On butterfly-points in βX, Tychonoff products and weak Lindelöf numbers

Sergei Logunov

Abstract. Let X be the Tychonoff product $\prod_{\alpha<\tau} X_\alpha$ of τ-many Tychonoff non-single point spaces X_α. Let $p \in X^*$ be a point in the closure of some $G \subset X$ whose weak Lindelöf number is strictly less than the cofinality of τ. Then we show that $\beta X \setminus \{p\}$ is not normal. Under some additional assumptions, p is a butterfly-point in βX. In particular, this is true if either $X = \omega^\tau$ or $X = R^\tau$ and τ is infinite and not countably cofinal.

Keywords: Butterfly-point; non-normality point; Čech–Stone compactification; Tychonoff product; weak Lindelöf number

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1. Introduction

Let $X^* = \beta X \setminus X$ be the remainder of the Čech–Stone compactification βX of the Tychonoff space X. One of the most classical and intriguing question in the theory of the countable discrete space $\omega = \{0, 1, 2, \ldots\}$ is the following, see [3]:

Is $\omega^* \setminus \{p\}$ not normal for any point p of ω^*?

Despite great efforts so far it was only partially solved, see for example [2], [1] and [9]. But it could be answered for crowded spaces, see for example [4], [5] and [8]. It is closely related to the following concept of B. Shapirovskij: a point p of X is called a b-point or a butterfly-point in X, if there are subsets F and G of $X \setminus \{p\}$ such that $\{p\} = [F] \cap [G]$, see [7]. We say that a point p of X^* is a b-point in βX if there are subsets F and G of $X^* \setminus \{p\}$ with the following properties: $\{p\} = [F] \cap [G]$ and $[F \cup G] \subset X^*$. It clearly implies that $\beta X \setminus \{p\}$ is not normal. In [6] the following results were obtained:

Theorem. *Let a space $X = \prod_{\alpha<\tau} X_\alpha$ be the Tychonoff product of τ-many non-single point Tychonoff spaces X_α. Let a point $p \in X^*$ be in the closure of some subset $G \subset X$ with $C(G) < cf(\tau)$. Then $\beta X \setminus \{p\}$ is not normal.*

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We denote by $cf(\tau)$ the cofinality of τ, $d(X)$ the density and $C(X)$ the Suslin number of the space X. By the weak Lindelöf number, denoting it by $wL(X)$, we mean the minimal cardinal τ with the following property: every open cover $\mathcal{P}$ of X contains subfamily $\mathcal{P}'$ of cardinality at most τ with $\left[\bigcup \mathcal{P}'\right] = X$. Clearly, $wL(X) \leq C(X)$. By $\Psi^*(p, X)$ we denote the minimal cardinal τ with the following property: there is a family of τ open in βX sets $\{V_\alpha : \alpha < \tau\}$ such that

$$p \in \bigcap_{\alpha<\tau} V_\alpha \subset X^*.$$

We put $\Psi^*(X) = \sup\{\Psi^*(p, X) : p \in X^*\}$. Now we obtain

Theorem 1. *Let the space X be the Tychonoff product $\prod_{\alpha<\tau} X_\alpha$ of τ-many non-single point Tychonoff spaces X_α. Let a point $p \in X^*$ be in the closure of some $G \subset X$ with $wL(G) < cf(\tau)$. Then $\beta X \setminus \{p\}$ is not normal.*

Theorem 2. *Let the space X be the Tychonoff product $\prod_{\alpha<\tau} X_\alpha$ of τ-many non-single point Tychonoff spaces X_α. Let a point $p \in X^*$ be in the closure of some $G \subset X$ with $wL(G) < cf(\tau)$ and $\Psi^*(p, X) < cf(\tau)$. Then p is a butterfly-point in βX. Hence $\beta X \setminus \{p\}$ is not normal.*

Corollary 1. *Every point $p \in (\omega^\tau)^*$ is a butterfly-point in $\beta(\omega^\tau)$, if τ has uncountable cofinality.*

Corollary 2. *Every point $p \in (R^\tau)^*$ is a butterfly-point in $\beta(R^\tau)$, if τ has uncountable cofinality.*

Corollary 3. *Every point $p \in (X^\tau)^*$ is a butterfly-point in $\beta(X^\tau)$, if $d(X) + \Psi^*(X) < cf(\tau)$.*

By [6], p is a non-normality point of βX^τ under the assumptions of Corollaries 1–3.

2. Proofs

First, we prove Theorem 2 using its conditions and notation. Then we can easily prove Theorem 1 by omitting some unnecessary facts. By the Hewitt–Marczevski–Pondiczery theorem and its corollary on the Suslin number of products we obtain $C(X) < cf(\tau)$ in Corollaries 1–3. Therefore Theorem 2 implies these corollaries by Lemma 2.

