The category of compactifications and its coreflections

Anthony W. Hager, Brian Wynne

Abstract. We define "the category of compactifications", which is denoted **CM**, and consider its family of coreflections, denoted **corCM**. We show that **corCM** is a complete lattice with bottom the identity and top an interpretation of the Čech–Stone β. A $c \in$**corCM** implies the assignment to each locally compact, noncompact Y a compactification minimum for membership in the "object-range" of c. We describe the minimum proper compactifications of locally compact, noncompact spaces, show that these generate the atoms in **corCM** (thus **corCM** is not a set), show that any $c \in$**corCM** not the identity is above an atom, and that β is not the supremum of atoms.

Keywords: compactification; coreflection; atom in a lattice

Classification: 54B30, 54C10, 54D35, 06B23, 18A40

1. Introduction

The coreflections (and reflections) of various categories of topological spaces have been much studied (just for example, [9], [12, Chapter 9]). We introduce here a category **CM** of compactifications and show that the collection **corCM** of its coreflections is a proper class, though the only one previously known is the famous Čech–Stone β. Moreover, we show that **corCM** possesses the structure of a complete lattice and we characterize its atoms.

While the present paper is about compactification theory, it is representative of more general ideas applicable, and under study by the authors, in a variety of situations. Most closely related concerns covers of compact spaces (a subject with considerable history, e.g. [5], [6], [12]): our paper [8] gives a characterization of the atoms in the lattice of "covering operators". But also, "dually" (the arrows go in the opposite direction), we are considering, first, in a forthcoming paper, lattice-ordered groups and rings, where the focus is on "essential extension operators", another subject with considerable history, see [2], and the atoms in that lattice. Further, we are developing these ideas in rather general categories in another forthcoming paper. Some comment on all this appears in more detail in Section 7 here, and in yet more detail in Section 5 of [8].

DOI 10.14712/1213-7243.2022.024

Turning back to the present paper, here is an outline of the contents.

Section 2: The complete lattice $\mathcal{C}(Y)$ of compactifications of the locally compact Y. A standard discussion after [4], then with some novelty, the atoms and strong atoms therein. In particular, K is the strong atom in $\mathcal{C}(Y)$ if and only if $|K - Y| = 2$ and $K = \beta Y$.

Section 3: The category **CM**. This has objects (X, M) with X compact Hausdorff and M a closed nowhere dense subset of X, i.e., $X \in \mathcal{C}(X - M)$; and maps the $(X, M) \xleftarrow{f} (Z, N)$ which express $X - M = Z - N \equiv Y$ and $X \leq Z$ in $\mathcal{C}(Y)$.

Section 4: **corCM**, the coreflections in **CM**. Here "coreflection" takes the standard categorical meaning, see [10]. The Čech–Stone β is one (the only one we knew of prior to the present paper). It is shown that **corCM** is a complete lattice, with β the top.

Section 5: Atoms in **corCM**. These are determined as generated one-to-one from the strong atoms from Section 2. There are (obviously) so many of the latter that we see that **corCM** is a proper class.

Section 6: Atoms below $c \in$ **corCM**. Using Section 5, it follows that for every $c > \mathrm{Id}$ in **corCM**, there is an atom $a \leq c$, and via more detail from Section 5, β is not the supremum of all atoms (raising questions such as "What can be said about the latter?").

Section 7: Remarks. Mostly the present paper vs. [8].

2. Compactifications of a locally compact space

We first sketch out the basics, more or less borrowing from [4], then identify the atoms and strong atoms in the lattice $\mathcal{C}(Y)$ of compactifications of the locally compact Y, the latter being crucial to the sequel.

Comp denotes the category of compact Hausdorff spaces with continuous maps. For Y Tychonoff (locally compact or not), a compactification of Y is a $Y \xrightarrow{f} fY$, $fY \in$ **Comp** and f a dense embedding of Y in fY, and $\mathcal{C}(Y)$ is a family of these. One usually writes $(fY, f) \in \mathcal{C}(Y)$.

In addition, $\mathcal{C}(Y)$ is ordered as: $(fY, f) \leq (gY, g)$ means there is continuous $fY \xleftarrow{h} gY$ with $hg = f$, and they are *equivalent* if the h is a homeomorphism; that occurs if and only if $(fY, f) \leq (gY, g)$ and $(gY, g) \leq (fY, f)$. Reducing $\mathcal{C}(Y)$ by this equivalence creates a set (instead of a proper class). Then $\mathcal{C}(Y)$ will usually denote this set, with $Y \xrightarrow{f} fY$ constructed as $f(y) = y$ (all compactifications contain Y). When we do this, we will drop the f and just write "$K \in \mathcal{C}(Y)$", meaning $K \in$ **Comp** and K contains Y densely. If $K, L \in \mathcal{C}(Y)$, then $K \leq L$ means the corresponding pairs obey that relation in $\mathcal{C}(Y)$, and $K < L$ means $K \leq L$ and $L \nleq K$.

Proposition 2.1. *Then $(\mathcal{C}(Y), \leq)$ is a complete upper semi-lattice with top the Čech–Stone $(\beta Y, \beta)$.*

In addition, $(\mathcal{C}(Y), \leq)$ is a lattice (which is complete) if and only if Y is locally compact, and then the bottom of the lattice is the one-point compactification of Y, denoted $\dot{Y} = Y \cup \{\infty\}$.

