A Marchaud type inequality

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Abstract. We present a new Marchaud type inequality in $\mathbb{L}^p$ spaces.

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1. Introduction

For $1 \le p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π-periodic, pth power Lebesgue integrable (class of) functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

We also set $C_{2\pi}$ for the family of all 2π-periodic continuous functions on $\mathbb{R}$ with the norm

$$\|f\|_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|.$$

Moreover

$$X^p = \begin{cases} \mathbb{L}^p, & \text{if } 1 \le p < \infty, \\ C_{2\pi}, & \text{if } p = \infty. \end{cases}$$

Notice that X^∞ is not the space of essentially bounded functions.

For $m \in \mathbb{N}$ and $1 \le r \le p \le \infty$, $r \ne \infty$, set

$$W^m_{p,r} = \{f \in X^p \colon f = \varphi \ \text{ a.e. } \varphi, \varphi^{(1)}, \dots, \varphi^{(m-1)} \in AC,\ \varphi^{(m)} \in X^r\}. \tag{1}$$

In the case $p = r = \infty$, we set $W^m_{\infty,\infty} = C^m_{2\pi}$ (functions m-times continuously differentiable).

For $m \in \mathbb{N}$, $f \in X^p$, $1 \le p \le \infty$, and $t > 0$ the usual modulus of continuity (smoothness) of order m of f is defined by

$$\omega_m(f,t)_p = \sup_{|h|\le t} \|\Delta^m_h f\|_p, \tag{2}$$

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where

$$\Delta_h^m f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x+kh).$$

The Marchaud inequality appeared for the first time in [7], but there are various extensions. Let us recall a few.

Theorem 1 (H. Johnen, [5, page 302])**.** *Assume* $1 \le p < \infty$*. If* $f \in \mathbb{L}^p$ *and*

$$\int_0^1 \frac{\omega_{m+n}(f,u)_p}{u^{k+1}}\,\mathrm{d}u < \infty \tag{3}$$

for some $k \in \mathbb{N}$, $1 \le k \le m-1$*, then* $f \in W^k_{p,p}$*, and for* $0 < t \le 1$ *and* $n \in \mathbb{N}$,

$$\omega_m(f^{(k)},t)_p \le C_{r,m}\left(t^m \int_t^2 \frac{\omega_{m+n}(f,u)_p}{u^{m+k+1}}\,\mathrm{d}u + \int_0^t \frac{\omega_{m+n}(f,u)_p}{u^{k+1}}\,\mathrm{d}u\right). \tag{4}$$

Another kind of estimate was given by H. Johnen and K. Scherer in [6]. If $1 \le k \le m-1$ and (3) holds, then $f \in W^k_{p,p}$ and

$$\omega_{m-k}(f^{(k)},t)_p \le C \int_0^t \frac{\omega_m(f,u)_p}{u^{k+1}}\,\mathrm{d}u, \qquad f \in X^p.$$

A proof was also included in [3, pages 178–179].

Recall that if $1 \le r < p \le \infty$ and $f \in X^p$, then $f \in X^r$ and $\|f\|_r \le \|f\|_p$. Therefore $\omega_m(f,t)_r \le \omega_m(f,t)_p$.

In this note we show that a result similar to (4) holds, if we replace $\omega_m(f,u)_p$ by $\omega_m(f,u)_r$, with $1 \le r < p$. In fact, we prove the following theorem:

Theorem 2. *Assume* $1 \le r < p < \infty$ *and* $m \in \mathbb{N}$, $m > 2$*. There exists a constant* C *such that, if* $f \in \mathbb{L}^p$, $1 \le k < m-1$*, and*

$$\int_0^1 \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\,\mathrm{d}s < \infty, \tag{5}$$

then $f \in W^k_{p,p}$ *and for* $0 < t \le 1/2$,

$$\omega_{m-k}(f^{(k)},t)_p \le C\left(t^{m-1-k}\|f\|_r + \int_0^t \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\,\mathrm{d}s + \frac{\omega_m(f,t)_r}{t^{1+k}}\right). \tag{6}$$

