

On Beurling measure algebras

ROSS STOKKE

Abstract. We show how the measure theory of regular compacted-Borel measures defined on the δ -ring of compacted-Borel subsets of a weighted locally compact group (G, ω) provides a compatible framework for defining the corresponding Beurling measure algebra $\mathcal{M}(G, \omega)$, thus filling a gap in the literature.

Keywords: weighted locally compact group; group algebra; measure algebra; Beurling algebra

Classification: 43A10, 22D15, 43A05, 43A20, 43A60, 28C10

Throughout this article, G denotes a locally compact group and $\omega: G \rightarrow (0, \infty)$ is a continuous *weight function* satisfying

$$\omega(st) \leq \omega(s)\omega(t), \quad s, t \in G, \quad \text{and} \quad \omega(e_G) = 1;$$

the pair (G, ω) is called a *weighted locally compact group*. Let λ denote a fixed Haar measure on G , with respect to which the group algebra $L^1(G)$ and $L^\infty(G) = L^1(G)^*$ are defined in the usual way. The Beurling group algebra, $L^1(G, \omega)$, is composed of all functions f such that ωf belongs to $L^1(G)$, with $\|f\|_{1, \omega} := \|\omega f\|_1$ and convolution product. If $\mathcal{S}(G)$ is a closed subspace of $L^\infty(G)$, $\psi \in \mathcal{S}(G, \omega^{-1})$ exactly when $\psi/\omega \in \mathcal{S}(G)$; putting $\|\psi\|_{\infty, \omega^{-1}} = \|\psi/\omega\|_\infty$, $\mathcal{S}(G, \omega^{-1})$ is a Banach space and $S: \mathcal{S}(G, \omega^{-1}) \rightarrow \mathcal{S}(G): \psi \mapsto \psi/\omega$ is an isometric linear isomorphism. The Beurling group algebra $L^1(G, \omega)$ has become a classical object of study that has received significant research attention over the years, see the monographs [3], [11], [15] and the references therein; a sample of relevant articles include [6], [5], [7], [8], [17], [18], [20]. When ω is the trivial weight $\omega \equiv 1$ — the “non-weighted case” — $L^1(G, \omega) = L^1(G)$, the study of which is intimately linked with the measure algebra $M(G)$ of complex, regular, Borel measures on G , which contains $L^1(G)$ as a closed ideal.

The above definition of $L^1(G, \omega)$ is valid for any weight ω . As in the non-weighted case, it is desirable to have a Beurling measure algebra $M(G, \omega)$ that shares the same relationship with $L^1(G, \omega)$ that $M(G)$ shares with $L^1(G)$. In

the literature, $M(G, \omega)$ is usually defined as the collection of all complex regular measures ν defined on $\mathfrak{B}(G)$, the σ -algebra of Borel subsets of G , such that $\int \omega(t) d|\nu|(t) < \infty$, and the identification $M(G, \omega) = C_0(G, \omega^{-1})^*$ through $\langle \nu, \psi \rangle_\omega = \int \psi d\nu$ is required. This implies that the dual map, S^* , of the isometric isomorphism $S: C_0(G, \omega^{-1}) \rightarrow C_0(G)$ is itself a linear isometric isomorphism of $M(G)$ onto $M(G, \omega)$. Validity of this definition of $M(G, \omega)$ thus requires that for each $\mu \in M(G)$, $\nu = S^*\mu \in M(G, \omega)$ is a complex Borel measure defined on all of $\mathfrak{B}(G)$ — the near-universal requirement of “Borel measures” in abstract harmonic analysis — satisfying

$$(1) \quad \int \psi d\nu = \langle \nu, \psi \rangle_\omega = \left\langle \mu, \frac{\psi}{\omega} \right\rangle = \int \frac{\psi}{\omega} d\mu, \quad \psi \in C_0(G, \omega^{-1}).$$

However, when ω is not bounded away from zero, it can happen that no such complex measure on $\mathfrak{B}(G)$ exists.

To see this, consider (G, ω) where $G = (\mathbb{Z}, +)$ and $\omega(n) = 2^{-n}$, $n \in \mathbb{Z}$, and assume the above definition of $M(G, \omega)$ is sound. Since $\mu_1, \mu_2 \in l^1(\mathbb{Z})^+ = M(\mathbb{Z})^+$ and $\mu = \mu_1 - \mu_2 \in M(\mathbb{Z})$, where

$$\mu_1(n) = \begin{cases} 2^{-n}, & n \in 2\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mu_2(n) = \begin{cases} 2^{-n}, & n \in \mathbb{N} \setminus 2\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

$\nu_1 = S^*(\mu_1)$, $\nu_2 = S^*(\mu_2)$, and $\nu = S^*(\mu) = \nu_1 - \nu_2$ are then required to be complex measures on $\mathfrak{B}(G) = \wp(\mathbb{Z})$ satisfying (1). Hence, for each $n \in \mathbb{Z}$,

$$\begin{aligned} \nu_1(\{n\}) &= \int \chi_{\{n\}} d\nu_1 = \left\langle \mu_1, \frac{\chi_{\{n\}}}{\omega} \right\rangle = \begin{cases} 1, & n \in 2\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \\ \nu_2(\{n\}) &= \begin{cases} 1, & n \in \mathbb{N} \setminus 2\mathbb{N}, \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

hence, $\nu_1(2\mathbb{N}) = \sum_{k \in \mathbb{N}} \nu_1(\{2k\}) = \infty$ and $\nu_2(\mathbb{N} \setminus 2\mathbb{N}) = \sum_{k \in \mathbb{N}} \nu_2(\{2k - 1\}) = \infty$. Thus, ν_1, ν_2 do not map into \mathbb{C} . Moreover, (although ν_1, ν_2 can be viewed as positive measures), if $\nu = \nu_1 - \nu_2$ were a measure, additivity would give

$$\nu(\mathbb{N}) = \nu(2\mathbb{N}) + \nu(\mathbb{N} \setminus 2\mathbb{N}) = \nu_1(2\mathbb{N}) - \nu_2(\mathbb{N} \setminus 2\mathbb{N}) = \infty - \infty.$$

We conclude that functionals in $C_0(G, \omega^{-1})^*$ cannot necessarily be identified with complex Borel measures in the standard sense. It is perhaps for this reason that many authors assume the additional condition $\omega \geq 1$, since this guarantees containment of $M(G, \omega)$ in $M(G)$ and, thus, the essential properties of $M(G)$ also hold for $M(G, \omega)$, e.g., see [3]. Letting $\mathfrak{S}(G)$ denote the δ -ring of “compactified-Borel sets” — i.e., the δ -ring of all Borel subsets of G with compact closure — a *compactified-Borel measure* on G is a countably additive complex-valued function

on $\mathfrak{S}(G)$ in the sense of [4, Definitions II.1.2 and II.8.2]¹. For non-compact G , there are positive regular measures μ, ν on $\mathfrak{B}(G)$ such that $\mu(G) = \nu(G) = \infty$ (e.g., Haar measures), and therefore $\mu - \nu$ is not defined on $\mathfrak{B}(G)$; however, these same measures are real-valued on $\mathfrak{S}(G)$, so $\mu - \nu$ is well-defined on $\mathfrak{S}(G)$. This is one benefit to studying measure theory over $\mathfrak{S}(G)$, rather than on all of $\mathfrak{B}(G)$.

The purpose of this article is to show that the theory of complex regular compacted-Borel measures, as developed in [4] (also see paragraph two of the “Notes and Remarks” section of Chapter II of [4] for additional references), can be used to provide a rigorous definition of $M(G, \omega)$, thus providing a solid foundation for all the papers in which $M(G, \omega)$ is employed without the requirement that $\omega \geq 1$; moreover, we hope this reduces the number of instances in which the $\omega \geq 1$ assumption is required going forward. To stress that we are using the theory of complex regular compacted-Borel measures, we will use the notation $\mathcal{M}(G, \omega)$ — inspired by [4] — rather than $M(G, \omega)$. Beyond identifying the correct collection of measures to employ, work is required to establish the needed theory. As measure theory can be quite finicky in general; because the study of compacted-Borel measures introduces different technicalities than those encountered in the Borel measure situation; and because a lot of research already depends on the results found herein, we have included a careful treatment of our development of $\mathcal{M}(G, \omega)$. There are numerous detailed classical expositions of the basic theory $M(G)$, and we believe the same is required for $\mathcal{M}(G, \omega)$.

We restrict ourselves to developing only the most standard properties of $\mathcal{M}(G, \omega)$: we provide a careful definition of its elements and show that with convolution product it is a dual Banach algebra containing a copy of the Beurling group algebra $L^1(G, \omega)$ as a closed ideal. Beyond this, we only show that $\mathcal{M}(G, \omega)$ embeds via a strict-to-weak* continuous isometric isomorphism as a subalgebra of the universal enveloping dual Banach algebra of $\mathcal{L}^1(G, \omega)$, $WAP(L^\infty(G, \omega))^*$, a result needed in [12]. The inspiration for this paper was our need to work with $\mathcal{M}(G, \omega)$ in [12].

1. $\mathcal{M}(G, \omega)$: definition and basic properties

Unless explicitly indicated otherwise, all references are to statements in Sections 1, 2, 5, 7–10 of Chapter II and Sections 10 of Chapter III of [4]. We will mostly adhere to the notation found therein. In particular, $\mathcal{M}(G)$ is the linear space composed of all regular complex compacted-Borel measures on G (Sections II.8 and III.10) and $\mathcal{M}_r(G)$ is the Banach space of *bounded* measures

¹In [4], for the sake of brevity, the authors refer to compacted-Borel measures simply as Borel measures. To our knowledge, with the exception of [4], Borel measures in abstract harmonic analysis are always defined on $\mathfrak{B}(G)$.

in $\mathcal{M}(G)$ (Sections II.1 and II.8). Let $\mathfrak{C}(G)$ denote the directed set of compact subsets of G , and denote the space of continuous functions on G with compact support by $C_{00}(G)$, the space of continuous functions on G vanishing at infinity by $C_0(G)$, and the space of continuous functions on G supported on $K \in \mathfrak{C}(G)$ by $C_K(G)$; unless the context requires otherwise, these spaces are taken with the uniform norm $\|\cdot\|_\infty$.