If $(L, \leq)$ is any lattice with bottom 0 and top 1, an *atom* in L is an $a > 0$ for which $0 \leq x \leq a$ implies $x = 0$ or $x = a$. G. Birkhoff in [1] states the definition of a *strictly meet irreducible* element $m \in L$ as the minimum in $\{x \in L \colon 0 < x\}$; we call such m the strong atom of L. Such m may not exist; if it does it is unique, and the unique atom; not every atom is strong.

Henceforth, $\mathcal{C}(Y)$ refers to the complete lattice of compactifications for Y locally compact and not compact. We now describe the atoms and strong atoms in $\mathcal{C}(Y)$ (apparently novel here).

Lemma 2.2. *Suppose $K \in \mathcal{C}(Y)$. Then $\dot{Y} \leq K$, which we express as $\dot{Y} \overset{\varphi}{\twoheadleftarrow} K$.*

If $p \neq q$ in K and $\varphi(p) = \varphi(q)$ (thus $p, q \notin Y$), let $K_{pq} \overset{\varphi_{pq}}{\twoheadleftarrow} K$ denote the continuous quotient map which identifies only p and q. Then, $K_{pq} \in \mathcal{C}(Y)$ and $K_{pq} < K$.

Proof: Routine. □

Proposition 2.3. (a) *We have that A is an atom in $\mathcal{C}(Y)$ if and only if $|A - Y| = 2$.*

(b) *For some Y there are such A, and other Y not. Sometimes such A is the unique atom, and sometimes not.*

Proof: (a) If $K \in \mathcal{C}(Y)$ has $|K - Y| \geq 3$, then Lemma 2.2 shows some p, q yield $\dot{Y} < K_{pq} < K$, so K is not an atom.

If $|A - 2| = 2$, then Lemma 2.2 shows $\dot{Y}$ is unique for $\dot{Y} < A$, so A is an atom.

(b) For example, $Y = \mathbb{R}$ (the real line) has unique $A = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and $Y = \mathbb{R} \times \mathbb{R}$ has no A. Furthermore, $Y = \mathbb{N}$ has many A with $|A - \mathbb{N}| = 2$ (write $\mathbb{N} = N_1 \cup N_2$ disjoint with each N_i infinite, and let $A = \dot{N_1} \cup \dot{N_2}$ disjointly). See [11] for more about 2-point compactifications. □

Here is a construction, notation, and terminology which will persist throughout the paper.

Let X be any Tychonoff space for which there are $p \neq q$ in $\beta X - X$. (If there are no such p, q, i.e., $|\beta X - X| \leq 1$, then X is called *almost-compact*. See [5].) Then, $Y = \beta X - \{p, q\}$ is locally compact, and in the notation of Lemma 2.2 and Proposition 2.3, $(\beta X)_{pq} = \dot{Y}$. Then, $\beta X = \beta Y$ is the unique 2-point compactification of Y, and we have $\dot{Y} \overset{\gamma}{\twoheadleftarrow} \ddot{Y} = \beta Y$ (here γ is the obvious φ_{pq} from Lemma 2.2). We call such a construct a "γ".

Note that, for many X there are many resulting γ's, and there is a proper class of X's yielding γ's which are very different (in the terminology introduced in Section 3 below, "not isomorphic in **CM**").

We call $L \in \mathcal{C}(Y)$ *proper* just in case $\dot{Y} < L$ in $\mathcal{C}(Y)$. Now, for general noncompact locally compact Y, there is the possibility of a *minimum proper compactification*: a proper $K \in \mathcal{C}(Y)$ such that $K \leq L$ for every proper $L \in \mathcal{C}(Y)$.

Theorem 2.4. *Suppose $K \in \mathcal{C}(Y)$. These are equivalent.*

(a) *K is the strong atom in $\mathcal{C}(Y)$.*
(b) *K is the minimum proper compactification of Y.*
(c) *$|K - Y| = 2$ and $K = \beta Y$.*

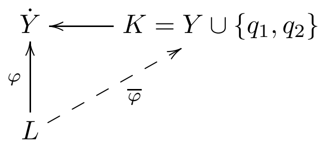
When these obtain, we express $\dot{Y} \leftarrow K$ as $\dot{Y} \xleftarrow{\gamma} \ddot{Y} = \beta Y$, per previous remarks.

Proof: It is evident that minimum proper compactifications and strong atoms are the same, so (a) $\Leftrightarrow$ (b). And, it is evident that (c) $\Rightarrow$ (b), since $\mathcal{C}(Y) = \{\dot{Y}, \beta Y\}$ when (c) holds.

We show (b) $\Rightarrow$ (c). Suppose (b) holds. We shall use Lemma 2.2 and its notation.

First, $|K - Y| = 2$: if there are distinct $p, q, r \in K - Y$, then K_{pq} is proper and $K_{pq} < K$.

Now write $K = Y \cup \{q_1, q_2\}$, and suppose $L \in \mathcal{C}(Y)$ is proper. Then $K \leq L$ by (b), and the situation is expressed as



from which we see that $L - Y = \varphi^{-1}(\{\infty\})$ has only two points: if not, then there are different $p, q, r \in \varphi^{-1}(\{\infty\})$ with $\overline{\varphi}(p) = \overline{\varphi}(q) = q_1$ and $\overline{\varphi}(r) = q_2$. But then $K \not\leq L_{pr}$, contradicting (b).

