Counterexamples to Hedetniemi's conjecture and infinite Boolean lattices

Claude Tardif

Abstract. We prove that for any $c \geq 5$, there exists an infinite family $(G_n)_{n\in\mathbb{N}}$ of graphs such that $\chi(G_n) > c$ for all $n \in \mathbb{N}$ and $\chi(G_m \times G_n) \leq c$ for all $m \neq n$. These counterexamples to Hedetniemi's conjecture show that the Boolean lattice of exponential graphs with K_c as a base is infinite for $c \geq 5$.

Keywords: categorical product; graph colouring; Hedetniemi's conjecture

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1. Introduction

The *categorical product* of two graphs G and H is the graph $G\times H$ with vertex-set $V(G \times H) = V(G) \times V(H)$, whose edges are the pairs $\{(g_1, h_1), (g_2, h_2)\}$ such that $\{g_1, g_2\}$ is an edge of G and $\{h_1, h_2\}$ is an edge of H. Hedetniemi's conjecture of 1966 in [11] states that the chromatic number of a categorical product of graphs is equal to the minimum of the chromatic numbers of the factors. In 2019, Y. Shitov in [12] refuted the conjecture by constructing counterexamples for very large chromatic numbers.

Shitov's construction was subsequently adapted and modified by many authors. The asymptotic bounds on the difference $\min\{\chi(G), \chi(H)\} - \chi(G \times H)$ and on the ratio $\chi(G \times H)/\min\{\chi(G), \chi(H)\}$ were investigated in [18], [10], [23]; in [23] it is shown that the ratio $\chi(G \times H)/\min\{\chi(G), \chi(H)\}$ can get arbitrarily close to $1/2$. In another direction, the sizes and chromatic numbers of counterexamples were gradually decreased in [24], [16], [20]. By now it is known that for any $n \geq 5$, there exists pairs of $(n+1)$-chromatic graphs whose product is n-chromatic.

Following the work [4] of M. El-Zahar and N. Sauer, it is known that the product of n-chromatic graphs is n-chromatic for any $n \leq 4$. So, one outstanding unsolved question is whether the chromatic number of a product of 5-chromatic graphs can be 4. Also, the asymptotic behaviour of the so-called *Poljak–Rödl*

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function f is not yet understood. This function is defined by

$$f(n) = \min\{\chi(G \times H)\colon \chi(G) = \chi(H) = n\}.$$

It is now known that $\limsup_{n\to\infty}(f(n)/n) \leq 1/2$, but it is not yet known whether f is bounded or unbounded. In this paper, we expand on the known counterexamples to Hedetniemi's conjecture, but not towards either of these two problems. Our main result is the following.

Theorem 1. *For any $c \geq 5$, there exists an infinite family $(G_n)_{n\in\mathbb{N}}$ of graphs such that $\chi(G_n) > c$ for all $n \in \mathbb{N}$ and $\chi(G_m \times G_n) \leq c$ for all $m \neq n$.*

The reason to concoct this new style of disproof of Hedetniemi's conjecture is connected to the reason why Hedetniemi's conjecture was appealing in 1966. The identity $\chi(G\times H) = \min\{\chi(G), \chi(H)\}$ remains valid in many cases. In particular, it is not hard to show that a categorical product of nonbipartite graphs remains nonbipartite, by identifying an odd cycle in a categorical product of odd cycles. The structure of the critical graphs for higher chromatic numbers is not as well understood. However in 1966, Hajós' construction looked promising, and NP-completeness had not yet been formulated. It was reasonable to hope that the structure of the critical graphs might become sufficiently clear, and that the odd cycle argument could be adapted to higher chromatic numbers.

But things did not turn out that way. Instead of a general understanding of critical graphs, various lower bounds on the chromatic number have been devised over time. Among these, we find topological bounds, fractional chromatic numbers and much more. Each bound can be tight or not, depending on the class of graphs considered. Now, adaptations of Hedetniemi's conjecture can be formulated for each lower bound on the chromatic number. Indeed many of these have been proved over the years, see [14], [22], [5], [1]. However, the list of useful lower bounds on the chromatic number is not exhausted; where will others be found?

