Three small results on normal first countable linearly H-closed spaces

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Abstract. We use topological consequences of PFA, $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ and PFA(S)[S] proved by other authors to show that normal first countable linearly H-closed spaces with various additional properties are compact in these models.

Keywords: linearly H-closed space; normal space; first countable space; forcing axiom

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1. Statements

In this small note, we prove three modest results about the following problem we raised in [3, Question 2.12], using topological consequences of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ (MA – Martin's axiom), PFA(S)[S] and PFA (PFA – proper forcing axiom) due to other authors.

Problem 1. Is there in ZFC (Zermelo–Fraenkel set theory with axiom of choice) a noncompact normal linearly H-closed space which is first countable (or equivalently has G_δ-points)?

By *space* we mean *topological Hausdorff space*, hence *regular* and *normal* imply *Hausdorff*. A *cover* of a space always means a cover by open sets, and a cover is a *chain cover* if it is linearly ordered by the inclusion relation. A space is *linearly H-closed* provided any chain cover has a dense member. There are various equivalent definitions, see for instance Lemmas 2.2–2.3 in [3] or Theorems 2.5, 2.12–2.14 in [2]. A space is *H-closed* if and only if any cover has a finite subfamily with a dense union. Regular H-closed spaces are compact.

We strongly suspect that solving Problem 1 is not ranked very high on mankind's priorities list but hope that our results are of some interest for researchers in the field. To help enhancing their curiosity (and for context), we start by listing known relevant results. Recall that a space is *feebly compact* if and only

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if every locally finite family of nonempty open sets is finite or equivalently (for Hausdorff spaces) if and only if any countable cover has a finite subfamily with a dense union. A linearly H-closed space is feebly compact. A feebly compact space is *pseudocompact* (that is, any real valued continuous function is bounded) and the converse holds for Tychonoff spaces. (We do not include regularity in the definitions of pseudocompactness and feeble compactness.) Recall also that a pseudocompact normal space is countably compact.

In the list below, we tried to give the name of the person(s) who first constructed the example (in general in contexts unrelated to linearly H-closed spaces) or proved the result. Each fact given without a reference is explained in [3, Example 2.9], where links to the original sources can be found. We assume that the reader knows what the axioms CH (continuum hypothesis), $\mathsf{MA} + \neg\mathsf{CH}$, PFA, $\diamondsuit$, $\mathfrak{p} = \omega_1$ mean. Recall that a space is *countably tight* if and only if for each subset E and each point x in its closure, there is a countable $A \subset E$ whose closure contains x. A first countable space is of course countably tight. For other undefined terms see the given references. Fact A explains the equivalence in the statement of Problem 1.

A list of known facts:

A. A regular feebly compact space with G_δ points is first countable, see J. R. Porter and R. G. Woods [13, Proposition 2.2].

B. A weakly linearly Lindelöf (in particular weakly Lindelöf or ccc – countable chain condition) feebly compact space is linearly H-closed, see O. T. Alas, L. R. Junqueira and R. G. Wilson [2, Theorem 2.13].

C. There are noncompact linearly H-closed spaces which are moreover perfect, Tychonoff and first countable (for instance the famous space Ψ due to J. R. Isbell and T. S. Mrówka), or Fréchet–Urysohn and collectionwise normal (for instance a Σ-product of 2^{ω_1}).

D. Under CH, a normal first countable linearly H-closed space has cardinality less than or equal to $\aleph_1$ and is weakly Lindelöf, see A. Bella [6, Theorem 4.4 & Corollary 4.6].

E. If $\mathfrak{p} = \omega_1$ (in particular under CH), there is a noncompact normal separable first countable locally compact locally countable linearly H-closed space, namely a space of the type $\gamma\mathbb{N}$ due originally to S. P. Franklin and M. Rajagopalan and studied in detail by other authors, in particular by P. J. Nyikos.