Notice that the case $k = m-1$ is not included in Theorem 2. It is done because in such a case the term corresponding to $\|f\|_r$ does not go to zero when $t \to 0$. The result can be improved. As we show in Proposition 2, if we assume condition (5), then $f \in W^k_{r,r}$. A stronger property holds, but it requires to consider fractional derivatives and fractional moduli of smoothness (for definitions see [1]). In fact, it can be proved that under condition (5) there exists $\beta \in (0,1)$ such that the fractional derivative $D^{k+\beta}f$ exists a.e. This kind of problems goes beyond the scope of the paper.

2. Known results

In this section we recall some known facts.

The classes $W^m_{p,p}$ can be described in terms of strong derivatives. A function $f\in\mathbb{L}^p$ has a strong derivative, if there exists $g\in\mathbb{L}^p$ such that

$$\lim_{h\to 0+}\left\|\frac{f(\cdot+h)-f(\cdot)}{h}-g\right\|_p=0.$$

In such a case we denote $g=D^{(1)}_s f$. For $m\in\mathbb{N}$, the strong derivative is defined by $D^{(m)}_s(f)=D^{(1)}_s(D^{(m-1)}_s f)$. It is known that $f\in W^m_{p,p}$ if and only if f has a strong derivative $D^{(m)}_s f$, see [2, Theorem 10.1.12]. Moreover, if $f\in W^m_{p,p}$, then $D^{(m)}_s f=\varphi^{(m)}$, where φ is associated to f as in (1). In what follows we identify f and φ. Moreover, if $f\in W^m_{p,p}$, all strong derivatives of lower order exist, see [2, Theorem 10.1.6].

It is known that $D^m\colon W^m_{p,p}\to\mathbb{L}^p$, defined by $D^m f=D^{(m)}_s f$ is a closed linear operator, see [1, Lemma 2].

In the next result, for $1\le s<\infty$, $\mathbb{L}^s[a,b]$ denotes the usual Lebesgue space.

Theorem 3 (V. N. Gabushin, [4]). *Assume $p,q,r\ge 1$ are real numbers, $0\le k<m$, $k,n\in\mathbb{N}$, and*

$$\frac{m-k}{q}+\frac{k}{r}\ge\frac{m}{p}. \tag{7}$$

There exists a constant A such that, if $f\in\mathbb{L}^q[a,b]$ and $f^{(m)}\in\mathbb{L}^r[a,b]$, and $0<\delta\le(b-a)$, then

$$\delta^k\|D^k f\|_p\le A(\delta^{1/p-1/q}\|f\|_q+\delta^{m+1/p-1/r}\|D^m f\|_r).$$

Proposition 1 (see [3, page 45]). *If $1\le p\le\infty$, $f\in\mathbb{L}^p$, $m,n\in\mathbb{N}$ and $s>0$, then*

$$\omega_m(f,ns)_p\le n^m\,\omega_m(f,s)_p, \tag{8}$$

and

$$\omega_m(f,t)_p\le 2\,\omega_{m-1}(f,t)_p. \tag{9}$$

Theorem 4 (see [3, page 177]). *If $1 \le r < \infty$ and $m \in \mathbb{N}$, there exist positive constants M_1 and M_2 such that for $f \in \mathbb{L}^r$ and $0 < t$*

$$M_1 \omega_m(f,t)_r \le K_m(f,t)_r \le M_2 \omega_m(f,t)_r, \tag{10}$$

where

$$K_m(f,t)_r = \inf_{g \in W^m_{p,p}} \{\|f-g\|_r + t^m \|D^m g\|_r\}.$$

For $n \in \mathbb{N}_0$, let $\mathbb{T}_n$ be the family of all trigonometric polynomials of degree not greater than n. For $f \in \mathbb{L}^r$ define