The natural context of Theorem 1 is that of the Boolean exponential lattices $K_c^{\mathcal{G}}$ that will be presented in the next section. Theorem 1 is a reformulation of the fact that $K_c^{\mathcal{G}}$ is infinite for any $c \geq 5$. Any lower bound β on the chromatic number that satisfies the identity $\beta(G \times H) = \min\{\beta(G), \beta(H)\}$ corresponds to a filter in $K_c^{\mathcal{G}}$. Therefore, understanding the structure of $K_c^{\mathcal{G}}$ may be relevant.

2. Exponential graphs

A *homomorphism* from a graph G to a graph H is a map φ from the vertex-set of G to that of H such that if $\{u, v\}$ is an edge of G, then $\{\varphi(u), \varphi(v)\}$ is an edge of H. For graphs G and H, we write $G \to H$ if there exists a homomorphism from G to H, and $G \leftrightarrow H$ if $G \to H$ and $H \to G$. Let $\mathcal{G}$ be the class of finite graphs. The relation "$\to$" is transitive on $\mathcal{G}$ and its quotient by the equivalence "$\leftrightarrow$" gives rise to a distributive lattice order on $\mathcal{G}/\leftrightarrow$, with "$\times$" as meet and the disjoint union "$+$" as join. The chromatic number of G is the least integer c such that $G \to K_c$, where K_c is the complete graph on the vertex-set $\{1, \ldots, c\}$. Hedetniemi's conjecture states that every complete graph is meet-irreducible in $\mathcal{G}/\leftrightarrow$. Indeed if $G \times H \to K_c$, then $K_c \leftrightarrow (G + K_c) \times (H + K_c)$; meet-irreducibility then implies $G + K_c \to K_c$ or $H + K_c \to K_c$.

For graphs K, G, the *exponential graph* K^G is the graph whose vertices are the functions from the vertex-set of G to that of K; and whose edges are the pairs $\{f_1, f_2\}$ of functions such that for every edge $\{g_1, g_2\}$ of G, $\{f_1(g_1), f_2(g_2)\}$ is an edge of K. The properties of exponentiation are well-known, and exposed in [3], [7]. We list some of the most relevant properties here:

(i) $G \times H \to K$ if and only if $H \to K^G$. In particular, K^G contains a loop if and only if $G \to K$.
(ii) If $G \to H$, then $K^H \to K^G$.
(iii) $K^{K^{K^G}} \leftrightarrow K^G$.
(iv) $K^{G+H} \leftrightarrow K^G \times K^H$.
(v) The constant maps induce a copy of K in K^G.

In particular, the identity $G \times K_c^G \to K_c$ always holds. Hence Hedetniemi's conjecture is equivalent to the statement that if $\chi(G) > c$, then $\chi(K_c^G) \leq c$, see [4].

We denote $K^{\mathcal{G}}$ the class of exponential graphs K^G, $G \in \mathcal{G}$, ordered by "$\to$" and quotiented by "$\leftrightarrow$". Its minimal element is K, and its maximal element is the single vertex with a loop, denoted $\mathring{1}$. (Note that we commit an abuse of notation and talk of graphs as elements of $K^{\mathcal{G}}$ even though formally, the elements of $K^{\mathcal{G}}$ are equivalence classes of graphs.) As a subposet of $\mathcal{G}$ ordered by "$\to$", $K^{\mathcal{G}}$ inherits the meet "$\times$" from $\mathcal{G}$, since $K^G \times K^H$ is homomorphically equivalent to the exponential graph K^{G+H}. However, $K^{\mathcal{G}}$ is not a sublattice of $\mathcal{G}$ because it is not closed under "$+$". However it turns out that $K^{\mathcal{G}}$ has its own join: $K^G \vee K^H = K^{K^{G+H}}$. It is well known, see [3], that $K^{\mathcal{G}}$ is a Boolean lattice for any K. Hedetniemi's conjecture states that for any c, $K_c^{\mathcal{G}}$ is the two-element lattice $\{K_c, \mathring{1}\}$. Theorem 1 is equivalent to the statement that for $c \geq 5$, $K_c^{\mathcal{G}}$ is an infinite Boolean lattice, see [17].

