10. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. If $b \in \mathbb{N}$, then*

$$\mathrm{cl}_{\mathbb{N}}(P(a,b)) = \mathbb{N} \cap \left(\bigcap_{i=1}^{k} [M(p_i) \cup P_F(p_i^{\alpha_i}, b)] \right). \tag{29}$$

Proof: Combining (19), (25) and (26) we have

$$\begin{aligned} \mathrm{cl}_{\mathbb{N}}(P(a,b)) &= \mathrm{cl}_{\mathbb{N}}\left(\bigcap_{i=1}^{k} P(p_i^{\alpha_i}, b) \right) = \bigcap_{i=1}^{k} \mathrm{cl}_{\mathbb{N}}(P(p_i^{\alpha_i}, b)) \\ &= \bigcap_{i=1}^{k} [M(p_i) \cup (P_F(p_i^{\alpha_i}, b) \cap \mathbb{N})] = \mathbb{N} \cap \left(\bigcap_{i=1}^{k} [M(p_i) \cup P_F(p_i^{\alpha_i}, b)] \right). \end{aligned}$$

□

5.3 Totally Brown subsets of the Golomb space. In this subsection we describe some subsets of $\mathbb{N}$ which are totally Brown, and hence connected in $(\mathbb{N}, \tau_G)$. We will use the following families of arithmetic progressions:

$$\mathcal{M} = \{M(a) \colon a \in \mathbb{N}\} \qquad \text{and} \qquad \mathcal{P} = \{P(a,b) \colon (a,b) \in \mathbb{N} \times \mathbb{N}\}.$$

Note that $\mathcal{M} \subset \mathcal{P}$ and by (16), $\mathcal{M}$ is closed under finite intersections. The following result shows that the union of any collection $\mathcal{A}$ of arithmetic progressions in $\mathbb{N}$, that contains at least one member of $\mathcal{M}$, is totally Brown in $(\mathbb{N}, \tau_G)$. When the family $\mathcal{A}$ is countable this result was proved slightly different in [2, Proposition 2.3, page 425]. As a particular case we obtain [20, Lemma 3.2, page 431] which claims that $M(p) \cup P(p,1)$ is connected in $(\mathbb{N}, \tau_G)$.

Theorem 5.11. *Let $\mathcal{A} \subset \mathcal{P}$ be such that $\mathcal{A} \cap \mathcal{M} \neq \emptyset$. Then $W = \bigcup \mathcal{A}$ is totally Brown in $(\mathbb{N}, \tau_G)$. In particular, any union of members of $\mathcal{M}$ is totally Brown in $(\mathbb{N}, \tau_G)$.*

Proof: Let $c \in \mathbb{N}$ be such that $M(c) \in \mathcal{A}$. Hence $M(c) \subset W$. Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of W. For each $i \in \{1, 2, \ldots, n\}$, let U_i be an open subset of $(\mathbb{N}, \tau_G)$ so that $O_i = W \cap U_i$ and take $b_i \in O_i$. Then there exist $a_i, c_i, d_i \in \mathbb{N}$ so that $\langle a_i, b_i \rangle = 1$, $P(a_i, b_i) \subset U_i$ and $b_i \in P(c_i, d_i) \in \mathcal{A}$. Then $P(c_i, d_i) \subset W$ and $P(c_i, d_i) \cap P(a_i, b_i) \neq \emptyset$ for every $i \in \{1, 2, \ldots, n\}$. Combining

(24) and (26) we have

$$\begin{aligned}
\emptyset &\neq M(c) \cap \left(\bigcap_{i=1}^{n} [\mathrm{cl}_{\mathbb{N}}(P(c_i, d_i)) \cap \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))] \right) \\
&= M(c) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(c_i, d_i) \cap P(a_i, b_i)) \right) \subset W \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(W \cap U_i) \right) \\
&= W \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(O_i) \right).
\end{aligned}$$

□

Now we characterize the arithmetic progressions $P(a,b)$ that are totally Brown in $(\mathbb{N}, \tau_G)$. To do this for each $a \in \mathbb{N}$ we define

$$\Theta(a) = \{p \in \mathbb{P} \colon p|a\}.$$

Note that $\Theta(a) = \emptyset$ if and only if $a = 1$. In [18, Theorem 3.3, page 901] it is shown that $P(a,b)$ is connected in $(\mathbb{N}, \tau_G)$ if and only if $\Theta(a) \subset \Theta(b)$. From the results that we have presented, the proof of 3) implies 4) in the next result is simpler than the one that appears in [18, Theorem 3.3, page 901]. Clause 7) of the next result was also proved in [2, Corollary 2.4, page 425].