$$E_{n,r}(f) = \inf_{T \in \mathbb{T}_n} \|f - T\|_r.$$

3. Two results related with strong derivatives

Theorem 5. *Let r be a real number, $1 \le r < \infty$, and $k, m \in \mathbb{N}$. If $f \in \mathbb{L}^r$ and*

$$\int_0^1 \frac{\omega_m(f,s)_r}{s^{1+k}}\, \mathrm{d}s < \infty, \tag{11}$$

then $f \in W^j_{r,r}$ for every j, $0 \le j \le k$. Moreover if $\{g_n\}_{n\in\mathbb{N}}$ is a sequence satisfying $g_n \in W^m_{r,r}$ and

$$\|f - g_n\|_r + \frac{1}{n^m}\|D^m g_n\|_r \le C \omega_m\Big(f, \frac{1}{n}\Big)_r, \qquad n \in \mathbb{N}, \tag{12}$$

with a constant C which depends not on f or n (whose existence is guaranteed by Theorem 4), then

$$(D^j f)(x) = (D^j g_n)(x) + \sum_{i=1}^{\infty} (D^j (g_{n2^i} - g_{n2^{i-1}}))(x) \qquad \text{a.e.} \tag{13}$$

for each j, $0 \le j \le k$, and every $n \in \mathbb{N}$.

Proof: Let $\{g_n\}_{n\in\mathbb{N}} \subset W^m_{r,r}$ be a sequence such that (12) holds.

Fix j, $0 \le j \le k$, and for $n, \nu \in \mathbb{N}$ set

$$G_n = \sum_{i=1}^{\infty}(g_{n2^i} - g_{n2^{i-1}}), \qquad S_{\nu,n} = \sum_{i=1}^{\nu}(g_{n2^i} - g_{n2^{i-1}}),$$

and

$$H_{\nu,n,j} = \sum_{i=1}^{\nu}(D^j g_{n2^i} - D^j g_{n2^{i-1}}) = D^j(S_{\nu,n}).$$

If $j = 0$, it follows from (12) that

$$\lim_{\nu\to\infty} \|f - g_n - S_{\nu,n}\|_r = \lim_{\nu\to\infty} \|f - g_{n2^\nu}\|_r = 0.$$

Hence $f - g_n = G_n$ a.e. for each $n \in \mathbb{N}$. This proves (13) for $j = 0$.

In what follows we assume $1 \le j \le k$.

First, we prove that $H_{\nu,n,j}$ is a Cauchy sequence in $\mathbb{L}^r$. If $\nu, \sigma \in \mathbb{N}$, since $k < m$, from Theorem 3 (with $\delta = 1/(n2^i)$ and $p = q = r$) and (12), we obtain

$$\begin{aligned}
\|H_{\nu+\sigma,n,j} - H_{\nu,n,j}\|_r &\le \sum_{i=\nu+1}^{\nu+\sigma} \|D^j(g_{n2^i} - g_{n2^{i-1}})\|_r \\
&\le A \sum_{i=\nu+1}^{\nu+\sigma} (n2^i)^j \|g_{n2^i} - g_{n2^{i-1}}\|_r + A \sum_{i=\nu+1}^{\nu+\sigma} \frac{1}{(n2^i)^{m-j}} \|D^m(g_{n2^i} - g_{n2^{i-1}})\|_r \\
&\le A \sum_{i=\nu+1}^{\nu+\sigma} (n2^i)^j (\|g_{n2^i} - f\|_r + \|g_{n2^{i-1}} - f\|_r) \\
&\quad + AC \sum_{i=\nu+1}^{\nu+\sigma} \frac{(n2^i)^m}{(n2^i)^{m-j}} \omega_m\Big(f, \frac{1}{n2^i}\Big)_r \\
&\quad + AC \sum_{i=\nu+1}^{\nu+\sigma} \frac{(n2^i)^m}{(n2^i)^{m-j}} \frac{1}{2^m} \omega_m\Big(f, \frac{1}{n2^{i-1}}\Big)_r \\
&\le 2AC \sum_{i=\nu}^{\nu+\sigma} (n2^{i+1})^j \omega_m\Big(f, \frac{1}{n2^i}\Big)_r + 2AC \sum_{i=\nu}^{\nu+\sigma} (n2^i)^j \omega_m\Big(f, \frac{1}{n2^i}\Big)_r
\end{aligned}$$