F. Under $\diamondsuit$, there is a noncompact perfectly normal first countable hereditarily separable linearly H-closed space, for instance an Ostaszewski space, and even a manifold (M. E. Rudin, see P. J. Nyikos' exposition in [11, Exercise 3.14]). If one adds Cohen reals to the model, the manifold keeps these properties, see [5].

G. Under PFA, a normal linearly H-closed space which is either locally separable and countably tight or locally ccc, locally compact and first countable is compact, see [3, Theorem 2.13].

H. Under $\mathsf{MA} + \neg\mathsf{CH}$, a perfectly normal linearly H-closed space is compact. This result is also compatible with CH, see M. Weiss and T. Eisworth in [3, Lemma 2.1].

I. A monotonically normal linearly H-closed space is compact, see O. T. Alas, L. R. Junqueira and R. G. Wilson [2, Theorem 2.17].

Our new results are the following. They fit between Facts G and H, so to say. Recall that the *spread* of a space is the supremum of the cardinalities of its discrete subspaces. Hereditarily separable spaces (such as those in Fact F) are of countable spread.

Theorem 1.1. PFA *implies that if X is a normal linearly H-closed space which is locally of countable spread, then X is compact and first countable.*

Theorem 1.2. *In a particular model of* $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ *and in any model of* $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$, *a locally compact, hereditarily normal, linearly H-closed space with G_δ points is compact.*

Theorem 1.3. *In any model of* $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$, *a linearly H-closed hereditarily normal space such that each point is a G_δ and has an open Lindelöf neighborhood is compact.*

$\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ and $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$ are formally not forcing axioms but rather a powerful method for obtaining models starting from a coherent Suslin tree S, using iterated forcing to obtain weaker versions (called $\mathsf{MA}_{\omega_1}(\mathsf{S})$ and $\mathsf{PFA}(\mathsf{S})$) of MA_{ω_1} and PFA which preserve (the Suslinity of) S, and then forcing with S. The precise definitions shall not concern us here as we use only topological implications which hold in these models. The known proof of the consistency of $\mathsf{PFA}(\mathsf{S})$ needs inaccessible cardinals, while that of $\mathsf{MA}_{\omega_1}(\mathsf{S})$ does not. Hence when we write 'in a model of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$' it is implied that the model is obtained without inaccessibles. For details, see article [17] by F. D. Tall and references therein. We recall that $\mathsf{PFA} \Longrightarrow \mathsf{MA} + \neg\mathsf{CH}$ and $\mathsf{PFA}(\mathsf{S})[\mathsf{S}] \Longrightarrow \mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$.

We note that $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$ implies $\mathfrak{p} = \omega_1$, see the introduction of [17], hence there is a model of set theory where Theorems 1.2–1.3 and Fact E hold, that is, there are first countable linearly H-closed locally compact spaces which are normal, but none is hereditarily normal.

2. Proofs

Our proofs are short and look like patchworks, as we mostly blend together results found elsewhere. Since reading a sequence of references can be somewhat dull, we provide proofs (if short enough) of the results that do not appear as an explicit lemma or theorem somewhere else, or for which we have small variants in the argument. Let us start with the list. First, the following easy fact will be used several times.

Lemma 2.1 ([3, Lemma 2.3] or [2, Theorem 2.1])**.** *If X is linearly H-closed and $U \subset X$ is open, then $\overline{U}$ is linearly H-closed.*

We denote the following property by HL following [17]. Recall that an *S-space* is a regular hereditarily separable non-Lindelöf space, while an *L-space* is a regular hereditarily Lindelöf non-separable space.
HL: A first countable hereditarily Lindelöf regular space is hereditarily separable, that is, first countable L-spaces do not exist.

Lemma 2.2 ([15], [1, Theorem 2.1], [16, Lemma 11])**.** HL *holds in models of* $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ *and of* $\mathsf{MA} + \neg\mathsf{CH}$.

We shall also use the basic result below which is part of the folklore and is cited for instance in [8]. A detailed proof can be found online in Dan Ma's topology blog[1].