Applying the preceding to $L = \beta Y$, the natural map $K \leftarrow \beta Y$ must be one-to-one, i.e. $\beta Y = K$. $\square$

3. The category CM of compactifications

Definition 3.1. The category **CM** has

Objects: (X, M) with $X \in$ **Comp** and M a closed nowhere dense subset of X ($X \in \mathcal{C}(X - M)$), and

Morphisms (called compactification maps): $(X,M) \xleftarrow{f} (Y,N)$, where $X \xleftarrow{f} Y$ is a continuous surjection with $f^{-1}(M) = N$, and f restricted to $Y - N$ is a homeomorphism onto $X - M$ (so $X \leq Y$ is witnessed by f in $\mathcal{C}(X - M)$).

We sometimes write "$f \in \mathbf{CM}$" to indicate that f is a $\mathbf{CM}$-morphism.

Proposition 3.2. *If $f, g \in \mathbf{CM}$, then $fg \in \mathbf{CM}$ if fg is defined.*

We have that $\mathbf{CM}$ is a category with identity morphisms the $(X,M) \xleftarrow{\mathrm{id}} (X,M)$ given by $\mathrm{id}(x) = x$ for every $x \in X$.

Proof: Evident. □

Proposition 3.3. *Suppose (X,M) and (Y,N) are in $\mathbf{CM}$, and $X \xleftarrow{h} Y$ is a function. These are equivalent.*

(a) *h is a homeomorphism with $h^{-1}(M) = N$.*
(b) *$(X,M) \xleftarrow{h} (Y,N)$ is an isomorphism in $\mathbf{CM}$.*
(c) *The restriction $X - M \xleftarrow{h^\circ} Y - N$ of h is a homeomorphism, and h witnesses the equivalence of X, Y in $\mathcal{C}(X - M)$.*

Proof: (a) $\Leftrightarrow$ (c) is immediate from the definition of equivalent compactifications.

(a) $\Rightarrow$ (b) Assuming (a), then $(X,M) \xleftarrow{h} (Y,N)$ is in $\mathbf{CM}$, and clearly $(X,M) \xrightarrow{h^{-1}} (Y,N)$ satisfies (a) also. Thus $h^{-1} \in \mathbf{CM}$. Thus (b).

(b) $\Rightarrow$ (a) Then h is invertible in $\mathbf{CM}$ and (a) follows. □

The following will be used later to establish the uniqueness of certain maps.

Proposition 3.4. *Every map in $\mathbf{CM}$ is $\mathbf{CM}$-epic and $\mathbf{CM}$-monic.*

Proof: Epic (or monic) have the categorical meaning of right (left, respectively) cancellable, see [10].

Every map in $\mathbf{CM}$ is a surjection, thus $\mathbf{Sets}$-epic and therefore $\mathbf{CM}$-epic.

Now consider

$$(Z,P) \xleftarrow[m]{} (X,M) \underset{g}{\overset{f}{\leftleftarrows}} (Y,N)$$

in $\mathbf{CM}$ with $mf = mg$. With $(\cdot)^\circ$ denoting restriction as in 3.3 (c), and noting $m^\circ f^\circ = (mf)^\circ = (mg)^\circ = m^\circ g^\circ$, all these $(\cdot)^\circ$ are homeomorphisms. In particular, m° is, thus, one-to-one, thus $\mathbf{Sets}$-monic, so $m^\circ f^\circ = m^\circ g^\circ$ implies $f^\circ = g^\circ$. Then, f and g are continuous extensions of the same map, whence $f = g$. □

Remarks 3.5. (a) $(X,M) \xleftarrow{f} (Y,N)$ in $\mathbf{CM}$ implies $X \xleftarrow{f} Y$ is a cover per [8], and in the category of compact spaces with only covers as maps, all maps are monic, see [6], so Proposition 3.4 follows. The proof above is simpler.

(b) We are not clear on what it means for

$$(X, M) \underset{g}{\overset{f}{\leftleftarrows}} (Y, N)$$

in **CM**. This does not imply $f = g$: Given $(X, M) \overset{f}{\leftarrow} (Y, N)$, suppose $Y \overset{h}{\leftarrow} Y$ is a homeomorphism that is not the identity with $h(Y - N) = Y - N$. If $g = fh$, then $f \neq g$ (but still, (Y, N) and $(Y, h^{-1}N)$ correspond to equivalent compactifications of $X - M$).

4. Coreflections in CM

A subcategory **A** of a category **B** is *coreflective* if the inclusion functor $\mathbf{A} \to \mathbf{B}$ has the right adjoint $\mathbf{A} \overset{a}{\leftarrow} \mathbf{B}$. Then a is called the *coreflection* of **B** onto **A**. This means: for every **B**-object B, there is $B \overset{a_B}{\leftarrow} aB$ such that whenever A is an **A**-object and $B \overset{f}{\leftarrow} A$, there is a unique $aB \overset{\bar{f}}{\leftarrow} A$ such that $a_B \bar{f} = f$, as

$$\begin{array}{ccc} B & \xleftarrow{a_B} & aB \\ {\scriptstyle f}\uparrow & \nearrow{\scriptstyle \bar{f}} & \\ A & & \end{array}$$

If all a_B are monic, then **A** is called *monocoreflective.*

The topic of this paper is coreflections in **CM**, which is a peculiar category. We first note the situation in the related but less peculiar categories of Tychonoff spaces with continuous maps (**Tych**) and compact Hausdorff spaces with continuous maps (**Comp**). By virtue of their completeness properties in each case, monocoreflective means closed under coproducts and quotients.