Theorem 5.12. *Let $a, b \in \mathbb{N}$. Then the following assertions are equivalent:*

1) *$P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$;*
2) *$P(a,b)$ is Brown in $(\mathbb{N}, \tau_G)$;*
3) *$P(a,b)$ is connected in $(\mathbb{N}, \tau_G)$;*
4) *$\Theta(a) \subset \Theta(b)$.*

In particular

5) *$M(c)$ is totally Brown in $(\mathbb{N}, \tau_G)$ for every $c \in \mathbb{N}$;*
6) *if $\langle a, b\rangle = 1$, then $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$ if and only if $a = 1$;*
7) *$(\mathbb{N}, \tau_G)$ is totally Brown.*

Proof: We have seen that totally Brown spaces are Brown and that Brown spaces are connected so 1) implies 2) and 2) implies 3). Now assume that $P(a,b)$ is connected in $(\mathbb{N}, \tau_G)$. If $\Theta(a) \not\subset \Theta(b)$ then $a \in \mathbb{N}_2$ and there is $p \in \Theta(a) \setminus \Theta(b)$. Then $\langle p, b\rangle = 1$ so, by Theorem 5.1, $P(p,b)$ is totally separated. Since $p|a$ we have $P(a,b) \subset P(p,b)$ so $P(a,b)$ is totally separated too. Since this contradicts the fact that $P(a,b)$ is connected, we infer that $\Theta(a) \subset \Theta(b)$, so 3) implies 4).

Now assume that $\Theta(a) \subset \Theta(b)$. Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of $P(a,b)$. For each $i \in \{1, 2, \ldots, n\}$, let U_i be an open subset of $(\mathbb{N}, \tau_G)$ so that $O_i = P(a,b) \cap U_i$ and take $b_i \in O_i$. Then $a|(b - b_i)$ and there is $a_i \in \mathbb{N}$ with $\langle a_i, b_i \rangle = 1$ and $P(a_i, b_i) \subset U_i$. Note that $P(a, b) \cap P(a_i, b_i) \neq \emptyset$. Assume that $\langle a, a_i \rangle \neq 1$ for some $i \in \{1, 2, \ldots, n\}$ and let $p \in \mathbb{P}$ be such that $p|a$ and $p|a_i$. Then $p|b$ and since $a|(b - b_i)$ we have $p|b_i$. Since this contradicts the fact that $\langle a_i, b_i \rangle = 1$, we infer that $\langle a, a_i \rangle = 1$ for each $i \in \{1, 2, \ldots, n\}$ and then $\langle a, a_1 a_2 \cdots a_n \rangle = 1$. This implies, by Corollary 4.8, that $P(a, b) \cap M(a_1 a_2 \cdots a_n) \neq \emptyset$. By Theorem 5.5 and (26) we have

$$
\begin{aligned}
\emptyset \neq P(a,b) \cap M(a_1 a_2 \cdots a_n) &\subset P(a,b) \cap \left(\bigcap_{i=1}^{n} M(a_i) \right) \\
&\subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i)) \right) \\
&= P(a,b) \cap \left(\bigcap_{i=1}^{n} [\mathrm{cl}_{\mathbb{N}}(P(a,b)) \cap \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))] \right) \\
&= P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a,b) \cap P(a_i, b_i)) \right) \subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(a,b) \cap U_i) \right) \\
&= P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(O_i) \right).
\end{aligned}
$$

This shows that $P(a, b)$ is totally Brown in $(\mathbb{N}, \tau_G)$. Hence 4) implies 1) and this completes the proof that assertions 1), 2), 3) and 4) are equivalent. To show 5) let $c \in \mathbb{N}$. Since $M(c) = P(c, c)$ and $\Theta(c) \subset \Theta(c)$, by 3) implies 1), $M(c)$ is totally Brown in $(\mathbb{N}, \tau_G)$.