Lemma 2.3. *If a regular space of countable spread is not hereditarily separable, it contains an L-space, and if it is not hereditarily Lindelöf, it contains an S-space.*

The next lemma is due to S. Todorčević.

Lemma 2.4 (S. Todorčević [18, Theorem 8.11])**.** (PFA) *A space of countable spread has G_δ points.*

It then follows that:

Lemma 2.5. (PFA) *If X is a regular linearly H-closed space which is locally of countable spread, then X is first countable and locally hereditarily separable.*

Proof: Let $U_x \ni x$ be an open neighborhood of countable spread. Since X is regular, we may choose an open V_x such that $x \in V_x \subset \overline{V_x} \subset U_x$. Then $\overline{V_x}$ is linearly H-closed by Lemma 2.1. Then by Lemma 2.4 $\overline{V_x}$ has G_δ points under PFA and hence is first countable by Fact A. If $\overline{V_x}$ is not hereditarily separable by Lemma 2.3 it contains an L-space, but since PFA implies $\mathsf{MA} + \neg\mathsf{CH}$, HL shows (Lemma 2.2) that it is impossible. Hence $\overline{V_x}$ is hereditarily separable. □

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[1] Available at `https://dantopology.wordpress.com/2018/10/15/a-little-corner-in-the-world-of-set-theoretic-topology/` when this note was written.

Theorem 1.1 follows almost immediately.

Proof of Theorem 1.1: Let X be normal, linearly H-closed and locally of countable spread. By Lemma 2.5, X is first countable and locally separable. The result follows by Fact G. $\square$

We now turn our attention to the other two theorems. We need the following easy lemma.

Lemma 2.6. *If X is locally compact, countably tight, linearly H-closed and noncompact, then X contains a σ-compact open set U such that $\overline{U}$ is linearly H-closed and noncompact.*

Proof: First, notice that a linearly H-closed Lindelöf space is H-closed and hence compact if regular. By Lemma 2.1 the closure of an open set in X is thus either compact or non-Lindelöf. We build open sets U_α with compact closure (indexed by countable ordinals) such that $\overline{U_\alpha} \subset U_\beta$ whenever $\alpha < \beta$ by induction, starting with some U_0. Given U_β for each $\beta < \alpha$, either $\overline{\bigcup_{\beta<\alpha} U_\beta}$ is non-Lindelöf and we are over, or it is compact. In the latter case, since X is noncompact, we may choose a point $x_\alpha \notin \overline{\bigcup_{\beta<\alpha} U_\beta}$. We then cover $\{x_\alpha\} \cup \overline{\bigcup_{\beta<\alpha} U_\beta}$ with finitely many open sets with compact closure whose union defines U_α. Notice that $\overline{U_\alpha} \subsetneq U_\beta$ when $\alpha < \beta$. If this goes on until ω_1, then $W = \bigcup_{\alpha<\omega_1} U_\alpha$ is clopen. Indeed, openness is immediate, and by countable tightness any point $x \in \overline{W}$ is in the closure of a countable subset of W which must be contained in some U_α, so $x \in \overline{U_\alpha} \subset U_{\alpha+1} \subset W$. By Lemma 2.1, W is linearly H-closed, which is impossible since none of the U_α is dense. Hence the process must stop before ω_1, that is, $\overline{\bigcup_{\beta<\alpha} U_\beta}$ is non-Lindelöf for some $\alpha < \omega_1$. $\square$

The next lemma also holds in ZFC and is due to P. J. Nyikos. Recall that scwH is a shorthand for *strongly collectionwise Hausdorff*. A space is $\aleph_1$-scwH if and only if any closed discrete subset of size less than or equal to $\aleph_1$ can be expanded to a discrete family of open sets. A space is ω_1-*compact* (or has *countable extent*) if and only if any closed discrete subspace is at most countable. Countably compact and Lindelöf spaces are ω_1-compact.