Tych has many monocoreflective subcategories, e.g., discrete spaces, P-spaces, and **Tych** itself, while **Comp** has only **Comp** (easily seen).

See [9] and [10] for elaboration on the above remarks.

CM has no apparent completeness properties, and all maps are monic. One may think of a coreflection a with range **A** in **CM** as an assignment to each (X, M) of a $(X, M) \leftarrow a(X, M)$ which is the minimum preimage with domain in **A** (which means minimum in the order in $\mathcal{C}(X - M)$).

Let **corCM** be the class of coreflection functors in **CM**, and for $a \in$ **corCM**, let

$$\mathrm{fix}(a) = \{(X, M) \colon (X, M) \xleftarrow{a_{(X,M)}} a(X, M) \text{ is a } \mathbf{CM}\text{-isomorphism}\}$$

(this is the object range of a, i.e., the associated coreflective subcategory). Note that fix(a) is closed under isomorphic copies: if $(X, M) \in \mathrm{fix}(a)$ and $(X, M) \leftarrow$

(Y,N) is a **CM**-isomorphism, then $(Y,N) \in \mathrm{fix}(a)$. The class **corCM** is ordered "pointwise" as: $a \leq b$ means that for every (X,M) there is $a(X,M) \overset{f}{\leftarrow} b(X,M)$ with $a_{(X,M)} f = b_{(X,M)}$; $a \leq b$ if and only if $\mathrm{fix}(a) \supseteq \mathrm{fix}(b)$.

Evidently, $\mathrm{Id} \leq a$ for every a ("Id" is $(X,M) \mapsto (X,M) \overset{\mathrm{id}}{\longleftarrow} (X,M)$, this latter id being $\mathrm{id}(x) = x$ for all $x \in X$). That is, Id is the bottom of **corCM**. We turn to the top. There the Čech–Stone β emerges, and we assume knowledge, see [5], [4], inter alia.

Call (K,N) *maximal* if $(K,M) \overset{f}{\leftarrow} (X,M)$ in **CM** implies f is a **CM**-isomorphism (i.e., $K \overset{f}{\leftarrow} X$ is a homeomorphism).

Proposition 4.1. *We have that (X,N) is maximal if and only if $X - N$ is C^*-embedded in X, i.e., $X = \beta(X-N)$ and $N = \beta(X-N) - (X-N)$.*

For every (X,M), there is $(X,M) \overset{f}{\leftarrow} (K,N)$ with (K,N) maximal, namely, $K = \beta(X-M)$ with $N = \beta(X-M) - (X-M)$, f witnessing β as maximum compactification.

Proof: Properties of β, especially that βZ is the maximum compactification of Z in **Tych**. □

Proposition 4.1 defines a function $(X,M) \mapsto (X,M) \overset{f}{\leftarrow} \beta(X,M)$, and describes $\mathrm{fix}(\beta)$.

Theorem 4.2. (a) *We have $\beta \in$ **corCM**, and*

(b) *$a \leq \beta$ for every $a \in$ **corCM**.*

Proof: (a) To see this, just check the items in the definition of coreflection given above.

(b) For every (X,M), $a(X,M)$ is a compactification of $X - M$, and $\beta(X,M)$ is the maximum such. □

We say that a family $\mathcal{F}$ of **CM**-objects has a *common* **CM**-*lower bound* just in case there is a locally compact not compact Z such that $X - M$ is homeomorphic to Z whenever $(X,M) \in \mathcal{F}$. Then, $X \in \mathcal{C}(Z)$ whenever $(X,M) \in \mathcal{F}$, so $Y = \bigwedge\{X : (X,M) \in \mathcal{F}\}$ exists in the complete lattice $\mathcal{C}(Z)$ and we write $\bigwedge \mathcal{F}$ for the **CM**-object $(Y, Y-Z)$. Note that for every $(X,M) \in \mathcal{F}$, there is a map $\bigwedge \mathcal{F} \leftarrow (X,M)$ in **CM** (i.e., the map witnessing $Y \leq X$ in $\mathcal{C}(Z)$).

Consider the following property that a collection **B** of **CM**-objects might enjoy.

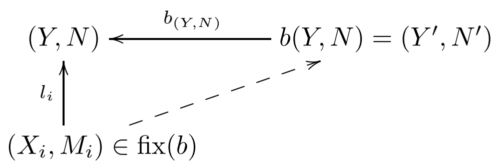
Property 4.3. **B** is closed under isomorphic copies and for every family $\mathcal{F} \subseteq$ **B**, if $\mathcal{F}$ has a common **CM**-lower bound, then $\bigwedge \mathcal{F} \in$ **B**.

Here is a characterization of sorts of the classes $\mathrm{fix}(a)$ for $a \in$ **corCM**.

Theorem 4.4. (a) *If* $b \in \mathbf{corCM}$*, then* $\mathrm{fix}(b)$ *has Property* 4.3.