To show 6) assume that $\langle a, b \rangle = 1$. Then $\Theta(a) \cap \Theta(b) = \emptyset$. If $P(a, b)$ is totally Brown in $(\mathbb{N}, \tau_G)$ then, by 1) implies 4), we have $\Theta(a) \subset \Theta(b)$ so $\Theta(a) = \emptyset$ and then $a = 1$. Conversely, if $a = 1$, then $\emptyset = \Theta(a) \subset \Theta(b)$ so by 4) implies 1) the set $P(a, b)$ is totally Brown in $(\mathbb{N}, \tau_G)$. This shows 6). Finally, since $\mathbb{N} = M(1)$ by 5) $(\mathbb{N}, \tau_G)$ is totally Brown. □

Corollary 5.13. *The Golomb space $(\mathbb{N}, \tau_G)$ is aposyndetic.*

Proof: Since $(\mathbb{N}, \tau_G)$ is totally Brown and T_2, the result follows from this and Theorem 3.6. □

Corollary 5.14. *Let $a, b \in \mathbb{N}$. Then $P(a, b)$ is totally separated if and only if $\Theta(a) \not\subset \Theta(b)$.*

Proof: Assume first that $P(a, b)$ is totally separated. Then $P(a, b)$ is not connected in $(\mathbb{N}, \tau_G)$ and, by Theorem 5.12, $\Theta(a) \not\subset \Theta(b)$. Now assume that $\Theta(a) \not\subset \Theta(b)$ and let $p \in \Theta(a) \setminus \Theta(b)$. Then $\langle p, b \rangle = 1$ and since $p|a$ we have $P(a, b) \subset P(p, b)$. By Theorem 5.1, $P(p, b)$ is totally separated and then $P(a, b)$ is totally separated too. □

Corollary 5.15. *For each $a, b \in \mathbb{N}$ the arithmetic progression $P(a,b)$ is either totally separated or totally Brown in $(\mathbb{N}, \tau_G)$.*

By Corollaries 5.3 and 5.13, $(\mathbb{N}, \tau_G)$ is aposyndetic at each of its points and connected im kleinen at none of its points.

We mention another application of Theorem 5.12. Given $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ let $q = \prod_{p \in \Theta(a)} p$. Note that q is square-free and if $\Theta(a) \subset \Theta(b)$ then $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$ and $P(a,b) \subset M(q)$. If $a = 1$, then $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$ and $P(a,b) \subset M(a)$. This shows that the members of $\mathcal{P}$ which are totally Brown in $(\mathbb{N}, \tau_G)$ are subsets of members of $\mathcal{M}$ of the form $M(q)$ with $q = 1$ or q square-free, which are totally Brown in $(\mathbb{N}, \tau_G)$ too.

By 7) of Theorem 5.12, $(\mathbb{N}, \tau_G)$ is connected. By (25)

$$\mathrm{cl}_{\mathbb{N}}(P(4,3)) = M(2) \cup [P_F(4,3) \cap \mathbb{N}] = M(2) \cup P(4,3).$$

Hence $\mathrm{cl}_{\mathbb{N}}(P_G(4,3))$ is a nonempty proper closed subset of $\mathbb{N}$ that contains a nonempty open subset of $(\mathbb{N}, \tau_G)$, namely $P(4,3)$. Moreover $\mathrm{cl}_{\mathbb{N}}(P(4,3))$ is not open in $(\mathbb{N}, \tau_G)$. Then $(\mathbb{N}, \tau_G)$ is not superconnected in the sense of [6].

It is worth to compare Corollary 5.3 with the comment mentioned in [18, page 901] in which it is said that "we can easily see that every base of the topology τ_G contains some disconnected arithmetic progressions". Then it is claimed that by this comment and the equivalence between 3) and 4) of Theorem 5.12, the space $(\mathbb{N}, \tau_G)$ is not locally connected.

Let $p \in \mathbb{P}$. We have seen that $M(p)$ is closed in $(\mathbb{N}, \tau_G)$. By 5) of Theorem 5.12, $M(p)$ is also connected in $(\mathbb{N}, \tau_G)$. Clearly $\mathrm{int}_{\mathbb{N}}(M(p)) = \emptyset$. Let $\mathcal{A} \subset \mathcal{P}$ be such that $\mathcal{A} \cap \mathcal{M} \neq \emptyset$. By Theorem 5.11 and Corollary 3.4 both $W = \bigcup \mathcal{A}$ and $\mathrm{cl}_{\mathbb{N}}(W)$ are totally Brown in $(\mathbb{N}, \tau_G)$. Since $(\mathbb{N}, \tau_G)$ is totally Brown, by Theorem 3.5 for any nonempty open subset U of $(\mathbb{N}, \tau_G)$, the set $\mathrm{cl}_{\mathbb{N}}(U)$ is totally Brown.

Despite the fact that, by Theorem 5.1, some arithmetic progressions are not totally Brown in $(\mathbb{N}, \tau_G)$, the next result shows that its closure is always totally Brown in $(\mathbb{N}, \tau_G)$.