Lemma 2.7 (P. J. Nyikos [12, Lemma 1.2])**.** *Let X be a hereditarily $\aleph_1$-scwH space, and $E \subset U \subset X$ be such that U is open, E is ω_1-compact and dense in U. Then $\overline{U} - U$ has countable spread.*

Proof: We give a very small variant of Nyikos' proof. Let D be a discrete subset of $\overline{U} - U$. Then D is closed discrete in the space $W = D \cup U$, because the closure of D does not intersect U. Since W is $\aleph_1$-scwH, we may let $\{V_d \colon d \in D\}$ be a discrete-in-W collection of open subsets, then $\{V_d \cap E\colon d \in D\}$ is a discrete-in-E collection of nonempty (by density of E) open sets. Since E is ω_1-compact, this collection is countable, and so is D. □

We also need some topological implications valid in (some) models of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$. The first one was originally proved by Z. Szentmiklóssy under $\mathsf{MA}+\neg\mathsf{CH}$. The fact that it holds under $\mathsf{PFA(S)[S]}$ is due to S. Todorčević, and in a model of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ by P. B. Larson and F. D. Tall, see [10], [17, Theorem 4.1] and references therein. We provide a short argument that it follows from the following statement which we call $\mathbf{\Sigma}^-$ (the terminology and the proof are taken from [10]). $\mathbf{\Sigma}^-$: In a compact countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.

Theorem 2.8 (In a model of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ and in every model of $\mathsf{PFA(S)[S]}$)**.** *A locally compact space of countable spread is hereditarily Lindelöf.*

Proof: It is shown in [17, Theorem 4.1] that $\mathbf{\Sigma}^-$ holds in a model of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ and in every model of $\mathsf{PFA(S)[S]}$.

If X is locally compact and of countable spread, its one point compactification X^* is countably tight (and has countable spread). Indeed, by a well known fact if X^* is not countably tight, it contains a perfect preimage of ω_1 (see e.g. [10, Lemma 4]), that is, a space Y with a closed onto map $p\colon Y \to \omega_1$ such that preimages of points are compact. But it is then easy to find an uncountable discrete subspace in Y. If X^* is not hereditarily Lindelöf, by Lemma 2.3 it contains an S-space and hence (by classical results, see e.g. [14, Corollary 3.2]) a subspace $Z = \{x_\alpha\colon \alpha \in \omega_1\}$ such that $\{x_\alpha\colon \alpha < \gamma\}$ is open in Z for each γ. But then $\mathbf{\Sigma}^-$ implies that X^* contains an uncountable discrete subset, a contradiction. □

We will also use the following result. Let us borrow again notation from [17] and define the property:
CW: Any first countable normal space is $\aleph_1$-scwH.

Theorem 2.9 ([9, Corollary 14])**.** CW *holds in any model obtained after forcing with a Suslin tree.*

We may now prove Theorem 1.2.

Proof of Theorem 1.2: As before, our assumptions imply that X is first countable. Assume that X is noncompact. Since X is linearly H-closed and normal, X is countably compact. By Lemma 2.6, we may assume that $X = \overline{U}$ where U is open and σ-compact. CW shows (Theorem 2.9) that X is hereditarily $\aleph_1$-scwH after forcing with a Suslin tree, hence in particular in any model of $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ or $\mathsf{PFA(S)[S]}$. Since U is open Lindelöf, Lemma 2.7 shows that

$\overline{U} - U = X - U$ has countable spread. Then Theorem 2.8 implies that $X - U$ is hereditarily Lindelöf. It follows that X is Lindelöf and hence compact. □

We now look at our last result. We will show that our assumptions imply that X is locally compact and apply Theorem 1.2. We need the following consequences of PFA(S)[S].

