On FI-mono-retractable modules

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Abstract. We introduce the notion of FI-mono-retractable modules which is a generalization of compressible modules. We investigate the properties of such modules. It is shown that the rings over which every cyclic module is FI-mono-retractable are simple Noetherian V-ring with zero socle or Artinian semisimple. The last section of the paper is devoted to the endomorphism rings of FI-retractable modules.

Keywords: retractable module; FI-mono-retractable module; compressible module; fully invariant submodule

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1. Introduction

Throughout this paper R denotes an arbitrary associative ring with identity and all modules are unitary right R-modules. For an R-module M, $S = \mathrm{End}(M)$ denotes the endomorphism ring of M. In addition, $\mathrm{E}(M)$, $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$ denote the injective hull, the socle and the Jacobson radical of M, respectively. Also $\mathrm{J}(R)$ stands for the Jacobson radical of R. Let M be a module and N be a nonzero submodule of M. Then N is called to be an *essential* submodule of M denoted by $N \leq_e M$ if $K \cap N \neq 0$ for every nonzero submodule K of M. A module M is called *uniform* if every nonzero submodule of M is essential in M. Recall that M is *singular* (*nonsingular*) provided that $Z(M) = M$ ($Z(M) = 0$) where $Z(M) = \{x \in M\colon\ xI = 0$ for some essential ideal I of $R\}$. A submodule N of M is called *fully invariant*, if for every $f \in \mathrm{End}(M)$, $f(N) \subseteq N$. Clearly 0 and M are fully invariant submodules of M. There are some well-known fully invariant submodules of a module M such as $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $Z(M)$. It is clear that the sum and the intersection of any collection of fully invariant submodules are also fully invariant. Thus the collection of fully invariant submodules of M is a sublattice of the complete modular lattice of all submodules of M. J. Zelmanowitz introduced the notion of compressible modules. A right R-module M is called *compressible* if for each nonzero submodule N of M there exists a monomorphism $f\colon M \to N$.

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For example if R is a domain, then the right R-module R is compressible. In [7] P. F. Smith and M. R. Vedadi study a generalization of compressible modules, essentially compressible modules, with focus on essential submodules. In this work we present another generalization of compressible modules namely, *FI-mono-retractable* modules by focusing just on nonzero fully invariant submodules. An R-module M is called FI-mono-retractable, if for any nonzero fully invariant submodule N of M there exists a monomorphism $f\colon M \to N$. Clearly compressible modules are FI-mono-retractable but the converse is not true in general.

In Section 2, we study some properties of FI-mono-retractable modules and we show that when the FI-mono-retractability implies the essentially retractability. Also we prove that the class of FI-mono-retractable modules is closed under direct sums and present some conditions to show when FI-retractability condition is preserved under taking submodules and homomorphic images. We prove that any finitely generated nonsingular FI-mono-retractable module that has a uniform submodule, has finite uniform dimension. In Section 3, we consider FI-mono-retractable modules over certain rings. Also we investigate rings over which every cyclic (cocyclic) module is FI-mono-retractable. Such rings are simple Noetherian V-ring with zero socle or Artinian semisimple. Section 3 is devoted to the endomorphism rings of FI-mono-retractable modules. We show that every finitely generated quasi-projective FI-mono-retractable module has a prime endomorphism ring and the endomorphism ring of an indecomposable quasi-injective FI-mono-retractable module is a field.

2. General properties

We first recall the following elementary well known facts about fully invariant submodules.

Proposition 2.1. *Let R be any ring and M be a nonzero R-module:*

(1) *Any sum or intersection of fully invariant submodules of M are again a fully invariant submodule.*
(2) *Let $K \leq N$ be submodules of M such that K is a fully invariant submodule of N and N is a fully invariant submodule of M. Then K is a fully invariant submodule of M.*
(3) *Let $M = \bigoplus_{i\in I} M_i$ and N be a nonzero fully invariant submodule of M. Then $N = \bigoplus_{i\in I}(N \cap M_i)$.*
(4) *Let $M = M_1 \oplus M_2$ be the direct sum of submodules M_1, M_2. Then M_1 is a fully invariant submodule of M if and only if $\operatorname{Hom}(M_1, M_2) = 0$.*