Theorem 5.16. *Let $\mathcal{A} \subset \mathcal{P}$ be such that $\mathcal{A} \neq \emptyset$. If $W = \bigcup \mathcal{A}$, then $B = \mathrm{cl}_{\mathbb{N}}(W)$ is totally Brown in $(\mathbb{N}, \tau_G)$. In particular, for each $a, b \in \mathbb{N}$ the set $\mathrm{cl}_{\mathbb{N}}(P(a,b))$ is totally Brown in $(\mathbb{N}, \tau_G)$.*

Proof: Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1, O_2, \ldots, O_n$ of B. For each $i \in \{1, 2, \ldots, n\}$ let U_i be an open subset of $(\mathbb{N}, \tau_G)$ so that $O_i = B \cap U_i$ and let $b_i \in O_i$. Take $a_i \in \mathbb{N}$ with $\langle a_i, b_i \rangle = 1$ and $P(a_i, b_i) \subset U_i$. Since $b_i \in \mathrm{cl}_{\mathbb{N}}(W)$ we have $W \cap P(a_i, b_i) \neq \emptyset$ so there exist $c_i, d_i, e_i \in \mathbb{N}$ such that $e_i \in P(c_i, d_i) \in \mathcal{A}$ and $e_i \in P(a_i, b_i)$. Hence $P(c_i, e_i) \cap P(a_i, b_i) \neq \emptyset$ and $P(c_i, e_i) \cap P(a_i, b_i) \subset$

$W \cap P(a_i, b_i)$. We also have

$$\begin{aligned}\mathrm{cl}_{\mathbb{N}}(P(c_i, e_i) \cap P(a_i, b_i)) &\subset \mathrm{cl}_{\mathbb{N}}(\mathrm{cl}_{\mathbb{N}}(P(c_i, e_i)) \cap P(a_i, b_i)) \\ &\subset \mathrm{cl}_{\mathbb{N}}(\mathrm{cl}_{\mathbb{N}}(W) \cap U_i) = \mathrm{cl}_{\mathbb{N}}(O_i).\end{aligned}$$

Applying (23) with the finite collection

$$\{P(c_1, e_1)\} \cup \{P(c_i, e_i) \colon i \in \{1, 2, \ldots, n\}\} \cup \{P(a_i, b_i) \colon i \in \{1, 2, \ldots, n\}\}$$

of arithmetic progressions in $\mathbb{N}$, as well as (26), we have

$$\begin{aligned}\emptyset \neq \mathrm{cl}_{\mathbb{N}}(P(c_1, e_1)) &\cap \left(\bigcap_{i=1}^{n} [\mathrm{cl}_{\mathbb{N}}(P(c_i, e_i)) \cap \mathrm{cl}_{\mathbb{N}}(P(a_i, b_i))]\right) \\ \subset B \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(P(c_i, e_i) \cap P(a_i, b_i))\right) &\subset B \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{\mathbb{N}}(O_i)\right).\end{aligned}$$

This shows that B is totally Brown in $(\mathbb{N}, \tau_G)$. □

Corollary 5.17. *Let* $a \in \mathbb{N}_2$ *and* $b \in \mathbb{N}$ *be such that* $\langle a, b\rangle = 1$. *Then* $P(a, b)$ *is totally separated and* $\mathrm{cl}_{\mathbb{N}}(P(a, b))$ *is totally Brown in* $(\mathbb{N}, \tau_G)$.

Proof: The result follows from Theorems 5.1 and 5.16. □

By Theorems 4.12 and 5.9 the intersection of the closure of finitely many members of $\mathcal{P}$ is totally Brown in $(\mathbb{N}, \tau_G)$. By Theorem 5.16 the union of the closure of finitely many members of $\mathcal{P}$ is also totally Brown in $(\mathbb{N}, \tau_G)$.

6. The Kirch space

In [14] A. M. Kirch considered the family

$$\mathcal{B}_K = \{P(a, b) \in \mathcal{B}_G \colon a \text{ is square-free}\},$$

which is a base of a topology τ_K on $\mathbb{N}$ so that $\tau_K \subset \tau_G$, and showed that the topological space $(\mathbb{N}, \tau_K)$ is connected, locally connected and Hausdorff. In [17, Examples 60 and 61, page 82] this topology is called the prime integer topology. However we call τ_K the *Kirch topology* and refer to the topological space $(\mathbb{N}, \tau_K)$ as the *Kirch space*. Clearly

$$\begin{aligned}\tau_K = \{\emptyset\} \cup \{U \subset \mathbb{N} \colon &\text{for each } b \in U \text{ there exists } a \in \mathbb{N}_2 \\ &\text{such that } P(a, b) \in \mathcal{B}_K \text{ and } P(a, b) \subset U\}.\end{aligned}$$