Theorem 1 will be proved by constructing an antichain from a chain in $K_c^{\mathcal{G}}$. We start with H_0 such that $\chi(H_0) > c$ and $\chi(K_c^{H_0}) > c$. It is easy to find an infinite sequence $(H_n)_{n \in \mathbb{N}}$ such that $H_{n+1} \to H_n$ and $\chi(H_n) > c$ for all $n \in \mathbb{N}$. We then have $K_c^{H_n} \to K_c^{H_{n+1}}$, so that $\chi(K_c^{H_n}) > c$ for all $n \in \mathbb{N}$. Consider the sequence $(G_n)_{n \in \mathbb{N}}$ defined by $G_n = H_n \times K_c^{H_{n+1}}$. It is not hard to show that

$(G_n)_{n\in\mathbb{N}}$ satisfies the conclusion of Theorem 1. Indeed for $m > n$ we have

$$G_m \times G_n \to H_m \times K_c^{H_{n+1}} \to H_{n+1} \times K_c^{H_{n+1}} \to K_c\,.$$

The difficulty is in proving $\chi(G_n) > c$, that is, $K_c^{H_{n+1}} \not\to K_c^{H_n}$. Even though it is easy to devise $(H_n)_{n\in\mathbb{N}}$ such that $H_n \not\to H_{n+1}$, the difficulty lies in having also $K_c^{H_{n+1}} \not\to K_c^{H_n}$, or equivalently $K_c^{K_c^{H_n}} \not\to K_c^{K_c^{H_{n+1}}}$. Our sequence $(H_n)_{n\in\mathbb{N}}$ will consist of some "universal graphs for wide colourings" presented in the next section.

3. Wide colourings

For a graph G and an odd integer $w = 2v+1$, the graph $\Gamma_w(G)$ has the same vertices as G, and two vertices are connected by an edge in $\Gamma_w(G)$ if they are endpoints of a walk of length w in G. Thus $\Gamma_w(G)$ contains loops only if the odd girth of G is at most w. Otherwise, $\Gamma_w(G)$ admits a proper vertex-colouring with sufficiently many colours. A proper colouring of $\Gamma_w(G)$ is called a $(v+1)$-*wide* colouring of G. (Here and below, we will use v for $\lfloor w/2 \rfloor$.)

The functor Γ_w has a right adjoint Ω_w which we describe next. For a graph H, $\Omega_w(H)$ is the graph whose vertices are the sequences $(X_0, X_1, \dots, X_v)$ of nonempty sets of vertices of H satisfying the following properties:

(i) X_0 is a singleton $\{x\}$;

(ii) for $i \in \{1, \dots, v\}$, every vertex of X_{i-1} is connected by an edge to every vertex of X_i;

(iii) for $i \in \{1, \dots, v-1\}$, $X_{i-1} \subseteq X_{i+1}$.

The edges of $\Omega_w(H)$ join the pairs $(X_0, \dots, X_v)$, $(Y_0, \dots, Y_v)$ satisfying the following properties:

(iv) For $i \in \{1, \dots, v\}$, $X_{i-1} \subseteq Y_i$ and $Y_{i-1} \subseteq X_i$;

(v) every vertex of X_v is connected by an edge to every vertex of Y_v.

Lemma 2 ([8], Theorem 3). *For two graphs G, H, we have $\Gamma_w(G) \to H$ if and only if $G \to \Omega_w(H)$.*

Thus a graph G admits a $(v+1)$-wide colouring with c colours if and only if $\Gamma_w(G) \to K_c$, that is, $G \to \Omega_w(K_c)$. The graphs $\Omega_w(K_c)$ are the "universal graphs for wide colourings". Note that their construction resembles that of Kneser graphs: adjacency in properties (ii) and (v) above is disjointness. As explained in [6], the existence of graphs that admit optimal colourings that are wide was conjectural at some point. This question is now settled:

Lemma 3 ([8], [19]). *For $c, v \geq 0$ and $w = 2v+1$, $\chi(\Omega_w(K_c)) = c$.*

In particular, $\Omega_w(K_c)$ is a c-chromatic graph which admits a $(v+1)$-wide colouring with c colours. (Note that the result is also implicit in [6], [2], [13], with alternative presentations of $\Omega_w(K_c)$.)