(5) *If* $N \leq L \leq M$ *such that* N *is a fully invariant submodule of* M *and* L/N *is a fully invariant submodule of* M/N*, then* L *is a fully invariant submodule of* M*.*

Proof: For proof of (1), (2), (3), (4) see [5, 2.1], [5, 1.9]. (5) Let $f\colon M \to M$ be a homomorphism. Then $f(N) \leq N$. Now, consider the homomorphism $\bar{f}\colon M/N \to M/N$ defined by $\bar{f}(m+N) = f(m)+N$ for all $m \in M$. So $\bar{f}(L/N) \leq L/N$. Clearly $\bar{f}(L/N) = (f(L)+N)/N$. Therefore $f(L) \leq L$. Hence L is a fully invariant submodule of M. □

Definition 2.2. An R-module M is called *FI-mono-retractable* provided for each nonzero fully invariant submodule N of M, there exists a monomorphism $f\colon M \to N$.

Remark 2.3. (1) Let R be a commutative ring. Following [1] an R-module M is multiplication if for each submodule N of M, there exists ideal I of R such that $N = MI$. Clearly, if M is multiplication, then M is compressible if and only if M is FI-mono-retractable.

(2) Following [8] an R-module M is called fully prime if for any nonzero fully invariant submodule K of M, M is K-cogenerated. And M is called prime if for any nonzero fully invariant submodule K of M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$. It is clear that every FI-mono-retractable module is fully prime. Also, every FI-mono-retractable module is a prime module. Because if M is FI-mono-retractable and N is a nonzero fully invariant submodule of M, then there exists monomorphism $f\colon M \to N$. Let $r \in \mathrm{Ann}_R(N)$. Then $f(Mr) \leq Nr = 0$. So $\mathrm{Ann}_R(N) \leq \mathrm{Ann}_R(M)$.

In the following result we present a condition in which the two concepts of compressible and FI-mono-retractable are equivalent. In [10] M has (*) condition if for any nonzero proper submodule K of M, there is an $r \in R \setminus \mathrm{Ann}_R(M)$ with $Mr \subset K$.

Proposition 2.4. *Any FI-mono-retractable module with* (*) *condition is compressible.*

Proof: Suppose that M is FI-mono-retractable and N any nonzero submodule of M. By (*) condition there exists $r \in R \setminus \mathrm{Ann}_R(M)$ such that $MrR \subset N$. Since MrR is fully invariant submodule of M, there exists a monomorphism $f\colon M \to MrR$ and so $i \circ f\colon M \to N$ is a monomorphism where i denoted the inclusion map $i\colon MrR \to N$. □

In general the class of FI-mono-retractable modules is not closed under taking submodules and factor modules. For example $\mathbb{Z}$ as $\mathbb{Z}$-module is FI-mono-retractable module but $\mathbb{Z}$-module $\mathbb{Z}/(4\mathbb{Z})$ is not FI-mono-retractable module retractable. However, there are some special cases, as follows.

Proposition 2.5. *Any fully invariant submodule of an FI-mono-retractable module is FI-mono-retractable.*

Proof: Suppose that M is FI-mono-retractable and N any nonzero fully invariant submodule of M. Let K be a nonzero fully invariant submodule of N. Then K is fully invariant submodule of M. By assumption there exists a monomorphism $f\colon M \to K$. Then $f \circ i\colon N \to K$ is a nonzero monomorphism where i denotes the inclusion map of N to M. □

Proposition 2.6. *Let N be a submodule of an FI-mono-retractable module M such that $\alpha(N) + \alpha^{-1}(N) \leq N$ for every monic epimorphism $\alpha \in \mathrm{End}(M)$. Then the module M/N is an FI-mono-retractable module.*

Proof: Let L/N be a nonzero fully invariant submodule of M/N. By Proposition 2.1 (5), L is a fully invariant submodule of M. By hypothesis, there exists monomorphism $f\colon M \to L$. Since $f(N) + f^{-1}(N) \leq N$, the induced mapping $\bar{f}\colon M/N \to L/N$ defined by $\bar{f}(m+N) = f(m)+N$ is a monomorphism. It follows that M/N is an FI-mono-retractable module. □