Remark 1.1. (a) Let $\mu \in \mathcal{M}(G)$. A Borel subset A of G belongs to \mathcal{E}_μ if A is contained in some open set U such that

$$\sup\{|\mu|(A') : A' \in \mathfrak{S}(G) \text{ and } A' \subseteq U\} < \infty;$$

\mathcal{E}_μ is a δ -ring containing $\mathfrak{S}(G)$ and for $A \in \mathcal{E}_\mu$ putting

$$(2) \quad \mu_e(A) := \lim_C \mu(C), \quad \text{where } C \in \mathfrak{C}(G), C \subseteq A,$$

we obtain a complex measure on \mathcal{E}_μ extending μ , called the maximal regular extension of μ (II.8.15). Observe that any Borel subset of a set in \mathcal{E}_μ is also in \mathcal{E}_μ , from which it readily follows that $h\chi_E$ is locally μ_e -measurable whenever $E \in \mathcal{E}_\mu$ and h is a Borel-measurable function on G .

(b) When $\mu \in \mathcal{M}_r(G)$, $\mathcal{E}_\mu = \mathfrak{B}(G)$ and $\mu_e \in M(G)$, where $M(G)$ denotes the usual measure algebra of regular complex Borel measures $\mu: \mathfrak{B}(G) \rightarrow \mathbb{C}$, e.g., see [2], [9], [14]. Thus, the measures in $\mathcal{M}_r(G)$ are in one-to-one correspondence with measures in $M(G)$ via $\mu \mapsto \mu_e$; moreover, it is clear from the results in Section III.10 (or Theorem 1.5, below, in the non-weighted case) that $\mu \mapsto \mu_e$ is a weak*-continuous isometric algebra isomorphism of $\mathcal{M}_r(G)$ onto $M(G)$. Thus, for the purposes of abstract harmonic analysis on (non-weighted) G , $\mathcal{M}_r(G)$ can be used in place of the usual $M(G)$, and, as shown in [4], provides some advantages.

For $\mu \in \mathcal{M}(G)$, let I_μ denote the linear functional $I_\mu(f) = \int f d\mu$ defined on $\mathcal{L}^1(\mu)$, or any subspace of $\mathcal{L}^1(\mu)$. Then

$$(3) \quad \mu \mapsto I_\mu: \mathcal{M}(G) \rightarrow \mathfrak{I}$$

is a linear bijection where \mathfrak{I} is the set of all linear functionals I on $C_{00}(G)$ such that $I \in C_K(G)^*$ for each $K \in \mathfrak{C}(G)$; (3) maps $\mathcal{M}(G)^+$ onto \mathfrak{I}^+ and $\mathcal{M}_r(G)$ onto $C_{00}(G)^* = C_0(G)^*$ (II.8.12).

Remark 1.2. It should be noted that when μ is a complex measure on a δ -ring \mathfrak{S} , $f \in \mathcal{L}^1(\mu)$ requires that f vanishes off a countable union of sets in \mathfrak{S} (II.2.5, paragraph 2). Thus, when $f \in \mathcal{L}^1(\mu)$ for $\mu \in \mathcal{M}(G)$, f must vanish off a σ -compact set, a technical issue requiring careful attention throughout this

note. Consider the case when $\mu \in \mathcal{M}_r(G)$. Then any $\phi \in C_0(G)$ vanishes off a σ -compact set and since ϕ is continuous and bounded, it is easy to see that $\phi \in \mathcal{L}^1(\mu)$. Assuming further that $\mu \geq 0$ and $\phi \geq 0$ and taking an increasing sequence (ϕ_n) in $C_{00}(G)^+$ such that $\|\phi_n - \phi\|_\infty \rightarrow 0$, $\lim I_\mu(\phi_n) = \lim \int \phi_n d\mu = \int \phi d\mu = I_\mu(\phi)$ (e.g., by MCT II.7), so I_μ is the unique continuous extension of I_μ on $C_{00}(G)$ to $C_0(G)$. Thus, $C_0(G)^* = \{I_\mu: \mu \in \mathcal{M}_r(G)\}$, so — in this theory and as usual — we can identify $\mathcal{M}_r(G)$ and $C_0(G)^*$ through the pairing $\langle \mu, \phi \rangle = \int \phi d\mu$.

Let $\nu \in \mathcal{M}(G)$, h a continuous function on G . Then h is locally ν -measurable (II.8.2) and for each $A \in \mathfrak{S}(G)$, $h\chi_A \in \mathcal{L}^1(\nu)$ since $|h|$ is bounded on A ; i.e., h is locally ν -summable. Therefore,

$$h\nu(A) := \int h\chi_A d\nu, \quad A \in \mathfrak{S}(G),$$

defines a complex measure on $\mathfrak{S}(G)$ (see II.7.2, where the notation $h d\nu$ rather than $h\nu$ is used); as $h\nu \ll \nu$ (II.7.8), $h\nu \in \mathcal{M}(G)$ (II.8.3). If $h > 0$, then $(1/h)(h\nu) \in \mathcal{M}(G)$ and a simple application of II.7.5 gives $(1/h)(h\nu) = \nu$.

Hence, $\omega\nu \sim \nu$ for each $\nu \in \mathcal{M}(G)$, and

$$\mathcal{M}(G) \rightarrow \mathcal{M}(G): \nu \mapsto \omega\nu$$

defines a linear isomorphism with inverse $\nu \mapsto (1/\omega)\nu$. We can thus define

$$\mathcal{M}(G, \omega) := \{\nu \in \mathcal{M}(G): \omega\nu \in \mathcal{M}_r(G)\}; \quad \text{letting } \|\nu\|_\omega = \|\omega\nu\|, \nu \in \mathcal{M}(G, \omega),$$

it follows that $\mathcal{M}(G, \omega)$ is a Banach space and $\nu \mapsto \omega\nu$ is an isometric linear isomorphism of $\mathcal{M}(G, \omega)$ onto $\mathcal{M}_r(G)$ with inverse map $\mu \mapsto (1/\omega)\mu$. (As shown in the introduction, this definition cannot, in general, be made with $M(G)$ replacing $\mathcal{M}_r(G)$.) Observe that by II.7.3, $\nu \in \mathcal{M}(G, \omega)$ exactly when $|\nu| \in \mathcal{M}(G, \omega)$, and $\|\nu\|_\omega = \| |\nu| \|_\omega$.

Proposition 1.3. *For each $\nu \in \mathcal{M}(G, \omega)$, $I_\nu \in C_0(G, \omega^{-1})^*$ and $\|I_\nu\| = \|\nu\|_\omega$; moreover,*

$$(4) \quad C_0(G, \omega^{-1})^* = \{I_\nu: \nu \in \mathcal{M}(G, \omega)\}.$$

We can thus make the identification $\mathcal{M}(G, \omega) = C_0(G, \omega^{-1})^*$ through the pairing

$$\langle \nu, \psi \rangle_\omega = \int \psi d\nu, \quad \nu \in \mathcal{M}(G, \omega), \psi \in C_0(G, \omega^{-1}).$$

With respect to this identification, the inverse isometric isomorphisms

$$\mathcal{M}(G, \omega) \rightarrow \mathcal{M}_r(G): \nu \mapsto \omega\nu \quad \text{and} \quad \mathcal{M}_r(G) \rightarrow \mathcal{M}(G, \omega): \mu \mapsto \frac{1}{\omega}\mu$$

are weak*-homeomorphisms.

PROOF: As noted above, $S: C_0(G, \omega^{-1}) \rightarrow C_0(G): \psi \mapsto (\psi/\omega)$ is an isometric isomorphism, so $S^*: \mathcal{M}_r(G) = C_0(G)^* \rightarrow C_0(G, \omega^{-1})^*$ is also an isometric isomorphism. Let $\nu \in \mathcal{M}(G, \omega)$. Then $\omega\nu \in \mathcal{M}_r(G)$ and for $\psi \in C_0(G, \omega^{-1})$, $(\psi/\omega) \in C_0(G) \subseteq \mathcal{L}^1(\omega\nu)$, see Remark 1.2; therefore by II.7.5, $\psi = (\psi/\omega)\omega \in \mathcal{L}^1(\nu)$ and

$$\langle I_\nu, \psi \rangle = \int \psi \, d\nu = \int \frac{\psi}{\omega} \, d(\omega\nu) = \langle \omega\nu, S(\psi) \rangle = \langle S^*(\omega\nu), \psi \rangle.$$

Hence, $C_0(G, \omega^{-1}) \subseteq \mathcal{L}^1(\nu)$, $I_\nu = S^*(\omega\nu) \in C_0(G, \omega^{-1})^*$, and therefore $\|I_\nu\| = \|S^*(\omega\nu)\| = \|\omega\nu\| = \|\nu\|_\omega$; since $S^*(\mu) = I_{\omega^{-1}\mu}$ and S^* maps onto $C_0(G, \omega^{-1})^*$, we have (4). Making the identification of ν and I_ν , $\mu \mapsto (1/\omega)\mu = S^*(\mu)$ is weak*-continuous, with (weak*-continuous) inverse map $\nu \mapsto \omega\nu$. \square

In Lemma 1.4, X is a locally compact Hausdorff space, $h: X \rightarrow (0, \infty)$ is a continuous function, and $\mu \in \mathcal{M}(X)^+$ is such that $h\mu \in \mathcal{M}_r(X)$. Observe that $\mathcal{E}_\mu \subseteq \mathfrak{B}(X) = \mathcal{E}_{h\mu}$; see Remark 1.1.