(here we have used (8) and it will be used again in the next inequality)

$$\begin{aligned}
&\le 2^{m+1} AC \sum_{i=\nu}^{\nu+\sigma} (n2^{i+1})^j \omega_m\Big(f, \frac{1}{n2^{i+1}}\Big)_r + 2AC \sum_{i=\nu}^{\nu+\sigma} (n2^i)^j \omega_m\Big(f, \frac{1}{n2^i}\Big)_r \\
&\le C_1 \sum_{i=\nu}^{\nu+\sigma+1} (n2^i)^j \omega_m\Big(f, \frac{1}{n2^i}\Big)_r \le C_2 \sum_{i=\nu}^{\nu+\sigma+1} \int_{1/n2^i}^{1/(n2^{i-1})} \frac{\omega_m(f,s)_r}{s^{j+1}}\,\mathrm{d}s \\
&\le C_2 \int_0^{1/(n2^\nu)} \frac{\omega_m(f,s)_r}{s^{j+1}}\,\mathrm{d}s.
\end{aligned}$$

Therefore, it follows from (11) that $\{H_{\nu,n,j}\}_{\nu=1}^\infty$ is a Cauchy sequence in $\mathbb{L}^r$. Thus there exists $g \in \mathbb{L}^r$ such that $\|H_{\nu,n,j} - g\|_r \to 0$, as $\nu \to \infty$.

Since $S_{\nu,n} \to f - g_n = G_n$, $H_{\nu,n}(f) = D^k(S_{\nu,n}) \to g$ in $\mathbb{L}^r$, as $\nu \to \infty$, and D^j is a closed linear operator, then $f - g_n = G_n \in W^j_{r,r}$ and $D^j(f - g_n) =$

$D^j f - D^j g_n = g$. Hence

$$D^j g_n + \sum_{i=1}^{\infty} (D^j g_{n2^i} - D^j g_{n2^{i-1}}) = g + D^j g_n = D^j f \qquad \text{a.e.}$$

□

Proposition 2. *If* $1 \leq r < p \leq \infty$, *and* $f \in \mathbb{L}^r$ *satisfies* (11) *with* $k \in \mathbb{N}$, *then* $f \in W^k_{r,r}$.

Proof: It is known, see [9, page 334], that if

$$\sum_{i=1}^{\infty} i^{k-1} E_{i,r}(f) < \infty,$$

then f is equivalent to a function $g \in X^k_{r,r}$. On the other hand, there exists a constant C such that for each $f \in \mathbb{L}^r$, see [9, page 325],

$$E_{n,r}(f) \leq C\omega_m\Big(f, \frac{1}{n+1}\Big)_r. \tag{14}$$

Hence

$$\sum_{i=1}^{\infty} i^{k-1} E_{i,r}(f) \leq C_1 \sum_{i=1}^{\infty} i^{k-1} \omega_m\Big(f, \frac{1}{i+1}\Big)_r \leq C_1 \sum_{i=1}^{\infty} \int_i^{i+1} s^{k-1} \omega_m\Big(f, \frac{1}{s}\Big)_r \, \mathrm{d}s$$
$$= C_1 \int_1^{\infty} s^{k-1} \omega_m\Big(f, \frac{1}{s}\Big)_r \, \mathrm{d}s = C_1 \int_0^1 \frac{\omega_m(f,t)_r}{t^{k+1}} \, \mathrm{d}t < \infty.$$