Corollary 2.7. *Let M be an FI-mono-retractable module. Then the module $M/Z(M)$ is an FI-mono-retractable module.*

Proof: Let N be a submodule of M containing $Z(M)$ such that $N/Z(M)$ is a fully invariant submodule of $M/Z(M)$. Since $Z(M)$ is fully invariant submodule of M then N is a fully invariant submodule of M. So there exists a monomorphism $f\colon M \to N$. Then f induces $\bar{f}\colon M/Z(M) \to N/Z(M)$ defined by $\bar{f}(m + Z(M)) = f(m) + Z(M)$. Since $f^{-1}(Z(M)) \leq Z(M)$, $\bar{f}$ is monomorphism. □

Proposition 2.8. *Let R be any ring and $M = \bigoplus_{i\in I} M_i$ be a direct sum of FI-mono-retractable module M_i. Then M is a mono-retractable module.*

Proof: Let N be any fully invariant submodule of M. Then by Proposition 2.1 (3), $N = \bigoplus_{i\in I}(N \cap M_i)$. Since $N \cap M_i$ is a fully invariant submodule of M_i, there exists monomorphism $f_i\colon M_i \to N \cap M_i$. Hence $f = \bigoplus_{i\in I} f_i\colon \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I}(N \cap M_i)$ is monomorphism. □

A module M is called *cocyclic* provided it contains an essential simple submodule.

Proposition 2.9. *Let M be FI-mono-retractable module. Then:*

(1) *M is either semisimple or* $\mathrm{Soc}(M) = 0$.
(2) *M is either singular or nonsingular.*
(3) *M is singular and semisimple or* $\mathrm{Soc}(M)$ *is projective, but not both.*
(4) $\mathrm{Ann}_R(M)$ *is a prime ideal of R.*
(5) *If M is finitely generated, then no submodule of M is an infinite direct sum of nonzero fully invariant submodules.*
(6) *If M is cocyclic, then M is simple.*
(7) *If M is quasi-injective, then M has no nontrivial fully invariant submodule.*

PROOF: (1) Suppose that $\mathrm{Soc}(M) \neq 0$. Then there exists a monomorphism $f\colon M \to \mathrm{Soc}(M)$. So M is semisimple.

(2) The proof is similar to (1).

(3) Suppose that M is FI-mono-retractable. If $Z(M) \cap \mathrm{Soc}(M) = 0$, then [4, 1.24] implies that $\mathrm{Soc}(M)$ is projective. If $Z(M) \cap \mathrm{Soc}(M) \neq 0$, then FI-mono-retractable condition on M implies that there exists a monomorphism $f\colon M \to Z(M) \cap \mathrm{Soc}(M)$. So M is singular and semisimple.

(4) Suppose that I, J are nonzero right ideals of R such that $MI \neq 0$, $MJ \neq 0$ and $MIJ = 0$. By FI-mono-retractable condition on M there exists monomorphism $f\colon M \to MI$. Then $f(MJ) \leq MIJ = 0$. So $MJ = 0$.

(5) Suppose that $N = N_1 \oplus N_2 \oplus \cdots$ is direct sum of fully invariant submodule of M. By assumption there exists a monomorphism $f\colon M \to N$. Since M is finitely generated $f(M) \leq N_1 \oplus \cdots \oplus N_k$ for some positive integer k. Also $f(N_{k+1} \oplus \cdots) \leq f(M) \cap (N_{k+1} \oplus \cdots) = 0$. Hence $N_{k+1} \oplus \cdots \leq \mathrm{Ker} f = 0$.

(6) Thist is trivial consequence of (1).

(7) Suppose that N is a nonzero fully invariant submodule of M. By assumption there exists a monomorphism $f\colon M \to N$. Then $M \cong K$ for some submodule $K \leq N$. Now the isomorphism $g\colon K \to M$ can be extended to $\bar{g}\colon M \to M$ such that $\bar{g}(K) = g(K)$. Therefore $M = g(K) = \bar{g}(K) \leq \bar{g}(N) \leq N$. Consequently $M = N$. □

A module M is said to have *finite uniform dimension* denoted by $\mathrm{u.dim}(M) < \infty$ if it does not contain an infinite direct sum of nonzero submodules.