We will use the fact that $\Omega_w(K_c)$ is connected, which is not hard to prove from its explicit description. In [9], Corollary 1 it is shown that the odd girth of $\Omega_w(K_c)$ is $w + 2\lceil w/(c-2)\rceil$. Thus, while we have $\Omega_{w+2}(K_c) \to \Omega_w(K_c)$ for any odd w, we also have $\Omega_w(K_c) \not\to \Omega_{w+2}(K_c)$. We use a few other properties of Γ_w and Ω_w:

Lemma 4 ([9], Lemma 4)**.** *For any graph G, $\Gamma_w(\Omega_w(G)) \leftrightarrow G$.*

Two further properties are folklore. We include a proof for convenience.

Lemma 5. *Let a, b be odd integers. Then*

(i) $\Omega_a(G \times H) \leftrightarrow \Omega_a(G) \times \Omega_a(H)$;
(ii) $\Omega_a(\Omega_b(H)) \leftrightarrow \Omega_{ab}(H)$.

Proof: Property (i) holds for any right adjoint. We have

$$\begin{aligned} K \to \Omega_a(G \times H) &\Leftrightarrow \Gamma_a(K) \to G \times H \to G \qquad (H\text{, respectively}) \\ &\Leftrightarrow K \to \Omega_a(G) \qquad (\Omega_a(H)\text{, respectively}) \\ &\Leftrightarrow K \to \Omega_a(G) \times \Omega_a(H). \end{aligned}$$

Property (ii) follows from the identity $\Gamma_b(\Gamma_a(G)) = \Gamma_{ab}(G)$, which is obvious from the definition. We have

$$\begin{aligned} G \to \Omega_{ab}(H) &\Leftrightarrow \Gamma_b(\Gamma_a(G)) = \Gamma_{ab}(G) \to H \\ &\Leftrightarrow \Gamma_a(G) \to \Omega_b(H) \\ &\Leftrightarrow G \to \Omega_a(\Omega_b(H)). \end{aligned}$$

□

4. Main results

The graphs $\Omega_w(K_c)$ are used in the construction of counterexamples to Hedetniemi's conjecture in [16], [20]. In particular, the proof of Theorem 4.1 of [20] contains an implicit proof of the following statement.

Lemma 6. *For $c \geq 5$ and $w \geq 13$, the connected component of the constant maps in $K_c^{\Omega_w(K_{2c-2})}$ has chromatic number larger than c.*

In the next section, we adapt the argument to prove our main auxiliary result:

Lemma 7. *For $c \geq 5$ and $w \geq 7$,*

$$K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}} \to K_c + \Omega_w(K_{2c-2}) \to K_c^{K_c^{\Omega_w(K_{2c-2})}}.$$

Thus for $w \geq 13$, $K_c + \Omega_w(K_{2c-2})$ is sandwiched between two nontrivial elements of $K_c^{\mathcal{G}}$. Perhaps the stronger statement $K_c^{K_c^{\Omega_w(K_{2c-2})}} \leftrightarrow K_c + \Omega_w(K_{2c-2})$ always holds. Proposition 8 presents consequences of this hypothesis. We first show how Lemma 7 is used to prove Theorem 1.

Proof of Theorem 1: Let $\omega(0) = 5$ and $\omega(k) = 2\omega(k-1) + 3$ for $k \geq 1$. By Lemma 7, we have

$$K_c^{K_c^{\Omega_{\omega(k)}(K_{2c-2})}} \to K_c + \Omega_{\omega(k-1)+2}(K_{2c-2}).$$