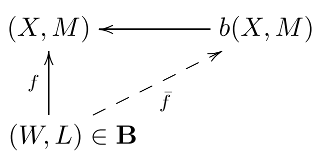
(b) *If* $\mathbf{B}$ *is a collection of* $\mathbf{CM}$*-objects that has Property* 4.3*, then there is* $b \in \mathbf{corCM}$ *with* $\mathrm{fix}(b) = \mathbf{B}$.

Proof: (a) Suppose $b \in \mathbf{corCM}$. Then, as was mentioned earlier, $\mathrm{fix}(b)$ is closed under isomorphic copies. Suppose $\mathcal{F} = \{(X_i, M_i)\}_{i \in I} \subseteq \mathrm{fix}(b)$ has a common $\mathbf{CM}$-lower bound. Then, one obtains $(Y, N) \xleftarrow{l_i} (X_i, M_i)$ in $\mathbf{CM}$ for every $i \in I$, where $(Y, N) = \bigwedge \mathcal{F}$. Let $b(Y, N) = (Y', N')$ and note that $b_{(Y,N)}$ witnesses $Y \leq Y'$ in $\mathcal{C}(Z)$. Applying the universal property for b yields



showing $Y' \leq X_i$ in $\mathcal{C}(Z)$ for all $i \in I$. But Y is the greatest lower bound of the X_i's, so $Y' \leq Y$ and hence Y and Y' are equivalent in $\mathcal{C}(Z)$. Thus, using Proposition 3.3, $b_{(Y,N)}$ is a $\mathbf{CM}$-isomorphism, i.e., $(Y, N) \in \mathrm{fix}(b)$.

(b) Suppose $\mathbf{B}$ has Property 4.3, let (X, M) be given, and define $b(X, M) = \bigwedge \mathcal{F} = (Y, Y - Z)$, where $Z = X - M$ and $\mathcal{F}$ is the family of $(W, L) \in \mathbf{B}$ such that $W - L$ is homeomorphic to Z and $X \leq W$ in $\mathcal{C}(Z)$ (here Z witnesses that $\mathcal{F}$ has a common $\mathbf{CM}$-lower bound); the map $b_{(X,M)}$ is taken to be the one that witnesses $X \leq Y$ in $\mathcal{C}(Z)$. Since $\mathbf{B}$ obeys Property 4.3, we know $b(X, M) \in \mathbf{B}$. Moreover, we have $\mathrm{fix}(b) = \mathbf{B}$. For the required universal mapping property:



because f means $(W, L) \in \mathcal{F}$, and $b(X, M) = \bigwedge \mathcal{F}$, so $\bar{f}$ exists by definition. $\square$

Corollary 4.5. $\mathbf{corCM}$ *is a complete lattice, with bottom* Id *and top* β*, see Proposition* 4.2.

Proof: Consider the oppositely ordered family $\{\mathrm{fix}(b) \colon b \in \mathbf{corCM}\}$. Using Theorem 4.4 for a subfamily $\{b_j\}_J$ of $\mathbf{corCM}$, each $\mathbf{B}_j = \mathrm{fix}(b_j)$ having Property 4.3, one checks that $\bigcap_J \mathbf{B}_j$ retains Property 4.3.

This gives $\bigwedge_J \mathbf{B}_j$, thus dually $\bigvee_J b_j$, in $\mathbf{corCM}$. $\square$

Remarks 4.6. (a) A "direct" construction of $c \equiv \bigvee_J b_j$ could proceed as follows.

Given (X, M) and writing $b_j(X, M) = (Y_j, N_j)$, one sees that $Y_j \leq \beta X$ in $\mathcal{C}(X - M)$ for all $j \in J$, see Theorem 4.2, so one may use $\bigvee_J Y_j$ in the complete lattice $\mathcal{C}(X - M)$ to define $c(X, M)$.

(b) Distinctly related to Theorem 4.4 and Corollary 4.5 are (1) [12, 8.4 (f), (t)], which concerns covers of spaces (the connection discussed further below in Section 7), and (2) [7, 2.4], which concerns a dual situation of hull operators in a class of lattice-ordered groups.

(c) A consequence, or interpretation, of the development here is: if $\mathbf{B} = \mathrm{fix}(b)$ for $b \in \mathbf{corCM}$, and Z is any locally compact not compact space, then in the complete lattice $\mathcal{C}(Z)$, there is the least compactification lying in $\mathbf{B}$, namely $b(\dot{Z}, \{\infty\})$ ($\dot{Z} = Z \cup \{\infty\}$, the one-point compactification).

5. Atoms in the coreflections

Recall from Section 2 the minimum proper compactification of some locally compact not compact Y, which are the $\dot{Y} \overset{\gamma}{\leftarrow} \ddot{Y} = \beta Y$, which we call a "$\gamma$". One more piece of notation: for any (X, M), we write $[(X, M)]$ for the class of **CM**-objects isomorphic to (X, M).