The fact that $\tau_K \subset \tau_G$ implies, by Theorem 3.2, that $(\mathbb{N}, \tau_K)$ is totally Brown. Moreover if $C \subset \mathbb{N}$ is totally Brown (Brown, connected, respectively) in $(\mathbb{N}, \tau_G)$, then C is totally Brown (Brown, connected, respectively) in $(\mathbb{N}, \tau_K)$. Since $(\mathbb{N}, \tau_K)$ is totally Brown and T_2, by Theorem 3.6, $(\mathbb{N}, \tau_K)$ is aposyndetic.

The inclusion $\tau_K \subset \tau_G$ also implies that for each $A \subset \mathbb{N}$ we have

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(A) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(A).$$

Hence many results presented in Subsection 5.2 remain valid in $(\mathbb{N}, \tau_K)$. Particularly all the results from Theorem 5.4 to Corollary 5.7 are valid in $(\mathbb{N}, \tau_K)$. Hence we have the following theorem.

Theorem 6.1. *Let* $a, b, a_1, b_1, a_2, b_2, \ldots, a_k, b_k \in \mathbb{N}$. *Then*

1) $P_F(a, b) \cap \mathbb{N} \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a, b))$;
2) $M(a) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a^n, b))$ *for each* $n \in \mathbb{N}$;
3) *for each nonempty open subset* U *of* $(\mathbb{N}, \tau_K)$, *there exists* $c \in \mathbb{N}$ *such that* $M(c) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(U)$;
4) $\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(1, b)) = \mathbb{N}$;
5) *for any finite collection* $\{P(a_i, b_i) \colon i \in \{1, 2, \ldots, k\}\}$ *of arithmetic progressions in* $\mathbb{N}$, *we have* $M([a_1, a_2, \ldots, a_k]) \subset \bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a_i, b_i))$ *and, in particular,* $\bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a_i, b_i)) \neq \emptyset$;
6) $M(c) \cap \left(\bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a_i, b_i))\right) \neq \emptyset$ *for each* $c \in \mathbb{N}$.

Assertion 4) of Theorem 6.1 is [23, Remark 4.1, page 676]. Concerning equality (25) its right side is contained in its left side, considering the closure in $(\mathbb{N}, \tau_K)$. However the proper way to obtain $\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b))$ is presented in the next result (compare the first part with [23, Theorem 4.4, page 676] and the second part with [23, Corollary 4.5, page 678]).

Theorem 6.2. *Let* $b \in \mathbb{N}$ *and* $p \in \mathbb{P}$. *Then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b)) = M(p) \cup [P_F(p, b) \cap \mathbb{N}] \qquad \text{for each } n \in \mathbb{N}. \tag{30}$$

In particular, $\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b)) = \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p, b))$ *for every* $n \in \mathbb{N}$.

Proof: Take $x \in \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b))$ and consider that $x \notin M(p)$. Then $\langle p, x\rangle = 1$. This implies, since $P(p, x)$ is an open subset of $(\mathbb{N}, \tau_K)$ that contains x, that $P(p, x) \cap P(p^n, b) \neq \emptyset$. Hence, by (7), $p|(x - b)$, so $x \in P_F(p, b) \cap \mathbb{N}$. This shows that x is in the right side of (30).

Now take $x \in P_F(p, b) \cap \mathbb{N}$ and let W be an open subset of $(\mathbb{N}, \tau_K)$ with $x \in W$. Then there exists $c \in \mathbb{N}_2$ square-free so that $\langle c, x\rangle = 1$ and $P(c, x) \subset W$. Hence either $\langle c, p^n\rangle = 1$ or $\langle c, p^n\rangle = p$. In any case since $p|(x - b)$ by (7) we have $P(c, x) \cap P(p^n, b) \neq \emptyset$. Then $W \cap P(p^n, b) \neq \emptyset$. This shows that $x \in \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b))$, so $P_F(p, b) \cap \mathbb{N} \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n, b))$. By 2) of Theorem 6.1 we have $M(p) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n,b))$. Hence the right side of (30) is contained in its left side. □

The same proof of Theorem 5.9 applies in $(\mathbb{N},\tau_K)$. Hence we have the following result.