Therefore $\Omega_{\omega(k-1)}(K_{2c-2}) \not\to K_c^{K_c^{\Omega_{\omega(k)}(K_{2c-2})}}$, since $\Omega_{\omega(k-1)}(K_{2c-2})$ is connected, it has chromatic number larger than that of K_c and odd girth smaller than that of $\Omega_{\omega(k-1)+2}(K_{2c-2})$. This implies

$$\chi\big(\Omega_{\omega(k-1)}(K_{2c-2}) \times K_c^{\Omega_{\omega(k)}(K_{2c-2})}\big) > c$$

for all $k \geq 1$. Hence the sequence $(G_k)_{k \geq 1}$ of graphs defined by

$$G_k = \Omega_{\omega(k-1)}(K_{2c-2}) \times K_c^{\Omega_{\omega(k)}(K_{2c-2})}$$

satisfies $\chi(G_k) > c$ for all k, and for $m > n$,

$$\begin{aligned} G_m \times G_n &\to \Omega_{\omega(m)}(K_{2c-2}) \times K_c^{\Omega_{\omega(n+1)}(K_{2c-2})} \\ &\to \Omega_{\omega(n+1)}(K_{2c-2}) \times K_c^{\Omega_{\omega(n+1)}(K_{2c-2})} \to K_c\,. \end{aligned}$$

□

The *weak Hedetniemi conjecture* states that the Poljak–Rödl function defined in the introduction is unbounded. That is, for every chromatic number c, there exists a bound $b(c)$ such that if G, H satisfy $\chi(G) \geq \chi(H) \geq b(c)$, then $\chi(G \times H) > c$. It is known, see [21], that the weak Hedetniemi conjecture is equivalent to the statement that $b(9)$ is well-defined.

If $K_c^{\mathcal{G}}$ were finite, then a fortiori $b(c)$ would be well-defined. Theorem 1 rules out this possibility for $c \geq 5$. Indeed, for the sequence $(G_k)_{k \geq 1}$ of graphs appearing in the proof of Theorem 1, $\big(K_c^{K_c^{G_k}}\big)_{k \geq 1}$ is an infinite antichain in $K_c^{\mathcal{G}}$. However, even though $K_c^{\mathcal{G}}$ is infinite, it is not known whether it contains graphs with arbitrarily large chromatic numbers. Again, if the chromatic numbers of elements of $K_c^{\mathcal{G}}$ were bounded, then $b(c)$ would be well-defined. Here we show that a strengthening of the conclusion of Lemma 7 would rule out this possibility as well.

Proposition 8. *Suppose that for* $c \geq 5$ *and* $w \geq 13$, *we have*

$$K_c^{K_c^{\Omega_w(K_{2c-2})}} \leftrightarrow K_c + \Omega_w(K_{2c-2}).$$

Then for all $c \geq 5$, $K_c^{\mathcal{G}}$ *contains graphs with arbitrarily large chromatic numbers.*

Proof: By Lemma 6, for all $c \geq 5$ and $w \geq 13$, we have

$$\Omega_w(K_{2c-2}) \times K_c^{\Omega_w(K_{2c-2})} \to K_c\,, \tag{1}$$

with both factors having chromatic number larger than c. Let $d \geq 2(2c-2)-2$ and $l \geq 13$ be integers such that

$$\Omega_l(K_d) \times K_{2c-2}^{\Omega_l(K_d)} \to K_{2c-2}, \tag{2}$$

with both factors having chromatic number larger than $2c-2$. By Lemma 5, applying Ω_w on both sides of (2) yields

$$\Omega_w\big(\Omega_l(K_d) \times K_{2c-2}^{\Omega_l(K_d)}\big) \leftrightarrow \Omega_w(\Omega_l(K_d)) \times \Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big) \to \Omega_w(K_{2c-2}).$$

We then multiply both sides by $K_c^{\Omega_w(K_{2c-2})}$ and use (1) to get

$$\Omega_w(\Omega_l(K_d)) \times \Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big) \times K_c^{\Omega_w(K_{2c-2})} \to \Omega_w(K_{2c-2}) \times K_c^{\Omega_w(K_{2c-2})} \to K_c\,,$$

where the leftmost factor is equivalent to $\Omega_{w\cdot l}(K_d)$ and the other two can be combined to give

$$\Omega_{w\cdot l}(K_d) \times \big(\Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big) \times K_c^{\Omega_w(K_{2c-2})}\big) \to K_c\,. \tag{3}$$

At first, this seems to show that $\Omega_{w\cdot l}(K_d)$ is a factor in counterexample to Hedetniemi's conjecture. However we have not yet shown that the second factor has chromatic number larger than c. We need to show that

$$\Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big) \times K_c^{\Omega_w(K_{2c-2})} \not\to K_c\,, \tag{4}$$

that is,

$$\Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big) \not\to K_c^{K_c^{\Omega_w(K_{2c-2})}}. \tag{5}$$

This is where the hypothesis is needed.