□

4. Proof of Theorem 2

Given $t \in (0, 1/2]$, choose $n \in \mathbb{N}$ such that

$$\frac{1}{1+n} < t \leq \frac{1}{n} < 2\pi.$$

Set $\tau(j) = (1+n)2^j$ and $\lambda_j = 1/\tau(j)$. If $f \in \mathbb{L}^p$, taking into account (10) for each $j \in \mathbb{N}_0$ we can fix $h_j \in W^m_{r,r}$ such that

$$\|f - h_j\|_r + \lambda_j^m \|D^m h_j\|_r \leq 2M_2 \omega_m(f, \lambda_j)_r. \tag{15}$$

Step 1: From Proposition 2 we know that $f \in W^k_{r,r}$. Hence there exists φ such that $f = \varphi$ a.e., $\varphi, \varphi^{(1)}, \ldots, \varphi^{(m-1)} \in AC$, and $\varphi^{(m)} \in \mathbb{L}^r$. Thus we only need to find $G \in \mathbb{L}^p$, such that

$$\varphi^{(m)} = G \qquad \text{a.e.} \tag{16}$$

Of course, we can assume that $f = \varphi$.

The conditions $h_j \in W^m_{r,r}$ and $0 \le k < m$ imply $D^k h_j \in \mathbb{L}^p$. In fact $D^k h_j$ is an (absolutely) continuous function.

Taking into account that $h_i \to f$ in the norm of $\mathbb{L}^r$, see (15),

$$f = h_0 + \sum_{i=0}^{\infty} (h_{i+1} - h_i) \qquad \text{a.e.} \tag{17}$$

Since $(D^k h_{i+1} - D^k h_i) \in \mathbb{L}^p$ for each $i \in \mathbb{N}$, if

$$\sum_{i=0}^{\infty} \|D^k h_{i+1} - D^k h_i\|_p < \infty \tag{18}$$

then the series

$$D^k h_0 + \sum_{i=0}^{\infty} (D^k h_{i+1} - D^k h_i) \tag{19}$$

converges in $\mathbb{L}^p$, see [8, page 109].

Let us verify that (18) holds. In order to simplify the notations we set $\alpha = 1/r - 1/p$.

In the case $q = r < p$, condition (7) holds. Since $\lambda_i^k < 2\pi$, Theorem 3 can be used (with $\delta = \lambda_i$, recall that we consider $q = r$) and it follows from (15) that

$$\begin{aligned}
\lambda_i^k \|D^k h_{i+1} - D^k h_i\|_p &\le A(\lambda_i^{-\alpha} \|h_{i+1} - h_i\|_r + \lambda_i^{m-\alpha} \|D^m h_{i+1} - D^m h_i\|_r) \\
&\le A\lambda_i^{-\alpha}(\|h_{i+1} - f\|_r + \|f - h_i\|_r + \lambda_i^m \|D^m h_i\|_r + 2^m \lambda_{i+1}^m \|D^m h_{i+1}\|_r) \\
&\le 2M_2 A \lambda_j^{-\alpha}(\omega_m(f, \lambda_i)_r + 2^m \omega_m(f, \lambda_{i+1})_r) \\
&\le 2AM_2(1 + 2^m)\lambda_i^{-\alpha} \omega_m(f, \lambda_i)_r.
\end{aligned}$$

Note that, since $\lambda_i = 2\lambda_{i+1}$,

$$1 \le 2 \int_{\lambda_{i+1}}^{\lambda_i} \frac{\mathrm{d}s}{s}.$$

Therefore, using (8), one has (recall that $k + \alpha > 0$)