Lemma 1.4. *The function h is locally μ_e -summable and for any set $A \in \mathcal{E}_\mu$, $h(\mu_e)(A) = (h\mu)_e(A)$.*

PROOF: Let $A \in \mathcal{E}_\mu$. Take $(C_n)_n$ to be an increasing sequence of compact subsets of A such that $\mu_e(A) = \lim_n \mu(C_n)$ and let $D = \bigcup_n C_n$. Observe that $D, A \setminus D \in \mathcal{E}_\mu$ and $\mu_e(D) = \lim \mu_e(C_n) = \lim \mu(C_n) = \mu_e(A)$; hence

$$(5) \quad \mu_e(A \setminus D) = 0.$$

It follows that for any compact subset C of $A \setminus D$, $\mu(C) = 0$ and therefore, since h is locally μ -summable and bounded on C , $h\mu(C) = 0$. Hence,

$$(6) \quad \lim(h\mu)_e(A \setminus D) = \lim\{(h\mu)(C): C \in \mathfrak{C}(X), C \subseteq A \setminus D\} = 0.$$

As noted in Remark 1.1, $h\chi_{A \setminus D}$ is locally μ_e -measurable and it follows from (5) and II.2.7 that

$$(7) \quad \int h\chi_{A \setminus D} \, d\mu_e = \lim_n \int (h \wedge n)\chi_{A \setminus D} \, d\mu_e = 0.$$

Also, since $h\mu$ is bounded, $\lim \int h\chi_{C_n} \, d\mu_e = \lim \int h\chi_{C_n} \, d\mu = \sup(h\mu)(C_n) < \infty$ (using II.8.15 Remark 3), and therefore by II.2.7,

$$(8) \quad \int h\chi_D \, d\mu_e = \lim \int h\chi_{C_n} \, d\mu_e = \lim(h\mu)(C_n) = \lim(h\mu)_e(C_n) = (h\mu)_e(D).$$

From (7) and (8), $h\chi_{A \setminus D}, h\chi_D \in \mathcal{L}^1(\mu_e)$, whence $h\chi_A \in \mathcal{L}^1(\mu_e)$. Hence, h is locally μ_e summable. Moreover, (8), (7) and (6) yield $h(\mu_e)(A) = (h\mu)_e(A)$. \square

Let $p: G \times G \rightarrow G: (s, t) \mapsto st$. Following III.10.2, we say that $\mu, \nu \in \mathcal{M}(G)$ are convolvable, or that $\mu * \nu$ exists, if p is $\mu \times \nu$ -proper in the sense of II.10.3, i.e., if $p^{-1}(A) \in \mathcal{E}_{\mu \times \nu}$ whenever $A \in \mathfrak{S}(G)$. In this case, $\mu * \nu \in \mathcal{M}(G)$, where for $A \in \mathfrak{S}(G)$,

$$\begin{aligned} \mu * \nu(A) &= p_*((\mu \times \nu)_e)(A) = (\mu \times \nu)_e(p^{-1}(A)) \\ &= \lim\{(\mu \times \nu)(C): C \subseteq p^{-1}(A), C \in \mathfrak{C}(G \times G)\}; \end{aligned}$$

see III.10.2, II.10.3, II.10.5, II.10.1. Equivalently, one can check that $\mu * \nu$ exists if and only if

$$\sup\{(|\mu| \times |\nu|)(C): C \subseteq p^{-1}(D), C \in \mathfrak{C}(G \times G)\} < \infty$$

for every compact subset D of G . (In our context, the definition of $\mu \times \nu \in \mathcal{M}(G \times G)$ and its properties are found in Section II.9.)

Theorem 1.5. *With respect to convolution product, $\mathcal{M}(G, \omega) = C_0(G, \omega^{-1})^*$ is a Banach algebra, i.e., $(\mu, \nu) \mapsto \mu * \nu$ is a well-defined associative operation on $\mathcal{M}(G, \omega)$ satisfying $\|\mu * \nu\|_\omega \leq \|\mu\|_\omega \|\nu\|_\omega$. Moreover, for $\mu, \nu \in \mathcal{M}(G, \omega)$ and $\psi \in C_0(G, \omega^{-1})$,*

$$\begin{aligned} \langle \mu * \nu, \psi \rangle_\omega &= \int \psi(st) \, d(\mu \times \nu)_e(s, t) \\ (9) \qquad &= \iint \psi(st) \, d\mu(s) \, d\nu(t) = \iint \psi(st) \, d\nu(t) \, d\mu(s). \end{aligned}$$

PROOF: Let $\mu, \nu \in \mathcal{M}(G, \omega)$, with $\mu, \nu \geq 0$. Let D be a compact subset of G , C a compact subset of $p^{-1}(D)$. The functions $1_C(x, y)$ and $g(x, y) = (1/(\omega(x)\omega(y)))1_C(x, y)$ are Borel measurable functions, and are therefore locally $(\sigma \times \varrho)$ -measurable for any pair of measures $\sigma, \varrho \in \mathcal{M}(G)$; moreover, since they are nonnegative, bounded and vanish off C , $1_C, g \in \mathcal{L}^1(\sigma \times \varrho)$. Applying the Fubini theorem (II.9.8) to these functions, and using II.7.5 twice — which also applies by II.9.8 — we obtain

$$\begin{aligned} \mu \times \nu(C) &= \iint 1_C(x, y) \, d\mu(x) \, d\nu(y) = \iint g(x, y)\omega(x) \, d\mu(x)\omega(y) \, d\nu(y) \\ &= \iint g(x, y) \, d\omega\mu(x) \, d\omega\nu(y) \\ &= \int_{G \times G} \frac{1}{\omega(x)\omega(y)} 1_C(x, y) \, d(\omega\mu \times \omega\nu)(x, y) \end{aligned}$$

$$\begin{aligned} &\leq \int_{G \times G} \frac{1}{\omega(xy)} 1_C(x, y) \, d(\omega\mu \times \omega\nu)(x, y) \\ &\leq \int_{G \times G} M_D 1_C(x, y) \, d(\omega\mu \times \omega\nu)_e(x, y), \end{aligned}$$

where $M_D = \sup_{z \in D} \omega(z)^{-1}$, since $C \subseteq p^{-1}(D)$, and we have used II.8.15 Remark 3. Observe that $p^{-1}(D) \in \mathfrak{B}(G \times G) = \mathcal{E}_{\omega\mu \times \omega\nu}$, since $\omega\mu \times \omega\nu \in \mathcal{M}_r(G \times G)$ — see II.9.14 — so

$$\begin{aligned} \mu * \nu(C) &\leq \int_{G \times G} M_D 1_{p^{-1}(D)} \, d(\omega\mu \times \omega\nu)_e \leq M_D \|\omega\mu \times \omega\nu\| \\ &= M_D \|\omega\mu\| \|\omega\nu\| = M_D \|\mu\|_\omega \|\nu\|_\omega. \end{aligned}$$

Hence, $\mu * \nu$ exists. We now show $\mu * \nu \in \mathcal{M}(G, \omega)$ and $\|\mu * \nu\|_\omega \leq \|\mu\|_\omega \|\nu\|_\omega$. Let $A \in \mathfrak{S}(G)$. Since ω is continuous on G and $\mu * \nu \in \mathcal{M}(G)$, ω is locally $\mu * \nu$ -summable and $\omega(\mu * \nu) \in \mathcal{M}(G)$. Hence, $\omega\chi_A \in \mathcal{L}^1(\mu * \nu) = \mathcal{L}^1(p_*(\mu \times \nu)_e)$. Therefore, II.10.2 gives $(\omega\chi_A) \circ p \in \mathcal{L}^1((\mu \times \nu)_e)$ and

$$\begin{aligned} \omega(\mu * \nu)(A) &= \int \omega\chi_A \, d(p_*((\mu \times \nu)_e)) = \int (\omega\chi_A) \circ p \, d(\mu \times \nu)_e \\ &= \int \omega \circ p \chi_{p^{-1}(A)} \, d(\mu \times \nu)_e \leq \int (\omega \times \omega)\chi_{p^{-1}(A)} \, d(\mu \times \nu)_e, \end{aligned}$$

where $(\omega \times \omega)(s, t) = \omega(s)\omega(t)$. By II.9.9 and II.9.3, $(\omega \times \omega)(\mu \times \nu) = \omega\mu \times \omega\nu$, which belongs to $\mathcal{M}_r(G \times G)$ by II.9.14. Observe that $\omega \times \omega$ is locally $(\mu \times \nu)_e$ -summable, by Lemma 1.4, and $p^{-1}(A) \in \mathcal{E}_{\mu \times \nu}$, since $\mu * \nu$ exists. Hence, the above inequality and Lemma 1.4 yield

$$\begin{aligned} \omega(\mu * \nu)(A) &\leq (\omega \times \omega)(\mu \times \nu)_e(p^{-1}(A)) = ((\omega \times \omega)(\mu \times \nu))_e(p^{-1}(A)) \\ &= (\omega\mu \times \omega\nu)_e(p^{-1}(A)) \leq \|\omega\mu \times \omega\nu\| = \|\omega\mu\| \|\omega\nu\| = \|\mu\|_\omega \|\nu\|_\omega. \end{aligned}$$

Hence, $\omega(\mu * \nu)$ is bounded, i.e., $\mu * \nu \in \mathcal{M}(G, \omega)$, and $\|\mu * \nu\|_\omega = \|\omega(\mu * \nu)\| \leq \|\mu\|_\omega \|\nu\|_\omega$.