Theorem 2.10 ([7, Theorem 3.5], [1, Theorem 2.1], [16, Lemma 11]). (PFA(S)[S]) *Separable, hereditarily normal, countably compact spaces are compact.*

(We note in passing that under PFA, the theorem holds without 'hereditarily', this is due to Z. Balogh, A. Dow, D. H. Fremlin and P. J. Nyikos [4, Corollary 2] and is the main ingredient in the proof of Fact G.)

Lemma 2.11. (PFA(S)[S]) *Let X be first countable, linearly H-closed and hereditarily normal. Let U be open and Lindelöf. Then $\overline{U}$ is compact and $\overline{U} - U$ is hereditarily separable.*

Proof: Space X and any closed subset of X are countably compact. By CW (Theorem 2.9), X is hereditarily $\aleph_1$-scwH. By Lemma 2.7, $\partial U = \overline{U} - U$ has countable spread. By Lemmas 2.2–2.3, ∂U is (hereditarily) separable. By Theorem 2.10, ∂U is compact and hence $\overline{U} = U \cup \partial U$ is Lindelöf and thus compact. □

The proof of Theorem 1.3 is now a formality.

Proof of Theorem 1.3: The space is first countable, hence by Lemma 2.11 it is locally compact and Theorem 1.2 does the rest of the job. □

This finishes the proofs, the section and the note.

References

[1] Abraham U., Todorčević S., *Martin's axiom and first-countable S- and L-spaces*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pages 327–346.

[2] Alas O. T., Junqueira L. R., Wilson R. G., *On linearly H-closed spaces*, Topology Appl. **258** (2019), 161–171.

[3] Baillif M., *Notes on linearly H-closed spaces and OD-selection principles*, Topology Proc. **54** (2019), 109–124.

[4] Balogh Z., Dow A., Fremlin D. H., Nyikos P. J., *Countable tightness and proper forcing*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 1, 295–298.

[5] Balogh Z., Gruenhage G., *Two more perfectly normal non-metrizable manifolds*, Topology Appl. **151** (2005), no. 1–3, 260–272.

[6] Bella A., *Observations on some cardinality bounds*, Topology Appl. **228** (2017), 355–362.

[7] Dow A., Tall F. D., PFA(S)[S] *and countably compact spaces*, available in ArXiv 1607.04368 [math.LO] (2016), 24 pages.

[8] Eisworth T., Nyikos P., Shelah S., *Gently killing S-spaces*, Israel J. Math. **136** (2003), 189–220.

[9] Larson P. B., Tall F. D., *Locally compact perfectly normal spaces may all be paracompact*, Fund. Math. **210** (2010), no. 3, 285–300.
[10] Larson P., Tall F. D., *On the hereditary paracompactness of locally compact, hereditarily normal spaces*, Canad. Math. Bull. **57** (2014), no. 3, 579–584.
[11] Nyikos P., *The theory of nonmetrizable manifolds*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pages 633–684.
[12] Nyikos P. J., *Applications of some strong set-theoretic axioms to locally compact* T_5 *and hereditarily scwH spaces*, Fund. Math. **176** (2003), no. 1, 25–45.
[13] Porter J. R., Woods R. G., *Feebly compact spaces, Martin's axiom and "diamond"*, Proc. of the 1984 Topology Conf., Auburn, Ala., 1984, Topology Proc. **9** (1984), no. 1, 105–121.
[14] Roitman J., *Basic S and L*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pages 295–326.
[15] Szentmiklóssy Z., *S-spaces and L-spaces under Martin's axiom*, Topology, Vol. II, Proc. Fourth Colloq., Budapest, 1978, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam, 1980, pages 1139–1145.
[16] Tall F. D., PFA(S)[S] *and the Arhangel'skiĭ–Tall problem*, Topology Proc. **40** (2012), 99–108.
[17] Tall F. D., PFA(S)[S] *for the masses*, Topology Appl. **232** (2017), 13–21.
[18] Todorčević S., *Partition Problems in Topology*, Contemporary Mathematics, 84, American Mathematical Society, Providence, 1989.

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