Theorem 5.1. *For every $\dot{Y} \overset{\gamma}{\leftarrow} \ddot{Y} = \beta Y$, there is a corresponding $a(\gamma) \in$ **corCM** given by*

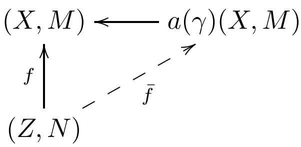
$$(X, M) \xleftarrow{a(\gamma)_{(X,M)}} a(\gamma)(X, M)$$
$$= \begin{cases} (\dot{Y}, \{\infty\}) \overset{\gamma}{\leftarrow} (\ddot{Y}, \ddot{Y} - Y) & \text{if } (X, M) = (\dot{Y}, \{\infty\}), \\ (X, M) \overset{\mathrm{id}}{\longleftarrow} (X, M) & \text{if } (X, M) \neq (\dot{Y}, \{\infty\}), \end{cases}$$

*(where "=" means "**CM**-isomorphic to") with*

$$\mathrm{fix}(a(\gamma)) = |\,\mathbf{CM}\,| - [(\dot{Y}, \{\infty\})].$$

Proof: Using Theorem 2.4, it is simple to check

(i) the universal mapping property: for every (X, M), and every $(X, M) \overset{f}{\leftarrow} (Z, N) \in \mathrm{fix}(a(\gamma))$, there exists $\bar{f}$ (unique by Proposition 3.4) making the diagram



commute.

(ii) $a(\gamma)(X,M) \in \operatorname{fix}(a(\gamma))$ for every (X,M) (since $\dot{Y}$ and $\ddot{Y}$ are not homeomorphic).
(iii) $\operatorname{fix}(a(\gamma)) = |\,\mathbf{CM}\,| - [(\dot{Y},\{\infty\})]$.

□

Theorem 5.2. *Suppose $a \in \mathbf{corCM}$.*

(a) *Then a is an atom in* **corCM** *if and only if there exists a unique (up to* **CM***-isomorphism) $(Z,M) \notin \operatorname{fix}(a)$.*
(b) *When* (a) *occurs, then*
   - *$a(Z,M)$ corresponds to the minimum proper compactification in $\mathcal{C}(Z-M)$, thus $a_{(Z,M)}$ corresponds to a γ (per remark after Theorem* 2.4*), and*
   - *$a = a(\gamma)$ (per* [10]*, this $=$ means "naturally equivalent").*

Proof: (a) For necessity, we prove the contrapositive. Assume that $(Z_1,M_1) \neq (Z_2,M_2)$ are not in $\operatorname{fix}(a)$. If Z_1-M_1 is homeomorphic to Z_2-M_2, then Z_1 and Z_2 cannot be equivalent in $\mathcal{C}(Z_1-M_1)$, so at least one "$\leq$" fails, say $Z_2 \nleq Z_1$.

Suppose $c \in \mathbf{corCM}$ is given by

$$(Y,N) \xrightarrow{c_{(Y,N)}} c(Y,N)$$
$$= \begin{cases} (Y,N) \xleftarrow{\text{id}} (Y,N) & \text{if } Y-N \cong Z_1-M_1 \text{ and } Y \leq Z_1, \\ (Y,N) \xleftarrow{a_{(Y,N)}} a(Y,N) & \text{if not,} \end{cases}$$

where "$\cong$" means "homeomorphic". If so, then $(Z_2,M_2) \notin \operatorname{fix}(c)$ because either $Z_1-M_1 \ncong Z_2-M_2$ or $Z_2 \nleq Z_1$ in $\mathcal{C}(Z_1-M_1)$, and therefore $c_{(Z_2,M_2)} = a_{(Z_2,M_2)}$, which is not a **CM**-isomorphism by hypothesis. It is clear that $(Z_1,M_1) \in \operatorname{fix}(c)$ and $c \leq a$ "pointwise". Thus $\mathrm{Id} < c < a$, and a is not an atom.

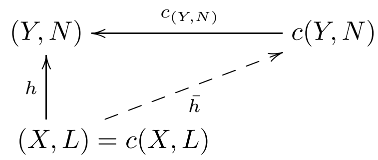
To complete the proof of necessity, we must verify that $c \in \mathbf{corCM}$. This can be done using Property 4.3, but we proceed directly.

First note that, from the definition, $(Y,N) \in \operatorname{fix}(c)$ if and only if either

(i) $Y-N \cong Z_1-M_1$ and $Y \leq Z_1$, or
(ii) $(Y,N) \in \operatorname{fix}(a)$.

Then c is idempotent, i.e., each $c(Y,N) \in \operatorname{fix}(c)$: In case (i), this is clear. In case (ii), $(Y,N) \in \operatorname{fix}(a)$, and since $c \leq a$, $(Y,N) \in \operatorname{fix}(c)$. This means $c(c(Y,N)) = c(Y,N)$ as desired.

As for the universal property, consider



The cases for (X, L) are again:

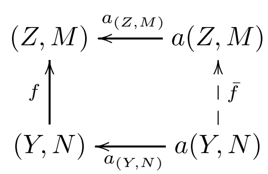
(i) $X - L \cong Z_1 - M_1$ and $X \leq Z_1$. Then $Y \leq Z_1$ (because of h), so $c_{(Y,N)} = \mathrm{id}$, so $\bar{h} = h$.

(ii) Here $(X, L) \in \mathrm{fix}(a)$, $c(Y, N) = a(Y, N)$, and $c_{(Y,N)} = a_{(Y,N)}$, so $\bar{h}$ exists because $a \in$ **corCM**.

Next, we prove sufficiency in (a). Suppose $a \in$ **corCM**, and there exists a unique (up to **CM**-isomorphism) $(Z, M) \notin \mathrm{fix}(a)$, i.e., $\mathrm{fix}(a) = |\,\mathbf{CM}\,| - [(Z, M)]$. We show that a is an atom.