Assume now that μ, ν are any two measures in $\mathcal{M}(G, \omega)$. As we have noted, $\sigma \in \mathcal{M}(G, \omega)$ exactly when $|\sigma| \in \mathcal{M}(G, \omega)$ and $\|\sigma\|_\omega = \| |\sigma| \|_\omega$, so it follows from III.10.3 and the positive case that $\mu * \nu$ exists and $|\mu * \nu| \leq |\mu| * |\nu|$. Hence, $\omega|\mu * \nu| \leq \omega|\mu| * \omega|\nu|$, so $\mu * \nu \in \mathcal{M}(G, \omega)$ and

$$\|\mu * \nu\|_\omega = \|\omega|\mu * \nu|\| \leq \|\omega|\mu| * \omega|\nu|\| = \| |\mu| * |\nu| \|_\omega \leq \| |\mu| \|_\omega \| |\nu| \|_\omega = \|\mu\|_\omega \|\nu\|_\omega.$$

Associativity of convolution in $\mathcal{M}(G, \omega)$ is now an immediate consequence of III.10.10. Since any $\psi \in C_0(G, \omega^{-1})$ vanishes off a σ -compact subset of G and any $\mu, \nu \in \mathcal{M}(G, \omega)$ are σ -bounded — since $\omega\mu$ and $\omega\nu$ are so, and $\omega\mu \sim \mu$, $\omega\nu \sim \nu$ — Remark III.10.8 applies to give (9). \square

Let $\lambda = \lambda_G$ be a fixed left Haar measure on G , $\mathcal{L}^1(G) = \mathcal{L}^1(\lambda)$. Then $\lambda \in \mathcal{M}(G)$ (Section III.7), so $\omega\lambda \in \mathcal{M}(G)$ as well and, since $\omega > 0$, $\omega\lambda \sim \lambda$, from which it follows that g is locally $\omega\lambda$ -measurable and vanishes off a σ -compact set if and only if $g\omega$ is locally λ -measurable and vanishes off a σ -compact set. Hence, if we define $\mathcal{L}^1(G, \omega) := \mathcal{L}^1(\omega\lambda)$, $g \in \mathcal{L}^1(G, \omega)$ exactly when $g\omega \in \mathcal{L}^1(G)$, and in this case $\int g d(\omega\lambda) = \int g\omega d\lambda$, by II.7.5. Thus,

$$\mathcal{L}^1(G, \omega) = \{g: g\omega \in \mathcal{L}^1(G)\} \quad \text{and} \quad \|g\|_\omega := \|g\|_{\mathcal{L}^1(\omega\lambda)} = \|g\omega\|_1$$

defines a Banach space norm on $\mathcal{L}^1(G, \omega)$. Moreover, $T: \mathcal{L}^1(G, \omega) \rightarrow \mathcal{L}^1(G): g \mapsto g\omega$ is an isometric linear isomorphism, with inverse $f \mapsto (1/\omega)f$, so $T^*: \mathcal{L}^\infty(G) = \mathcal{L}^1(G)^* \rightarrow \mathcal{L}^1(G, \omega)^*$ is a weak*-continuous isometric isomorphism given by $\langle T^*\phi, g \rangle_\omega = \langle \phi, \omega g \rangle = \int (\phi\omega)g d\lambda$. Letting

$$\mathcal{L}^\infty(G, \omega^{-1}) := \{\phi\omega: \phi \in \mathcal{L}^\infty(G)\} = \left\{ \psi: \frac{\psi}{\omega} \in \mathcal{L}^\infty(G) \right\},$$

where $\|\psi\|_{\infty, \omega^{-1}} := \left\| \frac{\psi}{\omega} \right\|_\infty$,

we can hence identify $\mathcal{L}^1(G, \omega)^*$ with $\mathcal{L}^\infty(G, \omega^{-1})$ via the pairing $\langle \psi, g \rangle_\omega = \int \psi g d\lambda$. Observe that $S = (T^*)^{-1}: \mathcal{L}^\infty(G, \omega^{-1}) \rightarrow \mathcal{L}^\infty(G): \psi \mapsto \psi/\omega$ is a weak*-homeomorphic isometric isomorphism. We note that $\mathcal{L}^\infty(G, \omega^{-1})$ is not usually the same space as $\mathcal{L}^\infty(\omega\lambda) (= \mathcal{L}^\infty(\lambda))$ because $\omega\lambda \sim \lambda$, which can also be identified with $\mathcal{L}^1(\omega\lambda)^* = \mathcal{L}^1(G, \omega)^*$ in the usual way by II.7.11. Note that because T^{-1} maps $C_{00}(G)$ onto itself, $C_{00}(G)$ is dense in $\mathcal{L}^1(G, \omega)$.

Let $g \in \mathcal{L}^1(G, \omega) = \mathcal{L}^1(\omega\lambda)$, $A \in \mathfrak{S}(G)$. Then $\omega g \in \mathcal{L}^1(\lambda)$ and $1/\omega$ is bounded on A , so $\chi_A g = ((1/\omega)\chi_A)\omega g \in \mathcal{L}^1(\lambda)$; hence, $g\lambda \in \mathcal{M}(G)$ is well-defined (II.7.2). Also, $\omega(g\lambda) = (\omega g)\lambda \in \mathcal{M}(G)$ by II.7.5 and, by II.7.9/III.11.3, $\|f\|_1 = \|f\lambda\|$ for $f \in \mathcal{L}^1(G)$ and

$$\mathcal{M}_a(G) = \{\mu \in \mathcal{M}_r(G): \mu \ll \lambda\} = \{f\lambda: f \in \mathcal{L}^1(G)\} = \{(\omega g)\lambda: g \in \mathcal{L}^1(G, \omega)\}.$$

Since $\omega\nu \sim \nu$ for any $\nu \in \mathcal{M}(G)$, it readily follows that $g \mapsto g\lambda: \mathcal{L}^1(G, \omega) \rightarrow \mathcal{M}_a(G, \omega)$ is a surjective linear isometry, where $\mathcal{M}_a(G, \omega) := \{\nu \in \mathcal{M}(G, \omega): \nu \ll \lambda\}$. We can thus identify $\mathcal{L}^1(G, \omega)$ with $\mathcal{M}_a(G, \omega)$ via $g \mapsto g\lambda$.

Proposition 1.6. *The Banach space $\mathcal{L}^1(G, \omega) = \mathcal{M}_a(G, \omega)$ is a closed ideal in $\mathcal{M}(G, \omega)$ and has a contractive approximate identity. Moreover, if $g \in \mathcal{L}^1(G, \omega)$ and $\nu \in \mathcal{M}(G, \omega)$, then $\nu * g, g * \nu \in \mathcal{L}^1(G, \omega)$ are given by the formulas, which hold for locally λ -almost all $t \in G$,*

$$(10) \quad \nu * g(t) = \int g(s^{-1}t) d\nu(s) \quad \text{and} \quad g * \nu(t) = \int \Delta(s^{-1})g(ts^{-1}) d\nu(s);$$

thus, $\mathcal{L}^1(G, \omega)$ is a Banach algebra with respect to the convolution product

$$(11) \quad f * g(t) = \int f(s)g(s^{-1}t) \, d\lambda(s).$$

PROOF: We have already noted that g is locally λ -summable and vanishes off a σ -compact set, and $(g\lambda) * \nu, \nu * (g\lambda)$ exist in $\mathcal{M}(G, \omega)$ by Theorem 1.5. Letting $h(t)$ and $k(t)$ be defined by the respective integral formulas on the left and right of (10), $\nu * (g\lambda) = h\lambda$ and $(g\lambda) * \nu = k\lambda$ by III.11.5. Thus, $h\lambda, k\lambda \in \mathcal{M}_a(G, \omega) = \{f\lambda: f \in \mathcal{L}^1(G, \omega)\}$, so the uniqueness part of the Radon–Nikodym theorem — see Remark 1 of II.7.8 — implies that $h, k \in \mathcal{L}^1(G, \omega)$. The formula (11) now follows quickly (or directly from III.11.6). Let \mathcal{I} be the neighbourhood system at e_G and for each $\alpha \in \mathcal{I}$, let $f_\alpha \in C_{00}(G)$ be chosen with $f_\alpha \geq 0$, $\|f_\alpha\|_1 = 1$ and support contained in α . Then $(f_\alpha)_\alpha$ is a bounded approximate identity for $\mathcal{L}^1(G)$. Letting $e_\alpha = \omega^{-1}f_\alpha$, $\|e_\alpha\|_\omega = 1$ and $\|e_\alpha\|_1 \rightarrow 1$, from which it easily follows that $(e_\alpha)_\alpha$ is also a bounded approximate identity for $\mathcal{L}^1(G)$; the proof of Lemma 2.1 in [6] now shows that $(e_\alpha)_\alpha$ is a contractive approximate identity for $\mathcal{L}^1(G, \omega)$. \square

Remark 1.7. Every Borel measurable function is locally λ -measurable and every $f \in L^1(G, \omega)$ — where $L^1(G, \omega)$ is defined in the usual sense (as in the introduction) — vanishes off a σ -compact set. It follows that the Banach algebra $\mathcal{L}^1(G, \omega)$, as we have defined it, exactly coincides with the usual definition of the Beurling group algebra $L^1(G, \omega)$, which, as noted in the introduction, is always valid. Going forward, we can therefore use any known result about $L^1(G, \omega) = \mathcal{L}^1(G, \omega)$ that was proved independently of $M(G, \omega)$.

2. The dual Banach algebra $\mathcal{M}(G, \omega)$ and the embedding map

The support of μ in $\mathcal{M}(G)$ is the set $s(\mu) = G \setminus \bigcup \{U \in \mathfrak{S}(G): U \text{ is open and } |\mu|(U) = 0\}$, see II.8.9. Let $\mathcal{M}_{cr}(G) = \{\mu \in \mathcal{M}(G): s(\mu) \text{ is compact}\}$.