Theorem 6.3. *Let $a_1, b_1, a_2, b_2, \dots, a_k, b_k \in \mathbb{N}$ be such that $\bigcap_{i=1}^k P(a_i,b_i) \neq \emptyset$. Then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}\left(\bigcap_{i=1}^k P(a_i,b_i)\right) = \bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a_i,b_i)). \tag{31}$$

The following result appears in both [3, Lemma 1, page 4] and [23, Theorem 4.6, page 678]. Our proof is shorter than the one presented in [23], though slightly different than the one that appears in [3].

Theorem 6.4. *If $a \in \mathbb{N}_2$ and $a = \prod_{i=1}^k p_i^{\alpha_i}$ is the standard prime decomposition of a, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) = \bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p_i^{\alpha_i},b)). \tag{32}$$

Proof: The result follows from (19) and (31). □

We are ready to present the formula for the closure in $(\mathbb{N},\tau_K)$ of an arithmetic progression in $\mathbb{N}$.

Theorem 6.5. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^k p_i^{\alpha_i}$ is the standard prime decomposition of a. If $b \in \mathbb{N}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) = \mathbb{N} \cap \left(\bigcap_{i=1}^k [M(p_i) \cup P_F(p_i,b)]\right). \tag{33}$$

Proof: By (30) and (32)

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) = \bigcap_{i=1}^k \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p_i^{\alpha_i},b)) &= \bigcap_{i=1}^k [M(p_i) \cup (P_F(p_i,b) \cap \mathbb{N})] \\ &= \mathbb{N} \cap \left(\bigcap_{i=1}^k [M(p_i) \cup P_F(p_i,b)]\right).\end{aligned}$$

□

Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^k p_i^{\alpha_i}$ is the standard prime decomposition of a. Define $c = p_1 p_2 \cdots p_k = \prod_{p \in \Theta(a)} p$ and for each $b \in \mathbb{N}$ let

$$A_b = \{d \leq c\colon \text{for each } i \in \{1,2,\dots,k\} \text{ either } p_i | d \text{ or } d \equiv b \pmod{p_i}\}.$$

Let $x \in \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b))$. Consider $n \in \mathbb{N}_0$ and $d \in \mathbb{N}$ so that $x = cn + d$ and $d \le c$. Given $i \in \{1,2,\dots,k\}$ we have $p_i | c$ and, by (33),

$$cn + d \in M(p_i) \cup P_F(p_i, b).$$

Hence either $p_i | d$ or $d \equiv b \pmod{p_i}$, so $d \in A_b$ and since $x \in P(c,d)$, it follows that $x \in \bigcup_{d\in A_b} P(c,d)$.

Now assume that $x \in \bigcup_{d\in A_b} P(c,d)$. Let $d \in A_b$ be so that $x \in P(c,d)$. Given $i \in \{1,2,\dots,k\}$ we have $p_i|c$ and, since $d \in A_b$ either $p_i|d$ or $d \equiv b \pmod{p_i}$. In the first case both c and d belong to $M(p_i)$, so $x \in M(p_i)$. In the second case we have $p_i|c$, $p_i|(d-b)$ and $c|(x-d)$, so $p_i|[(x-d)+(d-b)]$, i.e., $p_i|(x-b)$. Hence $x \in \mathbb{N} \cap P_F(p_i, b)$. This shows that

$$x \in \bigcap_{i=1}^{k} [M(p_i) \cup (P_F(p_i,b) \cap \mathbb{N})] = \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)).$$

Therefore we have the following result which is [23, Theorem 4.7, page 680].

Theorem 6.6. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. Define $c = p_1 p_2 \cdots p_k$. Then for each $b \in \mathbb{N}$*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) = \bigcup_{d\in A_b} P(c,d). \tag{34}$$

By (34) the right side of (33) is the union of finitely many arithmetic progressions in $\mathbb{N}$, all with the same common difference of successive members.

Theorem 6.7. *Let $a, b, c_1, d_1, c_2, d_2, \dots, c_n, d_n \in \mathbb{N}$ be such that $P(c_i, d_i) \in \mathcal{B}_K$ and $P(a,b) \cap P(c_i,d_i) \neq \emptyset$ for each $i \in \{1,2,\dots,n\}$. Then there exists $d \in \mathbb{N}$ so that for every $i \in \{1,2,\dots,n\}$,*

$$\emptyset \neq M(d) \cap P(a,b) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b) \cap P(c_i,d_i)).$$

Proof: For every $i \in \{1,2,\dots,n\}$ let $b_i \in P(a,b) \cap P(c_i,d_i)$. Then $a|(b_i - b)$, $c_i|(b_i - d_i)$ and, by (10),

$$P([a,c_i], b_i) \subset P(a,b) \cap P(c_i,d_i).$$

Since $P(c_i,d_i) \in \mathcal{B}_K$, each $c_i \in \mathbb{N}_2$ is square-free and $\langle c_i, d_i\rangle = 1$. Applying Theorem 2.1, there exists $q_i \in \mathbb{N}$ such that $[a,c_i] = aq_i$, $q_i|c_i$ and $\langle q_i, a\rangle = 1$. Let $d = q_1 q_2 \cdots q_n$. Since $\langle q_i, a\rangle = 1$ for each $i \in \{1,2,\dots,n\}$ we have $\langle d, a\rangle = 1$. Then, by Corollary 4.8, $M(d) \cap P(a,b) \neq \emptyset$.