Proposition 2.10. *Let M be a finitely generated nonsingular FI-mono-retractable module. Then M has finite uniform dimension if and only if M has a uniform submodule.*

PROOF: Suppose that M is a nonsingular FI-mono-retractable and U a uniform submodule of M. Let $N = \Sigma\{f(U)\colon f \in \mathrm{Hom}(U, M)\}$. Since N is fully invariant submodule of M, there exists monomorphism $g\colon M \to N$. Since M is finitely generated, there exists positive integer n and $f_i \in \mathrm{Hom}(U, M)$ such that $\mathrm{Im}\, g \leq f_1(U) + \cdots + f_n(U)$. Hence $g\colon M \to f_1(U) + \cdots + f_n(U)$ is a monomorphism. Now, define $\alpha\colon U^{(n)} \to f_1(U) + \cdots + f_n(U)$ by $\alpha(u_1, \cdots, u_n) = f_1(u_1) + \cdots + f_n(u_n)$. It is clear that α is an epimorphism. Then $U^{(n)}$ has finite uniform dimension. Also since $U^{(n)}/\mathrm{Ker}\,\alpha$ is nonsingular, $\mathrm{Ker}\,\alpha$ is closed in $U^{(n)}$. So [2, 5.10] implies that $U^{(n)}/\mathrm{Ker}\,\alpha$ and so $f_1(U_1) + \cdots + f_n(U_n)$ has finite uniform dimension. Consequently M has finite uniform dimension. □

3. FI-mono-retractable modules over certain rings

A ring R is called *right quasi-injective* if R_R is an injective module.

Proposition 3.1. *Let R be a right quasi-injective ring and M be a finitely generated nonsingular FI-mono-retractable module. If M has a uniform submodule, then M is semisimple, projective and injective.*

PROOF: Suppose that M has a uniform submodule. By Proposition 2.10, M has finite uniform dimension. Let $\mathrm{u.dim}(M) = n$. So there exists an essential submodule V of M such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ where $U_1, U_2, \ldots, U_n$ are uniform submodules of M. We prove that U_i is simple for each $1 \leq i \leq n$. Let $0 \neq x \in U_i$. Since xR is nonsingular and R is right quasi-injective, then $R/\mathrm{Ann}_R(x)$ and so xR is injective. Hence $xR = U_i$ because U_i is uniform. Therefore V is semisimple and injective. Now since V is an essential and injective submodule of M, $V = M$. Also by [4, 1.26], every nonsingular semisimple module is projective. □

A ring R is called right *V-ring* if every simple right R-module is injective.

Proposition 3.2. *Let R be any ring. Then R is V-ring if and only if every cocyclic R-module is FI-mono-retractable.*

PROOF: Suppose that R is V-ring and M a cocyclic R-module. Let N be simple and essential submodule of M. By assumption N is a direct summand of M. So $M = N$. Conversely, suppose that M is a simple R-module. It is clear that $M = \mathrm{Soc}(\mathrm{E}(M))$, hence $\mathrm{E}(M) = M$ by 2.9 (1). □

Proposition 3.3. *Let R be any ring such that every cyclic R-module is FI-mono-retractable. Then every semisimple R-module is injective.*

PROOF: Suppose that M is any semisimple R-module. Let $0 \neq x \in \mathrm{E}(M)$ then $0 \neq xR \cap M = \mathrm{Soc}(xR)$. By assumption xR is FI-mono-retractable. So Proposition 2.9 (1) implies xR is semisimple. So $xR \leq M$ and so $M = \mathrm{E}(M)$. □

Following [6] the ring R is right GV-ring in case each right simple R-module is injective or projective. Let R be any ring. In [6, 4.8] equivalent conditions are provided for R such that any semisimple R-module is injective.