Remark 2.1. 1. Observe that $s(\mu) = s(\mu_e) = G \setminus \bigcup \{V \in \mathcal{E}_\mu: V \text{ is open and } |\mu_e|(V) = 0\}$.

2. Since ω and $1/\omega$ are bounded on any set A in $\mathfrak{S}(G)$, $s(\mu) = s(\omega\mu) = s((1/\omega)\mu)$ for any $\mu \in \mathcal{M}(G)$.

3. By III.10.16, $\mathcal{M}_{cr}(G)$ is a dense subalgebra of $\mathcal{M}_r(G)$. From 2 above, the inverse linear isometries $\nu \mapsto \omega\nu$ and $\mu \mapsto (1/\omega)\mu$ between $\mathcal{M}(G, \omega)$ and $\mathcal{M}_r(G)$ map $\mathcal{M}_{cr}(G)$ onto itself, so $\mathcal{M}_{cr}(G)$ is also a dense subalgebra of $\mathcal{M}(G, \omega)$.

A measure σ on a δ -ring \mathfrak{S} is *concentrated* on a set F if for each $A \in \mathfrak{S}$, $A \cap F, A \setminus F \in \mathfrak{S}$ and $\sigma(A) = \sigma(A \cap F)$ or, equivalently, $\sigma(A \setminus F) = 0$. For $\mu \in \mathcal{M}(G)$ and a Borel set F , $A \cap F, A \setminus F \in \mathfrak{S}(G)$ ($A \cap F, A \setminus F \in \mathcal{E}_\mu$, respectively)

is automatic for any $A \in \mathfrak{S}(G)$ ($A \in \mathcal{E}_\mu$), and it is clear from (2) that μ is concentrated on F if and only if μ_e is concentrated on F . A function $\psi \in LUC(G, \omega^{-1})$ may fail to vanish off a σ -compact set and therefore, as noted in Remark 1.2, in this theory we cannot integrate ψ with respect to any μ in $\mathcal{M}(G)$. Lemma 2.2 allows us to move past this issue.

Lemma 2.2. (a) *Every μ in $\mathcal{M}(G)$ is concentrated on its support, $s(\mu)$.*

(b) *Let $\mu \in \mathcal{M}_r(G)$. Then μ (and therefore μ_e) is concentrated on a σ -compact subset F of G and, for any such F and any Borel measurable function $f \in \mathcal{L}^1(\mu_e)$, $f\chi_F \in \mathcal{L}^1(\mu) \cap \mathcal{L}^1(\mu_e)$ and*

$$\int f \, d\mu_e = \int f\chi_F \, d\mu_e = \int f\chi_F \, d\mu.$$

(c) *Any $\nu \in \mathcal{M}(G, \omega)$ is concentrated on a σ -compact set.*

PROOF: (a) Let $A \in \mathfrak{S}(G)$. Any compact subset of $A \setminus s(\mu)$ is covered by the collection of open sets $U \in \mathfrak{S}(G)$ with $|\mu|(U) = 0$, and is therefore $|\mu|$ -null; by regularity of μ (II.8.2(II)), $|\mu|(A \setminus s(\mu)) = 0$.

(b) Take $(C_n)_n$ to be an increasing sequence of compact subsets of $s(\mu)$ such that $|\mu|(C_n) > \|\mu\| - 1/n$ and let $F = \bigcup C_n$, where we have used (b). Then μ is concentrated on F because for $A \in \mathfrak{S}(G)$,

$$\begin{aligned} |\mu|(A \setminus F) &= |\mu|((A \setminus F) \cap s(\mu)) \leq |\mu|_e(s(\mu) \setminus F) = |\mu|_e(s(\mu)) - |\mu|_e(F) \\ &= \|\mu\| - \lim |\mu|(C_n) = 0. \end{aligned}$$

Suppose $\mu \geq 0$, F is any σ -compact set on which μ is concentrated, and $f \in \mathcal{L}^1(\mu_e)$ is a nonnegative Borel-measurable function. It is then clear (from II.2.2 and II.2.5) that $f\chi_F \in \mathcal{L}^1(\mu_e)$ and $\int f \, d\mu_e = \int f\chi_F \, d\mu_e$. Also, $f\chi_F$ is locally μ -measurable (II.8.2), vanishes off the σ -compact set F and, taking any sequence of nonnegative $\mathfrak{S}(G)$ -simple functions such that $h_n \uparrow f\chi_F$, II.2.2 gives

$$\int f\chi_f \, d\mu_e = \lim \int h_n \, d\mu_e = \lim \int h_n \, d\mu = \int f\chi_F \, d\mu.$$

(c) Since $\omega\nu \in \mathcal{M}_r(G)$ and $\nu \sim \omega\nu$, this follows from (b). □

Since $\mathcal{L}^1(G, \omega)$ is a closed ideal in $\mathcal{M}(G, \omega)$, through the operations

$$\begin{aligned} \langle \psi \cdot \nu, g \rangle &= \langle \psi, \nu * g \rangle \quad \text{and} \quad \langle \nu \cdot \psi, g \rangle = \langle \psi, g * \nu \rangle \\ &\text{for } \psi \in \mathcal{L}^\infty(G, \omega^{-1}), \nu \in \mathcal{M}(G, \omega), g \in \mathcal{L}^1(G, \omega), \end{aligned}$$

$\mathcal{L}^\infty(G, \omega^{-1}) = \mathcal{L}^1(G, \omega)^*$ is a dual $\mathcal{M}(G, \omega)$ -module. Observe that for $\psi \in \mathcal{L}^\infty(G, \omega^{-1})$ and $s \in G$,

$$\psi \cdot \delta_s(t) = \psi \cdot s(t) := \psi(st) \quad \text{and} \quad \delta_s \cdot \psi(t) = s \cdot \psi(t) := \psi(ts), \quad t \in G.$$

Recall that ψ belongs to $LUC(G, \omega^{-1})$ [$RUC(G, \omega^{-1})$] when ψ/ω belongs to $LUC(G)$ [$RUC(G)$]. For $LUC(G, \omega^{-1})$, the following is [8, Proposition 1.3] and [3, Propositions 7.15 and 7.17], (where no restrictions are needed on the weight ω); symmetric arguments establish the $RUC(G, \omega^{-1})$ case.

Lemma 2.3. *The following statements are equivalent:*

- (a) $\psi \in LUC(G, \omega^{-1})$ [$RUC(G, \omega^{-1})$];
- (b) $\psi \in l^\infty(G, \omega^{-1})$ and the map $G \rightarrow (l^\infty(G, \omega^{-1}), \|\cdot\|_{\infty, \omega^{-1}}) : s \mapsto \psi \cdot s$ [$s \cdot \psi$] is continuous;
- (c) $\psi \in \mathcal{L}^\infty(G, \omega^{-1})$ and the map $G \rightarrow (\mathcal{L}^\infty(G, \omega^{-1}), \|\cdot\|_{\infty, \omega^{-1}}) : s \mapsto \psi \cdot s$ [$s \cdot \psi$] is continuous;
- (d) $\psi \in \mathcal{L}^\infty(G, \omega^{-1}) \cdot \mathcal{L}^1(G, \omega)$ [$\psi \in \mathcal{L}^1(G, \omega) \cdot \mathcal{L}^\infty(G, \omega^{-1})$].

Remark 2.4. 1. Observe that condition (b) implies ψ is continuous on G , whence $\psi \in \mathcal{L}^\infty(G, \omega^{-1})$.

2. In the proof of [3, Proposition 7.15], the authors establish continuity of a function ψ satisfying (c) via Ascoli’s theorem. An alternative approach is to establish (i) and (ii) as follows:

(i) If $\phi \in \mathcal{L}^\infty(G, \omega^{-1})$ and $g \in \mathcal{L}^1(G, \omega)$, then $\phi \cdot g$ can be identified with the continuous function

$$(12) \quad (\phi \cdot g)(t) = \langle \phi, g * \delta_t \rangle \quad \text{for every } t \in G.$$

[Note that $H \in l^\infty(G, \omega^{-1})$ where $H(t) := \langle \phi, g * \delta_t \rangle$ and, since $t \mapsto g * \delta_t : G \rightarrow (\mathcal{L}^1(G, \omega), \|\cdot\|_\omega)$ is continuous — e.g., see [19, Lemma 3.1.5], which holds for any weight ω — H is continuous on G (and satisfies Lemma 2.3 (c)); in a standard way, one can check that for $f \in \mathcal{L}^1(G, \omega)$, $\langle \phi \cdot g, f \rangle = \langle H, f \rangle$.]

(ii) If ψ satisfies (c) and (e_i) is a bounded approximate identity for $\mathcal{L}^1(G, \omega)$, then $\|\psi \cdot e_i - \psi\|_{\infty, \omega^{-1}} \rightarrow 0$; since $CB(G, \omega^{-1})$ is closed in $\mathcal{L}^\infty(G, \omega^{-1})$, $\psi \in CB(G, \omega^{-1})$.

Proposition 2.5. *The spaces $LUC(G, \omega^{-1})$ and $RUC(G, \omega^{-1})$ are $\mathcal{M}(G, \omega)$ -submodules of $\mathcal{L}^\infty(G, \omega^{-1})$. Moreover, for $\nu \in \mathcal{M}(G, \omega)$, $\psi \in LUC(G, \omega^{-1})$*

$[\psi \in RUC(G, \omega^{-1})]$ and for every $s \in G$,

$$\begin{aligned} (\nu \cdot \psi)(s) &= \int (\psi \cdot s) \chi_{F_s} d\nu = \int \frac{\psi \cdot s}{\omega} \chi_{F_s} d(\omega\nu) = \int \frac{\psi \cdot s}{\omega} d(\omega\nu)_e \\ \left[(\psi \cdot \nu)(s) &= \int (s \cdot \psi) \chi_{F_s} d\nu = \int \frac{s \cdot \psi}{\omega} \chi_{F_s} d(\omega\nu) = \int \frac{s \cdot \psi}{\omega} d(\omega\nu)_e \right], \end{aligned}$$

where F_s is any σ -compact set on which ν is concentrated; F_s can be chosen to vary with $s \in G$.