Fix $i \in \{1,2,\dots,n\}$ and let $z \in M(d) \cap P(a,b)$ and U be an open subset of $(\mathbb{N},\tau_K)$ such that $z \in U$. Take $x \in \mathbb{N}_2$ square-free so that $\langle x, z\rangle = 1$ and

$P(x,z) \subset U$. Since $a|(z-b)$ and $a|(b_i-b)$ we have $a|[(z-b)-(b_i-b)]$, i.e., $a|(z-b_i)$. From $\langle x,z\rangle = 1$, $q_i|d$ and $d|z$ it follows that $\langle x,q_i\rangle = 1$. Hence

$$\langle x,[a,c_i]\rangle = \langle x,aq_i\rangle = \langle x,a\rangle\langle x,q_i\rangle = \langle x,a\rangle,$$

so $\langle x,[a,c_i]\rangle|(z-b_i)$. Therefore, by (7),

$$\emptyset \neq P(x,z) \cap P([a,c_i],b_i) \subset U \cap P(a,b) \cap P(c_i,d_i).$$

This shows that $z \in \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b) \cap P(c_i,d_i))$. □

6.1 Totally Brown subsets of the Kirch space. In this subsection we describe subsets of the Kirch space which are totally Brown. By Theorem 3.2 the results presented in Theorems 5.11 and 5.16 remain valid in $(\mathbb{N},\tau_K)$. By the same reason, the implication 4) implies 1) in Theorem 5.12 is valid in $(\mathbb{N},\tau_K)$, so we have the following result.

Theorem 6.8. *Let $a,b \in \mathbb{N}$ be such that $\Theta(a) \subset \Theta(b)$. Then $P(a,b)$ is totally Brown in $(\mathbb{N},\tau_K)$.*

By 4) of Theorem 6.1 and Theorem 6.8 for each $b \in \mathbb{N}$ the arithmetic progression $P(1,b)$ is totally Brown and dense in $(\mathbb{N},\tau_K)$. Note that if $P(a,b) \in \mathcal{B}_K$ then, since $a \neq 1$ and $\langle a,b\rangle = 1$ we have $\Theta(a) \not\subset \Theta(b)$. The following result generalizes [18, Theorem 3.5, page 901].

Theorem 6.9. *If $a,b \in \mathbb{N}$, then $P(a,b)$ is totally Brown in $(\mathbb{N},\tau_K)$.*

Proof: Fix $n \in \mathbb{N}_2$ as well as n nonempty open subsets $O_1,O_2,\ldots,O_n$ of $P(a,b)$. For each $i \in \{1,2,\ldots,n\}$ let U_i be an open subset of $(\mathbb{N},\tau_K)$ with $O_i = P(a,b) \cap U_i$ and take $d_i \in O_i$. Then there exists $c_i \in \mathbb{N}_2$ square-free such that $\langle c_i,d_i\rangle = 1$ and $P(c_i,d_i) \subset U_i$. Note that $P(c_i,d_i) \in \mathcal{B}_K$ and $P(a,b) \cap P(c_i,d_i) \neq \emptyset$ for each $i \in \{1,2,\ldots,n\}$. Then, by Theorem 6.7, there exists $d \in \mathbb{N}$ so that for every $i \in \{1,2,\ldots,n\}$,

$$\emptyset \neq M(d) \cap P(a,b) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b) \cap P(c_i,d_i)).$$

Thus

$$\begin{aligned}
\emptyset \neq M(d) \cap P(a,b) &= P(a,b) \cap M(d) \cap P(a,b) \\
&\subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b) \cap P(c_i,d_i)) \right) \\
&\subset P(a,b) \cap \left(\bigcap_{i=1}^{n} \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b) \cap U_i) \right) = P(a,b) \cap \left(\bigcap_{i=1}^{k} \mathrm{cl}_{(\mathbb{N},\tau_K)}(O_i) \right).
\end{aligned}$$

This shows that $P(a,b)$ is totally Brown in $(\mathbb{N},\tau_K)$. □

By Theorem 6.9 it follows that $(\mathbb{N}, \tau_K)$ is locally connected. We can also show that if $P(a,b) \in \mathcal{B}_K$ and $c \in P(a,b)$, then $P(a,b)$ is locally connected at c.