If $K_c^{K_c^{\Omega_w(K_{2c-2})}} \leftrightarrow K_c + \Omega_w(K_{2c-2})$, then any connected graph that admits a homomorphism to $K_c^{K_c^{\Omega_w(K_{2c-2})}}$ is either c colourable or admits a homomorphism to $\Omega_w(K_{2c-2})$. Consider the graph $\Omega_w(H)$, where H is the connected component of $K_{2c-2}^{\Omega_l(K_d)}$ containing the constant maps. These constant maps induce a copy of K_{2c-2} in H, and by Lemma 6, $\chi(H) > 2c-2$. The chromatic number of $\Omega_w(H)$ could be smaller than that of H, but we must have $\chi(\Omega_w(H)) \geq \chi(\Omega_w(K_{2c-2})) = 2c-2$. Therefore $\Omega_w(H)$ is not c colourable. Suppose that $\Omega_w(H) \to \Omega_w(K_{2c-2})$. By Lemma 4, we then have

$$H \leftrightarrow \Gamma_w(\Omega_w(H)) \to \Gamma_w(\Omega_w(K_{2c-2})) \leftrightarrow K_{2c-2}$$

which contradicts $\chi(H) > 2c-2$. Therefore, $\Omega_w(H)$, which is a subgraph of $\Omega_w\big(K_{2c-2}^{\Omega_l(K_d)}\big)$, does not admit a homomorphism to $K_c + \Omega_w(K_{2c-2})$ which is equivalent by hypothesis to $K_c^{K_c^{\Omega_w(K_{2c-2})}}$. This proves the statements (5) and (4) hence the second factor in (3) is not c-colourable. From this we conclude that $\chi\big(K_c^{\Omega_{w\cdot l}(K_d)}\big) > c$.

For c fixed, we use this result iteratively, putting $c(0) = c$, $w(0) = 13$ and for $k \geq 1$, $c(k) = 2c(k-1) - 2$ and $w(k) = 13w(k-1)$. Then for a fixed j, from

$$\chi\big(K_{c(i+1)}^{\Omega_{w(j)}(K_{c(i+j+2)})}\big) > c(i+1) \qquad \text{for all } i,$$

we get

$$\chi\big(K_{c(i)}^{\Omega_{w(j+1)}(K_{c(i+j+2)})}\big) > c(i) \qquad \text{for all } i.$$

Therefore, for all k, $K_c^{\mathcal{G}}$ contains the graph $K_c^{K_c^{\Omega_{w(k)}(K_{c(k+1)})}}$ which has chromatic number at least $c(k+1)$. □

It would be interesting to determine the triples c, w, n of parameters such that the identity $K_c^{K_c^{\Omega_w(K_n)}} \leftrightarrow K_c + \Omega_w(K_n)$ holds. In [15], a similar characterization of some double exponential directed graphs helps to characterize the exponential lattices with a complete graph as a base and finite directed graphs as exponents.

5. Proof of Lemma 7

To prove $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}} \to K_c + \Omega_w(K_{2c-2})$, we will show that the connected components of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ that are not c-colourable admit a $(v+1)$-wide colouring with $2c-2$ colours. Here, $w = 2v+1$, so that $2w-1 = 4v+1$. Hence the vertices of $\Omega_{2w-1}(K_{2c-2})$ are $(2v+1)$-tuples $(X_0, \ldots, X_{2v})$ of sets of vertices in K_{2c-2}. For a technical reason, it is best to represent the vertices of K_{2c-2} as pairs (x, q), with $x \in \{1, \ldots, c-1\}$ and $q \in \{1, 2\}$. The vertices of K_c are just the integers $1, \ldots, c$. The elements of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ will be represented by lower-case Greek letters, and those of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ by lower-case Latin letters. The functions on subsets of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ will be represented by upper-case

Greek letters; these include c-colourings of some components, automorphisms, and a $(v+1)$-wide colouring of some component with $2c-2$ colours.