PROOF: Letting $\nu \in \mathcal{M}(G, \omega)$, $\psi \in LUC(G, \omega^{-1})$, it is clear from Lemma 2.3 (d) that $\psi \cdot \nu, \nu \cdot \psi \in LUC(G, \omega^{-1})$. Since $\psi \cdot s/\omega \in LUC(G)$ and $\omega\nu \in \mathcal{M}_r(G)$,

$$H(s) = H_{\nu, \psi}(s) := \int \frac{\psi \cdot s}{\omega} d(\omega\nu)_e = \int \frac{\psi \cdot s}{\omega} \chi_{F_s} d(\omega\nu)$$

is well-defined, where we have used Lemma 2.2. The function $(\psi \cdot s) \chi_{F_s} \in l^\infty(G, \omega^{-1})$ is Borel measurable — and therefore locally ν -measurable — and vanishes off the σ -compact set F_s , so $(\psi \cdot s/\omega) \chi_{F_s} \in \mathcal{L}^1(\omega\nu)$. Therefore, by II.7.5, $(\psi \cdot s) \chi_{F_s} \in \mathcal{L}^1(\nu)$ and

$$\int (\psi \cdot s) \chi_{F_s} d\nu = \int \frac{\psi \cdot s}{\omega} \chi_{F_s} \omega d\nu = \int \frac{\psi \cdot s}{\omega} \chi_{F_s} d(\omega\nu) = H(s).$$

Since $|H(s)| \leq \|\psi \cdot s/\omega\|_\infty \|\omega\nu\| \leq \omega(s) \|\psi\|_{\infty, \omega^{-1}} \|\nu\|_\omega$, $H = H_{\nu, \psi} \in l^\infty(G, \omega^{-1})$ with $\|H_{\nu, \psi}\|_{\infty, \omega^{-1}} \leq \|\psi\|_{\infty, \omega^{-1}} \|\nu\|_\omega$. Hence, if $s_i \rightarrow s$ in G ,

$$\|(H_{\nu, \psi}) \cdot s_i - (H_{\nu, \psi}) \cdot s\|_{\infty, \omega^{-1}} = \|H_{\nu, \psi \cdot s_i - \psi \cdot s}\|_{\infty, \omega^{-1}} \leq \|\psi \cdot s_i - \psi \cdot s\|_{\infty, \omega^{-1}} \|\nu\|_\omega \rightarrow 0;$$

by Lemma 2.3, $H_{\nu, \psi} \in LUC(G, \omega^{-1})$. To show that $H_{\nu, \psi} = \nu \cdot \psi$, we can assume $\nu \geq 0$, $\psi \geq 0$ and take $F = F_s$ for each $s \in G$. Let $g \geq 0$ be a function in the dense subspace $C_{00}(G)$ of $\mathcal{L}^1(G, \omega)$. Since the maps $(s, t) \mapsto \psi(t) \Delta(s^{-1}) g(ts^{-1}) \chi_F(s)$, $\psi(ts) g(t) \chi_F(s)$ are Borel measurable — hence locally $(\nu \times \lambda)$ -measurable — and vanish off a σ -compact subset of $G \times G$, our applications of the Fubini theorem (II.9.8) are valid in the following calculation. Using (10):

$$\begin{aligned} \langle \nu \cdot \psi, g \rangle &= \langle \psi, g * \nu \rangle = \int \psi(t) \int \Delta(s^{-1}) g(ts^{-1}) d\nu(s) d\lambda(t) \\ &= \iint \psi(t) \Delta(s^{-1}) g(ts^{-1}) \chi_F(s) d\nu(s) d\lambda(t) \\ &= \iint \psi(t) \Delta(s^{-1}) g(ts^{-1}) \chi_F(s) d\lambda(t) d\nu(s) \end{aligned}$$

$$\begin{aligned}
 &= \iint \psi(ts)g(t)\chi_F(s) \, d\lambda(t) \, d\nu(s) \\
 &= \iint \psi \cdot t(s)\chi_F(s) \, d\nu(s) \, g(t) \, d\lambda(t) = \langle H_{\nu,\lambda}, g \rangle;
 \end{aligned}$$

since both functions are continuous, $\nu \cdot \lambda = H_{\nu,\lambda}$. □

Corollary 2.6. *The space $C_0(G, \omega^{-1})$ is a $\mathcal{M}(G, \omega)$ -submodule of $\mathcal{L}^\infty(G, \omega^{-1})$, and for $\nu \in \mathcal{M}(G, \omega)$, $\psi \in C_0(G, \omega^{-1})$ and $s \in G$,*

$$(13) \quad \nu \cdot \psi(s) = \int \psi \cdot s \, d\nu = \langle \nu, \psi \cdot s \rangle_\omega \quad \text{and} \quad \psi \cdot \nu(s) = \int s \cdot \psi \, d\nu = \langle \nu, s \cdot \psi \rangle_\omega.$$

PROOF: Let $\psi \in C_0(G, \omega^{-1})$ and let F be a σ -compact set on which ν is concentrated. Taking A_s to be a σ -compact set off which $\psi \cdot s$ and $s \cdot \psi$ vanish, and putting $F_s = F \cup A_s$, Proposition 2.5 gives $\nu \cdot \psi, \psi \cdot \nu \in (LUC \cap RUC)(G, \omega^{-1})$ and

$$\nu \cdot \psi(s) = \int (\psi \cdot s)\chi_{F_s} \, d\nu = \int \psi \cdot s \, d\nu \quad \text{and} \quad \psi \cdot \nu(s) = \int (s \cdot \psi)\chi_{F_s} \, d\nu = \int s \cdot \psi \, d\nu.$$

Observe that $\nu \cdot \psi$ is supported on $s(\psi)s(\nu)^{-1}$, which is compact when ν belongs to the dense subspace $\mathcal{M}_{cr}(G)$ of $\mathcal{M}(G, \omega)$ and ψ belongs to the dense subspace $C_{00}(G)$ of $C_0(G, \omega^{-1})$. It follows that $C_0(G, \omega^{-1})$ is a left (and similarly, right) $\mathcal{M}(G, \omega)$ -submodule of $\mathcal{L}^\infty(G, \omega^{-1})$. □

It follows that $\mathcal{M}(G, \omega) = C_0(G, \omega^{-1})^*$ is a dual $\mathcal{M}(G, \omega)$ -module with respect to the operations

$$\begin{aligned}
 \langle \mu \cdot_r \nu, \psi \rangle_\omega &= \langle \mu, \nu \cdot \psi \rangle_\omega \quad \text{and} \quad \langle \mu \cdot_l \nu, \psi \rangle_\omega = \langle \nu, \psi \cdot \mu \rangle_\omega, & \mu, \nu \in \mathcal{M}(G, \omega), \\
 & & \psi \in C_0(G, \omega^{-1}).
 \end{aligned}$$

However, from (9) and (13),

$$(14) \quad \mu \cdot_r \nu = \mu * \nu = \mu \cdot_l \nu,$$

so $(\mu, \nu) \mapsto \mu * \nu$ is separately weak*-continuous on $\mathcal{M}(G, \omega)$. Hence:

Corollary 2.7. *The Beurling measure algebra $\mathcal{M}(G, \omega)$ is a dual Banach algebra.*

Let A be a Banach algebra. Recall that a closed submodule $\mathcal{S}(A^*)$ of the dual A -bimodule A^* is left [right] introverted if for each $\mu \in \mathcal{S}(A^*)^*$ and $\phi \in \mathcal{S}(A^*)$, $\mu \square \phi \in \mathcal{S}(A^*)$ [$\phi \diamond \mu \in \mathcal{S}(A^*)$] where $\mu \square \phi, \phi \diamond \mu \in A^*$ are defined by

$$\langle \mu \square \phi, a \rangle_{A^*-A} = \langle \mu, \phi \cdot a \rangle_{\mathcal{S}^*-\mathcal{S}} \quad \text{and} \quad \langle \phi \diamond \mu, a \rangle_{A^*-A} = \langle \mu, a \cdot \phi \rangle_{\mathcal{S}^*-\mathcal{S}};$$

in this case, $\mathcal{S}(A^*)^*$ is a Banach algebra with respect to its left [right] Arens product

$$\langle \mu \square \nu, \phi \rangle = \langle \mu, \nu \square \phi \rangle \quad [\langle \mu \diamond \nu, \phi \rangle = \langle \nu, \phi \diamond \mu \rangle], \quad \mu, \nu \in \mathcal{S}(A^*)^*, \phi \in \mathcal{S}(A^*).$$

The map $\eta_S: A \rightarrow \mathcal{S}(A^*)^*$ defined by $\langle \eta_S(a), \phi \rangle = \langle \phi, a \rangle$ is a bounded homomorphism with weak*-dense range and, when A is left introverted, η_S maps into the topological centre of $(\mathcal{S}(A^*)^*, \square)$, $Z_t(\mathcal{S}(A^*)^*) = \{ \mu \in \mathcal{S}(A^*)^*: \nu \mapsto \mu \square \nu \text{ is } \text{wk}^* - \text{wk}^* \text{ continuous on } \mathcal{S}(A^*)^* \}$. For this see, e.g., [3].