A description of totally Brown subsets of the *Szczuka space* $(\mathbb{N}, \tau_S)$, defined by P. Szczuka in [19], where τ_S is called the common division topology on $\mathbb{N}$, is made in [1]. In the same paper there is also a description of totally Brown subsets of the *Rizza space* $(\mathbb{N}, \tau_R)$, defined by G. B. Rizza in [16], where τ_R is called the division topology on $\mathbb{N}$.

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References

[1] Alberto-Domínguez J. C., Acosta G., Madriz-Mendoza M., *The common division topology on* $\mathbb{N}$, accepted in Comment. Math. Univ. Carolin.

[2] Banakh T., Mioduszewski J., Turek S., *On continuous self-maps and homeomorphisms of the Golomb space*, Comment. Math. Univ. Carolin. **59** (2018), no. 4, 423–442.

[3] Banakh T., Stelmakh Y., Turek S., *The Kirch space is topologically rigid*, Topology Appl. **304** (2021), Paper No. 107782, 16 pages.

[4] Bing R. H., *A connected countable Hausdorff space*, Proc. Amer. Math. Soc. **4** (1953), 474.

[5] Clark P. L., Lebowitz-Lockard N., Pollack P., *A note on Golomb topologies*, Quaest. Math. **42** (2019), no. 1, 73–86.

[6] Dontchev J., *On superconnected spaces*, Serdica **20** (1994), no. 3–4, 345–350.

[7] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[8] Fine B., Rosenberger G., *Number Theory. An Introduction via the Density of Primes*, Birkhäuser/Springer, Cham, 2016.

[9] Furstenberg H., *On the infinitude of primes*, Amer. Math. Monthly **62** (1955), 353.

[10] Golomb S. W., *A connected topology for the integers*, Amer. Math. Monthly **66** (1959), 663–665.

[11] Golomb S. W., *Arithmetica topologica*, General Topology and Its Relations to Modern Analysis and Algebra, Proc. Sympos., Prague, 1961, Academic Press, New York, Publ. House Czech. Acad. Sci., Praha, 1962, 179–186 (Italian).

[12] Jones F. B., *Aposyndetic continua and certain boundary problems*, Amer. J. Math. **63** (1941), 545–553.

[13] Jones G. A., Jones J. M., *Elementary Number Theory*, Springer Undergraduate Mathematics Series, Springer, London, 1998.

[14] Kirch A. M., *A countable, connected, locally connected Hausdorff space*, Amer. Math. Monthly **76** (1969), 169–171.

[15] Nanda S., Panda H. K., *The fundamental group of principal superconnected spaces*, Rend. Mat. (6) **9** (1976), no. 4, 657–664.

[16] Rizza G. B., *A topology for the set of nonnegative integers*, Riv. Mat. Univ. Parma (5) **2** 1993, 179–185.

[17] Steen L. A., Seebach J. A., Jr., *Counterexamples in Topology*, Dover Publications, Mineola, New York, 1995.
[18] Szczuka P., *The connectedness of arithmetic progressions in Furstenberg's, Golomb's, and Kirch's topologies*, Demonstratio Math. **43** (2010), no. 4, 899–909.
[19] Szczuka P., *Connections between connected topological spaces on the set of positive integers*, Cent. Eur. J. Math. **11** (2013), no. 5, 876–881.
[20] Szczuka P., *The Darboux property for polynomials in Golomb's and Kirch's topologies*, Demonstratio Math. **46** (2013), no. 2, 429–435.
[21] Szczuka P., *Regular open arithmetic progressions in connected topological spaces on the set of positive integers*, Glas. Mat. Ser. III **49(69)** (2014), no. 1, 13–23.
[22] Szczuka P., *The closures of arithmetic progressions in the common division topology on the set of positive integers*, Cent. Eur. J. Math. **12** (2014), no. 7, 1008–1014.
[23] Szczuka P., *The closures of arithmetic progressions in Kirch's topology on the set of positive integers*, Int. J. Number Theory **11** (2015), no. 3, 673–682.
[24] Szyszkowska P., Szyszkowski M., *Properties of the common division topology on the set of positive integers*, J. Ramanujan Math. Soc. **33** (2018), no. 1, 91–98.

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