Proposition 3.4. *Let R be any ring. If every cyclic R-module is FI-mono-retractable, then R is either simple right Noetherian V-ring with zero socle or Artinian semisimple.*

PROOF: Suppose that R is a ring such that every cyclic R-module is FI-mono-retractable. Then by Proposition 3.2, every semisimple R-module is injective. Now the equivalent conditions in [6, 4.8] completes the proof. □

Proposition 3.5. *Let R be a commutative ring. If every cyclic R-module is FI-mono-retractable, then R is Artinian semisimple.*

PROOF: Suppose that every cyclic R-module is FI-mono-retractable. Proposition 3.3 implies that every semisimple R-module is injective. Now by [6, 4.9], R is Artinian semisimple. □

4. Endomorphism ring of certain FI-mono-retractable modules

Definition 4.1. A ring R is called *right FI-mono-retractable* if R_R is FI-mono-retractable.

Recall that an element $c \in R$ is *right regular* if $\mathrm{r.ann}_R(c) = 0$ where $\mathrm{r.ann}_R(c)$ denotes the right annihilator of c.

Lemma 4.2. *Let R be any ring. The following statements are equivalent:*

(1) *R is a right FI-mono-retractable ring.*
(2) *Every two-sided ideal of R has right regular element.*

PROOF: (1) $\Rightarrow$ (2) Suppose that R is FI-mono-retractable and I any two-sided ideal of R. There exists a monomorphism $f\colon R \to I$. So, $f(1_R)$ is right regular element of I.

(2) $\Rightarrow$ (1) Suppose I is any two-sided ideal of R and $x \in I$ a right regular element. Then the map $f\colon R \to I$ defined by $f(r) = xr$ is a monomorphism. □

Proposition 4.3. *Let M be a quasi-projective FI-mono-retractable module. If M is finitely generated, then $S = \mathrm{End}(M)$ is a right FI-mono-retractable ring.*

PROOF: Suppose that M is finitely generated quasi-projective FI-mono-retractable. Let I be any two-sided ideal of S. Then IM is a fully invariant submodule of M and so there exists a monomorphism $f\colon M \to IM$. Since M is finitely generated and quasi projective by [9, 18.4], $\mathrm{Hom}(M, IM) = I$. On the other hand, $\text{r.ann}_S(f) = \text{Hom}(M, \text{Ker} f) = 0$. Hence $f \in I$ is right regular element. The Lemma 4.2 implies that S is FI-mono-retractable. □

Corollary 4.4. *Let R be a right FI-mono-retractable ring. Then $M_n(R)$ is FI-mono-retractable for any n.*

Proof: Suppose that R is a right FI-mono-retractable ring. Then $R^{(n)}$ is a finitely generated R-module. Proposition 4.3 implies that $\text{End}(R^{(n)})$ is a right FI-mono-retractable ring and so $M_n(R)$ is a right FI-mono-retractable ring. □

Proposition 4.5. *Every right FI-mono-retractable ring is prime ring.*

Proof: Suppose that R is a right FI-mono-retractable ring. By Proposition 2.9 (4) $\text{Ann}_R(R)$ is a prime ideal and so R is prime. □

Corollary 4.6. *Let M be a quasi-projective FI-mono-retractable module. If M is finitely generated, then $S = \text{End}(M)$ is prime ring.*

Proof: The proof follows by Propositions 4.3 and 4.5. □

Proposition 4.7. *Let R be a right FI-mono-retractable ring. If R_R is quasi-injective, then R is simple.*

Proof: Suppose that R is a right FI-mono-retractable ring and R_R is quasi-injective. By Proposition 2.9 (7), R_R has no nontrivial fully invariant submodule. So R has no ideal other than the trivial ones. Therefore R is simple. □

Proposition 4.8. *Let R be a right hereditary right FI-mono-retractable ring. Then R is right Noetherian if and only if R has a right uniform ideal.*

Proof: Suppose that R has a right uniform ideal. Then R is right nonsingular ring because R is right hereditary. So by Proposition 2.10, R_R has finite uniform dimension. Now by [4, 5.20], the proof is completed. Conversely, it is clear. □

Proposition 4.9. *The endomorphism ring of an indecomposable quasi-injective FI-mono-retractable module is a field.*

Proof: Suppose that M is an indecomposable quasi-injective FI-mono-retractable module with $S = \text{End}(M)$. Since M is quasi-injective and FI-retractable, by Proposition 2.9 (7), M has no fully invariant submodule other than the trivial ones. Therefore [3, Exercise 29, page 183] implies that S is a field. □

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