Suppose $c < a$ so $\mathrm{fix}(c) \supsetneq \mathrm{fix}(a)$, i.e., there is $(Y, N) \in \mathrm{fix}(c) - \mathrm{fix}(a)$. Then $(Y, N) \notin \mathrm{fix}(a)$, so $(Y, N) = (Z, M)$. Therefore $\mathrm{fix}(c) = |\,\mathbf{CM}\,|$, $c = \mathrm{Id}$, and a is an atom.

(b) Now suppose $a \in$ **corCM** is an atom and $(Z, M) \notin \mathrm{fix}(a)$. Let $a(Z, M) = (Z', M')$. Then $Z' \in \mathcal{C}(Z - M)$ is proper because $(Z, M) \notin \mathrm{fix}(a)$. Given proper $Y \in \mathcal{C}(Z - M)$ and a map f witnessing $Z \leq Y$ in $\mathcal{C}(Z - M)$, let $N = Y - (Z - M)$ and consider



where $(Z, M) \overset{f}{\leftarrow} (Y, N)$ is not a **CM**-isomorphism (note $\bar{f}$ exists because $a \in$ **corCM**). Now $(Y, N) \neq (Z, M)$, so $(Y, N) \in \mathrm{fix}(a)$ and $a_{(Y,N)}$ is a **CM**-isomorphism. It follows that $Z' \leq Y$ in $\mathcal{C}(Z - M)$. Hence Z' is the minimum proper compactification in $\mathcal{C}(Z - M)$. So $a_{(Z,M)}$ is a γ by Theorem 2.4.

Finally, $a = a(\gamma)$ just because $\mathrm{fix}(a) = \mathrm{fix}(a(\gamma))$. □

6. Atoms below a coreflection

Theorem 6.1. *If* $\mathrm{Id} < c \in$ **corCM**, *then there is a* $\dot{Y} \overset{\gamma}{\leftarrow} \ddot{Y} = \beta Y$ *with* $c \geq a(\gamma)$.

PROOF: Let $\dot{Y} \overset{\gamma}{\leftarrow} \ddot{Y} = \beta Y$ be given, and recall

$$\mathrm{fix}(a(\gamma)) = |\,\mathbf{CM}\,| - [(\dot{Y}, \{\infty\})].$$

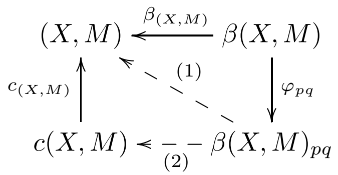
And recall that for every $c, d \in$ **corCM**, $c \leq d$ if and only if $\mathrm{fix}(c) \supseteq \mathrm{fix}(d)$.

Then $c \ngeq a(\gamma)$ for every γ if and only if $\dot{Y} \in \mathrm{fix}(c)$ for every $\dot{Y} \xleftarrow{\gamma} \ddot{Y} = \beta Y$.

From [6, 9.1] ([8, 3.4]), we have the following, which holds for covers and therefore for compactifications, stated in those terms with notation following Lemma 2.2.

Property 6.2. Suppose $(X, M) \xleftarrow{f} (Y, N)$ is in **CM**. If $(X, M) \xleftarrow{h} (Z, P)$ in **CM** has $(Z, P) \xleftarrow{d_{pq}} (Y_{pq}, P_{pq})$ for all $p, q \in Y$ with $f(p) = f(q)$, then h is a **CM**-isomorphism.

Fix $c \in$ **corCM**. Assume $c \ngeq a(\gamma)$ for every γ. Take (X, M). To complete the proof of Theorem 6.1, it suffices to show $(X, M) \in \mathrm{fix}(c)$. If $(X, M) \in \mathrm{fix}(\beta)$, then we are done. So suppose $(X, M) \xleftarrow{\beta_{(X,M)}} \beta(X, M)$ is not a **CM**-isomorphism. We use Property 6.2, with $(X, M) \xleftarrow{\beta_{(X,M)}} \beta(X, M)$ as $(X, M) \xleftarrow{f} (Y, N)$ and $(X, M) \xleftarrow{c_{(X,M)}} c(X, M)$ as $(X, M) \xleftarrow{h} (Z, P)$. Consider (here p, q are such that $\beta_{(X,M)}(p) = \beta_{(X,M)}(q)$)



where φ_{pq} is as in Lemma 2.2 and (1) is the obvious map j_{pq} such that $\beta_{(X,M)} = j_{pq}\varphi_{pq}$. Now, the map φ_{pq} is exactly a $\dot{Y} \xleftarrow{\gamma} \ddot{Y} = \beta Y$, and therefore $\beta(X, M)_{pq} \in \mathrm{fix}(c)$ since $c \ngeq a(\gamma)$. Thus the map (2) exists because $c \in$ **corCM**. Hence, the hypotheses of Property 6.2 are satisfied, $c_{(X,M)}$ is a **CM**-isomorphism, and $(X, M) \in \mathrm{fix}(c)$. □

The atoms of **corCM** are the $a(\gamma)$'s, β is the maximum coreflection, so $\beta \geq a(\gamma)$ for every γ. Thus $\beta \geq \bigvee_\gamma a(\gamma)$. But

Theorem 6.3. *We have* $\beta > \bigvee_\gamma a(\gamma)$.