$$\begin{aligned}
\sum_{j=0}^{\infty} \|D^k h_{j+1} - D^k h_j\|_p &\le 2(1 + 2^m)AM_2 \sum_{i=0}^{\infty} \lambda_i^{-\alpha} \frac{\omega_m(f, \lambda_i)_r}{\lambda_i^k} \\
&\le 4(1 + 2^m)AM_2 \sum_{i=0}^{\infty} \frac{2^m \omega_m(f, \lambda_{i+1})_r}{\lambda_i^{k+\alpha}} \int_{\lambda_{i+1}}^{\lambda_i} \frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&\le 4(1+2^m)\,A\,M_2\,2^m \sum_{i=0}^{\infty} \int_{\lambda_{i+1}}^{\lambda_i} \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\,\mathrm{d}s \\
&= 4(1+2^m)AM_2 2^m \int_0^{\lambda_0} \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\,\mathrm{d}s \qquad (20)\\
&\le 4(1+2^m)AM_2 2^m \int_0^{t} \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\,\mathrm{d}s < \infty,
\end{aligned}
$$

where we use condition (5). Thus the series (19) converges in $\mathbb{L}^p$. The limit of the partial sums of the series (19) will be denoted by M_k.

We will show that in (16) we can take $G = M_k$. Since h_0 can be chosen as g_{n+1} in (12) and h_i as $g_{(n+1)2^i}$, it follows from (13) that

$$
M_k = D^k g_{n+1} + \sum_{i=0}^{\infty} D^k\big(g_{(n+1)2^{i+1}} - g_{(n+1)2^i}\big) = D^k f \qquad \text{a.e.}
$$

We have proved that $f \in W^k_{p,p}$.

Step 2: We will need an estimate of $\|D^{m-1}h_0\|_p$ in terms of $\|f\|_r$ and $\omega_m(f,t)_r$. Take into account that

$$
\frac{1}{\lambda_0} = 1+n \le 2n \le \frac{2}{t}.
$$

We use again Theorem 3 (with $\delta = 1$) and (15) to obtain

$$
\begin{aligned}
\|D^{m-1}h_0\|_p &\le A(\|h_0\|_r + \|h_0^{(m)}\|_r) \le A(\|f\|_r + \|f-h_0\|_r + \|D^m h_0\|_r) \\
&\le A\Big(\|f\|_r + 2M_2\omega_m(f,\lambda_0)_r + \frac{1}{\lambda_0^m} 2M_2\omega_m(f,\lambda_0)_r\Big) \qquad (21)\\
&\le A\Big(\|f\|_r + 2M_2\frac{\omega_m(f,t)_r}{t^m} + 2\frac{2^m}{t^m} M_2\omega_m(f,t)_r\Big) \\
&\le A\Big(\|f\|_r + 2M_2(1+2^m)\frac{\omega_m(f,t)_r}{t^m}\Big).
\end{aligned}
$$

Step 3: Let us verify (6). From (8), (9) and (10), we know that

$$
\begin{aligned}
\omega_{m-k}(D^k f,t)_p &\le 2^{m-k}\omega_{m-k}\Big(D^k f, \frac{1}{1+n}\Big)_p \le 2^{1+m-k}\omega_{m-1-k}\Big(D^k f, \frac{1}{1+n}\Big)_p \\
&\le \frac{2^{1+m-k}}{M_1}\Big(\|D^k f - D^k h_0\|_p + \frac{1}{(1+n)^{m-1-k}}\|D^{m-1}h_0\|_p\Big) \qquad (22)\\
&\le \frac{2^{1+m-k}}{M_1}\Big(\|D^k f - D^k h_0\|_p + t^{m-1-k}\|D^{m-1}h_0\|_p\Big).
\end{aligned}
$$

From (22), (17), (20), and (15) we obtain

$$\omega_{m-k}(D^k f, t)_p \le C_2\left(\sum_{j=0}^{\infty} \|D^k h_{j+1} - D^k h_j\|_p + t^{m-1-k}\|D^{m-1}h_0\|_p\right)$$
$$\le C_2\left(\int_0^t \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\,\mathrm{d}s + t^{m-1-k}\|f\|_r + \frac{\omega_m(f,t)_r}{t^{1+k}}\right).$$

□

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