In our paper all spaces are Tychonoff spaces, R is a straight line, $\{E_\gamma : \gamma < \kappa\}$ is a family of cardinality κ and [ ] is the closure operator in βX. Moreover, x_{α_0} is the α_0th coordinate of the point $x = (x_\alpha)_{\alpha<\tau}$ of X and U_{α_0} is the α_0th factor of the product $U = \prod_{\alpha<\tau} U_\alpha$. All the ordinals are strictly less then the number of factors τ.

Considering pairwise products, if necessary, we can assume that each X_α contains at least three pairwise different points, let us call them a_α, b_α and c_α. Then the points $a = (a_\alpha)_{\alpha<\tau}$, $b = (b_\alpha)_{\alpha<\tau}$ and $c = (c_\alpha)_{\alpha<\tau}$ of the space X are of great importance in our construction. We will present it only for a, assuming it is completely similar for b and c.

We fix an arbitrary base $\mathcal{B}_\alpha$ in every X_α and assume that the base $\mathcal{B}$ of X consists of all products of the form $U = \prod_{\alpha<\tau} U_\alpha$, where $U_\alpha \neq X_\alpha$ for at most finitely many $\alpha < \tau$ for which $U_\alpha \in \mathcal{B}_\alpha$. For every $U \in \mathcal{B}$ we put

$$\lambda(U) = \max\{\alpha < \tau \colon U_\alpha \neq X_\alpha\}.$$

If $\alpha < \tau$, then

$$U(\alpha, a) = \prod_{\gamma\leq\alpha} U_\gamma \times \prod_{\gamma>\alpha} \{a_\gamma\}.$$

We denote by $\mathcal{O}$ all open neighbourhoods of the point p in βX. For each $O \in \mathcal{O}$ we fix both a unique $O' \in \mathcal{O}$ with $[O'] \subset O$ and a unique subfamily $F = F(O)$ of $\mathcal{B}$ with the following properties:

$$| F | \leq wL(G), \qquad \bigcup F \subset O \qquad \text{and} \qquad O' \cap G \subset \Big[\bigcup F \cap G\Big],$$

see Lemma 1. We put

$$\lambda(O) = \lambda(F) = \sup\{\lambda(U) \colon U \in F\}.$$

If $\alpha < \tau$, then

$$F(\alpha, a) = \{U(\alpha, a) \colon U \in F\}.$$

Since $| F | < cf(\tau)$, then $\lambda(O) < \tau$. We set $\mathcal{F} = \{F(O) \colon O \in \mathcal{O}\}$ and

$$\mathcal{F}(\alpha, a) = \{F(\alpha, a) \colon F \in \mathcal{F}\}.$$

We fix $\{V_\gamma \colon \gamma < \kappa_o\} \subset \mathcal{O}$ so that $\kappa_0 < cf(\tau)$ and $p \in \bigcap_{\gamma<\kappa_0} [V_\gamma] \subset X^*$. Then $\lambda_0 = \sup_{\gamma<\kappa_0} \lambda(V_\gamma)$ satisfies $\lambda_0 < \tau$.

Lemma 1. *If $wL(X) \leq \tau$ and $O \subset X$ is open, then $wL([O]) \leq \tau$.*

Proof: Let $\mathcal{P}$ be any open cover of $[O]$ and $U' \cap [O] = U$ for any $U \in \mathcal{P}$ and some open $U' \subset X$. Then the open cover $\mathcal{R} = \{U' \colon U \in \mathcal{P}\} \cup \{X \setminus [O]\}$ of X contains subfamily $\widetilde{\mathcal{R}}$ of cardinality at most τ with $\big[\bigcup \widetilde{\mathcal{R}}\big] = X$ and $\widetilde{\mathcal{P}} = \{U \in \mathcal{P} \colon U' \in \widetilde{\mathcal{R}}\}$ is as required. □

Lemma 2. *We have $\Psi^*(p, X) \leq \sup_{\alpha<\tau} \Psi^*(X_\alpha)$.*

Proof: Let $g \colon \beta X \to \prod_{\alpha<\tau} \beta X_\alpha$ be the continuous extension of the identity mapping $X \to X$. For any $x \in X$ there is $O \in \mathcal{O}$ with $x \notin [O]$. But $q = g(p)$ is in the closure of $O \cap X = g(O \cap X)$. Hence $q \neq x$ implies $q \notin X$, i.e. $q_{\alpha_0} \in X^*_{\alpha_0}$ for some $\alpha_0 < \tau$. For $\kappa = \Psi^*(X_{\alpha_0})$ we get $q_{\alpha_0} \in \bigcap_{\gamma<\kappa} E_\gamma \subset X^*_{\alpha_o}$ for some E_γ open in βX_{α_0}. Let $f\colon \beta X \to \beta X_{\alpha_0}$ be composition of g with orthogonal projection $\prod_{\alpha<\tau} \beta X_\alpha \to \beta X_{\alpha_0}$. Then $p \in \bigcap_{\gamma<\kappa} f^{-1}E_\gamma \subset X^*$. □

Lemma 3. *If $F \in \mathcal{F}$ and $\alpha \geq \lambda(F)$, then $\bigcup F(\alpha, a) \subset \bigcup F$.*