Proposition 2.8. *The subspace $C_0(G, \omega^{-1})$ of $\mathcal{L}^\infty(G, \omega^{-1}) = \mathcal{L}^1(G, \omega)^*$ is left and right introverted and $\mu * \nu = \mu \square \nu = \mu \diamond \nu$ for $\mu, \nu \in \mathcal{M}(G, \omega) = C_0(G, \omega^{-1})^*$.*

PROOF: By Corollary 2.6, $C_0(G, \omega^{-1})$ is a $\mathcal{L}^1(G, \omega)$ -submodule of $\mathcal{L}^\infty(G, \omega^{-1})$. Let $\mu, \nu \in \mathcal{M}(G, \omega)$, $\psi \in C_0(G, \omega^{-1})$. For $g \in \mathcal{L}^1(G, \omega)$, equation (14) gives

$$\langle \nu \square \psi, g \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} = \langle \nu, \psi \cdot g \rangle_\omega = \langle g * \nu, \psi \rangle_\omega = \langle g, \nu \cdot \psi \rangle_\omega = \langle \nu \cdot \psi, g \rangle_{\mathcal{L}^\infty - \mathcal{L}^1}.$$

Hence, $C_0(G, \omega)$ is left introverted and $\langle \mu \square \nu, \psi \rangle = \langle \mu, \nu \square \psi \rangle = \langle \mu, \nu \cdot \psi \rangle = \langle \mu * \nu, \psi \rangle$, where we have again used (14). Similarly, $C_0(G, \omega^{-1})$ is right introverted and $\mu * \nu = \mu \diamond \nu$. □

Let $\mathcal{S}(\omega^{-1})$ be a left introverted subspace of $\mathcal{L}^\infty(G, \omega^{-1})$ such that

$$C_0(G, \omega^{-1}) \preceq \mathcal{S}(\omega^{-1}) \preceq LUC(G, \omega^{-1})$$

and define

$$(15) \quad \Theta: \mathcal{M}(G, \omega) \rightarrow \mathcal{S}(\omega^{-1})^* \quad \text{by} \quad \langle \Theta(\nu), \psi \rangle_{\mathcal{S}^* - \mathcal{S}} = (\nu \cdot \psi)(e_G) = \int \psi \chi_{F_\nu} d\nu,$$

where $\nu \in \mathcal{M}(G, \omega)$, $\psi \in \mathcal{S}(\omega^{-1})$ and F_ν is any σ -compact set on which ν is concentrated. By Proposition 2.5, Θ is well-defined and $|\langle \Theta(\nu), \psi \rangle| \leq \|\nu \cdot \psi\|_{\infty, \omega^{-1}} \leq \|\nu\|_\omega \|\psi\|_{\infty, \omega^{-1}}$, so $\|\Theta(\nu)\| \leq \|\nu\|_\omega$; by equation (13), $\Theta(\nu)|_{C_0(G, \omega^{-1})} = \nu$, so $\|\Theta(\nu)\| = \|\nu\|_\omega$. Thus, Θ is a linear isometry.

Let so_l and so_r denote the left and right strict topologies on $\mathcal{M}(G, \omega)$ taken with respect to the ideal $\mathcal{L}^1(G, \omega)$, i.e., the locally convex topologies respectively generated by the semi-norms $p_g(\nu) = \|g * \nu\|$ and $q_g(\nu) = \|\nu * g\|$ for $g \in \mathcal{L}^1(G, \omega)$, $\nu \in \mathcal{M}(G, \omega)$. Since $\mathcal{L}^1(G, \omega)$ has a contractive approximate identity, (the unit ball of) $\mathcal{L}^1(G, \omega)$ is so_l/so_r -dense in (the unit ball of) $\mathcal{M}(G, \omega)$. Observe that when $\mathcal{S}(\omega^{-1}) \preceq LUC(G, \omega^{-1})$ is a $\mathcal{L}^1(G, \omega)$ -submodule of $\mathcal{L}^\infty(G, \omega^{-1})$, by Lemma 2.3 (d) and the Cohen factorization theorem [1, Theorem 11.10], $\mathcal{S}(\omega^{-1}) = \mathcal{S}(\omega^{-1}) \cdot \mathcal{L}^1(G, \omega)$. Also note that $LUC(G, \omega^{-1})$ is always left introverted in $\mathcal{L}^\infty(G, \omega^{-1})$ by Lemma 2.3 and [3, Proposition 5.9]. In the non-weighted case and when $\omega \geq 1$, the final statement in Proposition 2.9, which simplifies Arens product calculations, is [13, Lemma 3] and [3, Proposition 7.21], respectively.

Proposition 2.9. *Suppose that $\mathcal{S}(\omega^{-1})$ is a left [right] introverted subspace of $\mathcal{L}^\infty(G, \omega^{-1}) = \mathcal{L}^1(G, \omega)^*$ and*

$$C_0(G, \omega^{-1}) \preceq \mathcal{S}(\omega^{-1}) \preceq LUC(G, \omega^{-1}) [RUC(G, \omega^{-1})].$$

Then $\Theta: \mathcal{M}(G, \omega) \hookrightarrow \mathcal{S}(\omega^{-1})^*$ is a so_l -weak* [so_r -weak*] continuous isometric homomorphic embedding into $Z_t(\mathcal{S}(\omega^{-1})^*)$ that extends $\eta_{\mathcal{S}}: \mathcal{L}^1(G, \omega) \rightarrow \mathcal{S}(\omega^{-1})^*$. Moreover, $(n \square \psi)(s) = \langle n, \psi \cdot s \rangle$ for any $n \in \mathcal{S}(\omega^{-1})^*$, $\psi \in \mathcal{S}(\omega^{-1})$ and $s \in G$; hence, $\mathcal{S}(\omega^{-1})$ is introverted as a subspace of $l^\infty(G, \omega^{-1}) = l^1(G, \omega)^*$, the Arens product on $\mathcal{S}(\omega^{-1})^*$ agrees under either interpretation, and Θ also extends $\eta_{\mathcal{S}}: l^1(G, \omega) \hookrightarrow \mathcal{S}(\omega^{-1})^*$.

PROOF: If $g \in \mathcal{L}^1(G, \omega) = \mathcal{L}^1(\omega\lambda)$, g vanishes off a σ -compact set F_g , and therefore $g = g\lambda \in \mathcal{M}(G, \omega)$ is concentrated on F_g ; hence, for $\psi \in \mathcal{S}(\omega^{-1})$,

$$\langle \Theta(g), \psi \rangle_{\mathcal{S}^* - \mathcal{S}} = \int \psi \chi_{F_g} d(g\lambda) = \int \psi g d\lambda = \langle \psi, g \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} = \langle \eta_{\mathcal{S}}(g), \psi \rangle_{\mathcal{S}^* - \mathcal{S}}.$$

For $f \in \mathcal{L}^1(G, \omega)$, $\nu \in \mathcal{M}(G, \omega)$ and $\psi \in \mathcal{S}(\omega^{-1})$,

$$\begin{aligned} \langle \Theta(f) \square \Theta(\nu), \psi \rangle_{\mathcal{S}^* - \mathcal{S}} &= \langle \eta_{\mathcal{S}}(f), \Theta(\nu) \square \psi \rangle_{\mathcal{S}^* - \mathcal{S}} = \langle \Theta(\nu) \square \psi, f \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} \\ &= \langle \Theta(\nu), \psi \cdot f \rangle_{\mathcal{S}^* - \mathcal{S}} = \nu \cdot (\psi \cdot f)(e_G) = (\nu \cdot \psi) \cdot f(e_G) \\ &= \langle \nu \cdot \psi, f * \delta_{e_G} \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} = \langle \psi, f * \nu \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} \\ &= \langle \eta_{\mathcal{S}}(f * \nu), \psi \rangle_{\mathcal{S}^* - \mathcal{S}} = \langle \Theta(f * \nu), \psi \rangle_{\mathcal{S}^* - \mathcal{S}}, \end{aligned}$$

where we have used (12). Suppose that $\nu_i \rightarrow \nu$ with respect to so_l . Writing $\psi \in \mathcal{S}(\omega^{-1})$ as $\psi = \phi \cdot g$ for some $\phi \in \mathcal{S}(\omega^{-1})$ and $g \in \mathcal{L}^1(G, \omega)$,

$$\langle \Theta(\nu_i) - \Theta(\nu), \psi \rangle_{\mathcal{S}^* - \mathcal{S}} = \langle \Theta(g) \square \Theta(\nu_i - \nu), \phi \rangle_{\mathcal{S}^* - \mathcal{S}} = \langle \Theta(g * (\nu_i - \nu)), \phi \rangle_{\mathcal{S}^* - \mathcal{S}} \rightarrow 0.$$

Hence, Θ is so_l -weak* continuous. Let $\mu, \nu \in \mathcal{M}(G, \omega)$ and let (h_i) be a net in $\mathcal{L}^1(G, \omega)$ such that $so_l - \lim h_i = \mu$. Then $so_l - \lim h_i * \nu = \mu * \nu$, so

$$\Theta(\mu) \square \Theta(\nu) = \text{wk}^* - \lim \Theta(h_i) \square \Theta(\nu) = \text{wk}^* - \lim \Theta(h_i * \nu) = \Theta(\mu * \nu).$$

Identify the Banach algebra $\mathcal{M}(G, \omega)$ with its copy $\Theta(\mathcal{M}(G, \omega))$ in $\mathcal{S}(\omega^{-1})^*$. Since $\mathcal{S}(\omega^{-1}) = \mathcal{S}(\omega^{-1}) \cdot \mathcal{L}^1(G, \omega)$ is a right $\mathcal{M}(G, \omega)$ -module, $\mathcal{S}(\omega^{-1})^*$ is a left dual $\mathcal{M}(G, \omega)$ -module, and the proof of [7, Lemma 1.4] shows that $\mu \square n = \mu \cdot n$ for $\mu \in \mathcal{M}(G, \omega)$ and $n \in \mathcal{S}(\omega^{-1})^*$; hence, Θ maps into $Z_t(\mathcal{S}(\omega^{-1})^*)$. For $n \in \mathcal{S}(\omega^{-1})^*$, $\psi \in \mathcal{S}(\omega^{-1})$ and $s \in G$, $(n \square \psi)(s) = \langle \delta_s, n \square \psi \rangle = \langle \delta_s \square n, \psi \rangle = \langle \delta_s \cdot n, \psi \rangle = \langle n, \psi \cdot \delta_s \rangle = \langle n, \psi \cdot s \rangle$. The final line is now easily verified. \square

For a Banach algebra A , the space $WAP(A^*)$ of weakly almost periodic functionals on A is a left and right introverted subspace of A^* such that for every

$m, n \in WAP(A^*)^*$, $m \square n = m \diamond n$ [3, Proposition 3.11]. Thus, $WAP(A^*)^*$ is a dual Banach algebra. Moreover, $WAP(A^*)^*$ satisfies the following universal property [16, Theorem 4.10].