Proof: For Γ a family of γ's, $c = \bigvee_\Gamma a(\gamma)$ is the coreflection for

$$\bigcap_\Gamma \mathrm{fix}(a(\gamma)) = \bigcap_\Gamma (|\,\mathbf{CM}\,| - [(\dot{Y}_\gamma, \{\infty\})]) = |\,\mathbf{CM}\,| - \bigcup_\Gamma [(\dot{Y}_\gamma, \{\infty\})].$$

So, to show $\beta > \bigvee_\gamma a(\gamma)$ is exactly to show

$$\mathrm{fix}(\beta) \subsetneq |\,\mathbf{CM}\,| - \bigcup_\gamma [(\dot{Y}_\gamma, \{\infty\})],$$

i.e., to find $(X, M) \neq (\dot{Y}_\gamma, \{\infty\})$ for every γ and (X, M) is not any $\beta(Z, P)$. For example, $(X, M) = ([0, 1], \{1\})$ works. □

7. CM versus $\mathbf{Comp}^{\#}$, etc.

The present paper resembles our paper [8] in thrust and some details, hardly all. It is certainly not the case that the results of [8] imply or are implied by those here. We comment a bit on the situation.

Recall that the category $\mathbf{Comp}^{\#}$ has $|\mathbf{Comp}^{\#}| = |\mathbf{Comp}|$ (the same objects), but $X \stackrel{f}{\leftarrow} Y$ in $\mathbf{Comp}^{\#}$ means that f is a *cover* (continuous irreducible surjection). If $(X, M) \stackrel{f}{\leftarrow} (Y, N)$ is in **CM**, then $X \stackrel{f}{\leftarrow} Y$ is a cover (follows from [5], [4]), so **CM** may be viewed as a subcategory of $\mathbf{Comp}^{\#}$ (telling us little it appears).

The paper [8] is about $\mathbf{corComp}^{\#}$, and (inter alia) characterizes minimum proper covers as the $\dot{Y} \stackrel{\gamma}{\leftarrow} \ddot{Y} = \beta Y$ with βY (thus Y) extremally disconnected (thus also a minimum proper compactification and a "γ" in the present context), then shows (with proofs different, of necessity) that the similarly defined $a(\gamma)$ are the atoms in $\mathbf{corComp}^{\#}$. So, the atoms in $\mathbf{corComp}^{\#}$ are some, hardly all, of the atoms in **corCM**.

The similarity of minimum proper compactifications and minimum proper covers doubtless stems from the facts that in **corCM** ($\mathbf{corComp}^{\#}$, respectively) the top is β (the Gleason extremally disconnected cover operator, respectively). This idea bodes well for generalization, which we are attempting in a paper in preparation.

The section [8, Section 4] resembles Section 6 of the present paper, but reflects the considerable history of $\mathbf{corComp}^{\#}$ (though there is no prior history of its atoms), while **corCM** has no history (save the Čech–Stone β) of which we are aware (about atoms or not), in spite of the huge history of compactifications (just for example, [5], [4], [3]).

We also draw attention to [8, Section 5], which discusses the relation of $\mathbf{corComp}^{\#}$ to the study of hull operators in some categories of algebras, an ongoing and complicated project.

References

[1] Birkhoff G., *Lattice Theory*, American Mathematical Society Colloquium Publications, 25, American Mathematical Society, Providence, 1979.

[2] Carrera R., Hager A. W., *A classification of hull operators in archimedean lattice-ordered groups with unit*, Categ. Gen. Algebr. Struct. Appl. **13** (2020), no. 1, 83–103.

[3] Chandler R. E., *Hausdorff Compactifications*, Lecture Notes in Pure and Applied Mathematics, 23, Marcel Dekker, New York, 1976.

[4] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[5] Gillman L., Jerison M., *Rings of Continuous Functions*, Graduate Texts in Mathematics, 43, Springer, New York, 1976.
[6] Hager A. W., *Minimal covers of topological spaces*, Ann. New York Acad. Sci., 552, New York Acad. Sci., New York, 1989, pages 44–59.
[7] Hager A. W., Martinez J., *Hulls for various kinds of α-completeness in Archimedean lattice-ordered groups*, Order **16** (1999), no. 1, 89–103.
[8] Hager A. W., Wynne B., *Atoms in the lattice of covering operators in compact Hausdorff spaces*, Topology Appl. **289** (2021), Paper No. 107402, 9 pages.
[9] Herrlich H., *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Mathematics, 78, Springer, Berlin, 1968 (German).
[10] Herrlich H., Strecker G. E., *Category Theory*, Sigma Series in Pure Mathematics, 1, Heldermann Verlag, Berlin, 1979.
[11] Magill K. D., Jr., *N-point compactifications*, Amer. Math. Monthly **72** (1965), 1075–1081.
[12] Porter J. R., Woods R. G., *Extensions and Absolutes of Hausdorff Spaces*, Springer, New York, 1988.

A. W. Hager:
Department of Mathematics and Computer Science, Wesleyan University, Sience Tower, Room 655, 265 Church Street, Middletown, Connecticut, CT 06459, USA

E-mail: ahager@wesleyan.edu

B. Wynne:
Department of Mathematics, Lehman College, CUNY, Gillet Hall, Room 211, 250 Bedford Park Blvd West, Bronx, New York, 10468, USA

E-mail: brian.wynne@lehman.cuny.edu

(*Received* February 28, 2021)