Proof: If $U \in F$ and $U(\alpha, a)_\gamma \neq U_\gamma$, then $\gamma > \alpha \geq \lambda(U)$ implies $U_\gamma = X_\gamma$. Hence $U(\alpha, a) \subset U$ implies Lemma 3. □

Lemma 4. *For every $\alpha < \tau$ the family $\{\bigcup F(\alpha, a)\colon F \in \mathcal{F}\}$ is centered.*

Proof: Let $n \in N$ and $F_i \in \mathcal{F}$ for every $i < n$. Then $F_i = F(O_i)$ for some $O_i \in \mathcal{O}$ and $O_i' \cap G \subset [\bigcup F_i \cap G]$ by our construction. Since the nonempty $U = \bigcap_{i<n} O_i' \cap G$ is open in G, it is in the closure of every $\bigcup F_i \cap U$, which is open in U. There is a point $x = (x_\gamma)_{\gamma<\tau}$ of U with $x \in \bigcap_{i<n} (\bigcup F_i \cap U)$. Define a point $x' = (x'_\gamma)_{\gamma<\tau}$ of X as follows: $x'_\gamma = x_\gamma$ if $\gamma \leq \alpha$ and $x'_\gamma = a_\gamma$ otherwise. Then $x \in \bigcap_{i<n} U_i$ for some $U_i \in F_i$ implies

$$x' \in \bigcap_{i<n} U_i(\alpha, a) \subset \bigcap_{i<n} \bigcup F_i(\alpha, a).$$

□

For every $\alpha > \lambda_0$ we fix an arbitrary point $\xi_\alpha(a)$ in $\bigcap \{[\bigcup F]\colon F \in \mathcal{F}(\alpha, a)\}$ and put $A = \{\xi_\alpha(a)\colon \alpha > \lambda_0\}$.

Lemma 5. *If $O \in \mathcal{O}$ and $\alpha \geq \lambda(O)$, then $\xi_\alpha(a) \in [O]$.*

Proof: By Lemma 3 we obtain $\xi_\alpha(a) \in [\bigcup F(O)(\alpha, a)] \subset [\bigcup F(O)] \subset [O]$. □

Corollary 4. $p \in [A] \subset \bigcap_{\gamma<\kappa_0} [V_\gamma] \subset X^*$.

In the same way, b generates $B = \{\xi_\alpha(b)\colon \alpha > \lambda_0\}$ and c generates $C = \{\xi_\alpha(c)\colon \alpha > \lambda_0\}$, having the same properties as A. For every $\lambda > \lambda_0$ we put $A_\lambda = \{\xi_\alpha(a)\colon \alpha \in \lambda \setminus \lambda_0\}$, $B_\lambda = \{\xi_\alpha(b)\colon \alpha \in \lambda \setminus \lambda_0\}$ and $C_\lambda = \{\xi_\alpha(c)\colon \alpha \in \lambda \setminus \lambda_0\}$.

Lemma 6. *For every $\lambda > \lambda_0$ the closures of A_λ, B_λ and C_λ are pairwise disjoint.*

Proof: Let the continuous map $g\colon X_\lambda \to [0, 2]$ satisfy $g(a_\lambda) = 0$, $g(b_\lambda) = 1$ and $g(c_\lambda) = 2$. Its composition with the orthogonal projection $X \to X_\lambda$ has the continuous extension $f\colon \beta X \to [0, 2]$ ($f(x) = g(x_\lambda)$ for every $x \in X$). Then for any $\alpha \in \lambda \setminus \lambda_0$ and $F \in \mathcal{F}$ we obtain

$$f(\xi_\alpha(a)) \in f\Big[\bigcup F(\alpha, a)\Big] \subset \Big[f\Big(\bigcup F(\alpha, a)\Big)\Big] = \Big[f\Big(\bigcup_{U \in F} U(\alpha, a)\Big)\Big]$$
$$= \Big[\bigcup_{U \in F} f(U(\alpha, a))\Big] = \Big[\bigcup_{U \in F} g\{a_\lambda\}\Big] = \{O\}.$$

Hence $f(A_\lambda) = \{0\}$. Similarly, $f(B_\lambda) = \{1\}$ and $f(C_\lambda) = \{2\}$. □

Corollary 5. *At most one of the sets A, B and C contains p.*

Lemma 7. *The point p is a butterfly-point.*

Proof: Let $q \in X^*$ be not in the closure of some $O \in \mathcal{O}$. By Lemma 6 at most one of the sets $A_{\lambda(O)}$, $B_{\lambda(O)}$ and $C_{\lambda(O)}$ can contain q in its closure. By Lemma 5 the same is true for A, B and C. By Corollaries 4 and 5 our proof is complete. □

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S. Logunov:

Department for Algebra and Topology, Udmurt State University,
Universitetskaya 1, Izhevsk 426034, Russia

E-mail: olla209@yandex.ru

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