Theorem 2.10 (Runde). *If \mathfrak{B} is a dual Banach algebra and $\varphi: A \rightarrow \mathfrak{B}$ is a continuous algebra homomorphism, then there is a unique weak*-weak* continuous algebra homomorphism $\varphi_{WAP}: WAP(A^*)^* \rightarrow \mathfrak{B}$ such that $\varphi_{WAP} \circ \eta_{WAP} = \varphi$.*

Taking $A_\omega = \mathcal{L}^1(G, \omega)$, it follows that the embedding $\text{id}: \mathcal{L}^1(G, \omega) \hookrightarrow \mathcal{M}(G, \omega)$ determines a unique weak*-weak* continuous homomorphism $P: WAP(A_\omega^*)^* \rightarrow \mathcal{M}(G, \omega)$ such that $P \circ \eta_{WAP} = \text{id}$. Letting $P_*: C_0(G, \omega^{-1}) \rightarrow WAP(A_\omega^*)^* \preceq \mathcal{L}^\infty(G, \omega^{-1})$ be the predual mapping of P , $\langle P_*\psi, g \rangle_{\mathcal{L}^\infty - \mathcal{L}^1} = \langle P \circ \eta_{WAP}(g), \psi \rangle = \langle \psi, g \rangle_{\mathcal{L}^\infty - \mathcal{L}^1}$ for $\psi \in C_0(G, \omega^{-1})$, $g \in \mathcal{L}^1(G, \omega)$. Hence, $C_0(G, \omega^{-1}) \preceq WAP(A_\omega^*)^*$. Moreover, by [3, Proposition 3.12] and Lemma 2.3,

$$WAP(A_\omega^*)^* \preceq (LUC \cap RUC) \times (G, \omega^{-1}).$$

Hence, we have the following immediate corollary to Proposition 2.9.

Corollary 2.11. *The map $\Theta: \mathcal{M}(G, \omega) \hookrightarrow WAP(A_\omega^*)^*$, as defined in (15), is a so_l -weak* and so_r -weak* continuous isometric homomorphic embedding that extends $\eta_{WAP}: \mathcal{L}^1(G, \omega) \hookrightarrow WAP(A_\omega^*)^*$.*

As shown in [3], $WAP(A_\omega^*)^*$ may fail to equal $WAP(G, \omega^{-1}) = \{f: f/\omega \in WAP(G)\}$. Our final two results are needed in [12]. Corollary 2.12 improves [10, Theorem 5.6] in the case of $\mathcal{L}^1(G, \omega)$:

Corollary 2.12. *Let \mathfrak{B} be a dual Banach algebra, $\varphi: \mathcal{L}^1(G, \omega) \rightarrow \mathfrak{B}$ a bounded homomorphism. Then there is a unique so_l -weak* and so_r -weak* continuous homomorphic extension $\tilde{\varphi}: \mathcal{M}(G, \omega) \rightarrow \mathfrak{B}$ of φ .*

PROOF: Letting $\varphi_{WAP}: WAP(A_\omega^*)^* \rightarrow \mathfrak{B}$ be the weak*-weak* continuous extension of φ from Theorem 2.10 and $\Theta: \mathcal{M}(G, \omega) \hookrightarrow WAP(A_\omega^*)^*$ the so_l/so_r -weak* continuous embedding from Corollary 2.11, $\tilde{\varphi} := \varphi_{WAP} \circ \Theta$ is the desired extension; uniqueness follows from the so_l -density of $\mathcal{L}^1(G, \omega)$ in $\mathcal{M}(G, \omega)$. \square

Corollary 2.13. *Let \mathfrak{B} be a dual Banach algebra, $\varphi: \mathcal{M}(G, \omega) \rightarrow \mathfrak{B}$ a bounded homomorphism that is so_l -weak* continuous on the unit ball of $\mathcal{M}(G, \omega)$. Then φ is so_l -weak* and so_r -weak* continuous on all of $\mathcal{M}(G, \omega)$.*

PROOF: By Corollary 2.12, the restriction, φ_1 , of φ to $\mathcal{L}^1(G, \omega)$ has a so_l/so_r -weak* continuous extension $\tilde{\varphi}_1: \mathcal{M}(G, \omega) \rightarrow \mathfrak{B}$. As noted before, $\mathcal{L}^1(G, \omega)_{\|\cdot\| \leq 1}$ is so_l -dense in $\mathcal{M}(G, \omega)_{\|\cdot\| \leq 1}$, so $\varphi = \tilde{\varphi}_1$ on $\mathcal{M}(G, \omega)_{\|\cdot\| \leq 1}$ and therefore on $\mathcal{M}(G, \omega)$. \square

Remark 2.14. Suppose that (H, ω_H) is another weighted locally compact group and $\varphi: \mathcal{M}(G, \omega) \rightarrow \mathcal{M}(H, \omega_H)$ is a bounded algebra isomorphism. By [7, Lemma 3.3] — which applies, as written, to $\mathcal{M}(G, \omega)$ — φ is so_l -weak* continuous on bounded subsets of $\mathcal{M}(G, \omega)$. By Corollary 2.13, φ is so_l/so_r -weak* continuous on all of $\mathcal{M}(G, \omega)$.

Acknowledgement. The author is grateful to Fereidoun Ghahramani for helpful discussions regarding the topic of this paper.

REFERENCES

- [1] Bonsall F. F., Duncan J., *Complete Normed Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 80, Springer, New York, 1973.
- [2] Dales H. G., *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series, 24; Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.
- [3] Dales H. G., Lau A. T.-M., *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **177** (2005), no. 836, vi + 191 pages.
- [4] Fell J. M. G., Doran R. S., *Representations of *-algebras, Locally Compact Groups, and Banach *-algebraic Bundles, Vol. 1, Basic Representation Theory of Groups and Algebras*, Pure and Applied Mathematics, 125, Academic Press, Boston, 1988.
- [5] Ghahramani F., *Compact elements of weighted group algebras*, Pacific J. Math. **113** (1984), no. 1, 77–84.
- [6] Ghahramani F., *Weighted group algebra as an ideal in its second dual space*, Proc. Amer. Math. Soc. **90** (1984), no. 1, 71–76.
- [7] Ghahramani F., Zadeh S., *Bipositive isomorphisms between Beurling algebras and between their second dual algebras*, Canad. J. Math. **69** (2017), no. 1, 3–20.
- [8] Grønbaek N., *Amenability of weighted convolution algebras on locally compact groups*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 765–775.
- [9] Hewitt E., Ross K. A., *Abstract Harmonic Analysis, Vol. I: Structure of Topological Groups. Integration Theory, Group Representations*, Die Grundlehren der mathematischen Wissenschaften, 115, Academic Press, New York; Springer, Berlin, 1963.
- [10] Ilie M., Stokke R., *Weak*-continuous homomorphisms of Fourier–Stieltjes algebras*, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 107–120.
- [11] Kaniuth E., *A Course in Commutative Banach Algebras*, Graduate Texts in Mathematics, 246, Springer, New York, 2009.
- [12] Kroeker M. E., Stephens A., Stokke R., Yee R., *Norm-multiplicative homomorphisms of Beurling algebras*, J. Math. Anal. Appl. **509** (2022), no. 1, Paper No. 125935, 25 pages.
- [13] Lau A. T.-M., *Operators which commute with convolutions on subspaces of $L_\infty(G)$* , Colloq. Math. **39** (1978), no. 2, 351–359.
- [14] Palmer T. W., *Banach Algebras and the General Theory of *-algebras, Vol. I., Algebras and Banach algebras*, Encyclopedia of Mathematics and Its Applications, 49, Cambridge University Press, Cambridge, 1994.
- [15] Reiter H., Stegeman J. D., *Classical Harmonic Analysis and Locally Compact Groups*, London Mathematical Society Monographs, New Series, 22, The Clarendon Press, Oxford University Press, New York, 2000.
- [16] Runde V., *Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule*, Math. Scand. **95** (2004), no. 1, 124–144.

- [17] Samei E., *Weak amenability and 2-weak amenability of Beurling algebras*, J. Math. Anal. Appl. **346** (2008), no. 2, 451–467.
- [18] Shepelska V., Zhang Y., *Non-weakly amenable Beurling algebras*, Indiana Univ. Math. J. **67** (2018), no. 1, 119–150.
- [19] Zadeh S., *Isomorphisms of Banach Algebras Associated with Locally Compact Groups*, PhD Thesis, University of Manitoba, Canada, 2015.
- [20] Zadeh S., *Isometric isomorphisms of Beurling algebras*, J. Math. Anal. Appl. **438** (2016), no. 1, 1–13.

R. Stokke:

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WINNIPEG,
515 PORTAGE AVENUE, WINNIPEG, MANITOBA, MB R3B 2E9, CANADA

E-mail: r.stokke@uwinnipeg.ca

(Received February 16, 2021, revised April 14, 2021)