5.1 Constant maps. We will denote $\iota_x\colon \Omega_{2w-1}(K_{2c-2}) \to K_c$ the constant map with constant value x. The set $C = \{\iota_1, \ldots, \iota_c\}$ is a complete subgraph of size c in $K_c^{\Omega_{2w-1}(K_{2c-2})}$. The elements f of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ fall into two categories:

(i) Those whose restriction to C is bijective.
(ii) Those whose restriction to C contains a repeated colour.

If the restriction of f to C is bijective and g is adjacent to f, then the restriction of g to C coincides with that of f. Thus, any connected component H of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ either has all its elements bijective and identical on C, or all its elements nonbijective on C.

Let H be a component of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ whose elements are nonbijective on C. Let $\Xi\colon H \to K_c$ be defined by letting $\Xi(f)$ be a colour x such that there are distinct values $x_f, y_f \in \{1, \ldots, c\}$ of C with $f(\iota_{x_f}) = f(\iota_{y_f}) = x$. If f and g are adjacent in H, without loss of generality we have $x_f \neq y_g$, so $\Xi(f) = f(\iota_{x_f}) \neq g(\iota_{y_g}) = \Xi(g)$. This shows that Ξ is a proper colouring, hence $H \to K_c$.

Let H be a component whose elements are bijective on C. Let π be the permutation of $V(K_c)$ such that $f(\iota_x) = \pi(x)$ for every f in H and $\iota_x \in C$. Consider the automorphism Π of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ defined by $\Pi(f) = \pi^{-1} \circ f$. Then H is isomorphic to $\Pi(H)$, and for every $f \in V(H)$, $\Pi(f)$ maps every element of C to its constant value.

Thus we can restrict our attention to the subgraph of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ induced by the functions that map every element of C to its constant value. For any two such adjacent functions f, g for every $\lambda \in K_c^{\Omega_{2w-1}(K_{2c-2})}$ and for every x that is not in the image of λ, we have $f(\lambda)$ adjacent to $g(\iota_x) = x$. Let H_{id} be the subgraph of $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}}$ consisting of the functions that map every element of C to its constant value, and moreover are not isolated. Then for every f in H_{id} and $\lambda \in K_c^{\Omega_{2w-1}(K_{2c-2})}$, $f(\lambda)$ is in the image of λ. We will construct a $(2v+1)$-wide colouring $\Phi\colon H_{\mathrm{id}} \to K_{2c-2}$.

5.2 Elements of $K_c^{\Omega_{2w-1}(K_{2c-2})}$. Recall that the elements of K_{2c-2} are denoted as pairs (x, q), with $x \in \{1, \ldots, c-1\}$ and $q \in \{1, 2\}$. Let γ be the element of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ defined by

$$\gamma(\{(x, q)\}, X_1, \ldots, X_{2v}) = x.$$

For every f in H_{id}, $f(\gamma) \in \{1, \ldots, c-1\}$. We define $\Phi_0(f) = f(\gamma)$. The $(v+1)$-wide colouring $\Phi\colon H_{\mathrm{id}} \to K_{2c-2}$ will be of the form $\Phi(f) = (\Phi_0(f), \Phi_1(f))$ for a suitably defined Φ_1.

For $x \in \{1, \ldots, c-1\}$, $l \in \{1, \ldots, 2v-1\}$ and $i, j \in \{1, \ldots, c\}$, the *two-colouring* $\tau_{i,j}^{x,l}$ is the element of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ defined by

$$\tau_{i,j}^{x,l}(X_0, \ldots, X_{2v}) = \begin{cases} i & \text{if } \{(x,1),(x,2)\} \cap X_l \neq \emptyset, \\ j & \text{otherwise.} \end{cases}$$

For $x \in \{1, \ldots, c-1\}$, we then have γ adjacent to $\tau_{c,x}^{x,1}$ in $K_c^{\Omega_{2w-1}(K_{2c-2})}$. Indeed, let $(X_0, \ldots, X_{2v})$ and $(Y_0, \ldots, Y_{2v})$ be neighbours in $\Omega_{2w-1}(K_{2c-2})$. We need to show that $\gamma(X_0, \ldots, X_{2v}) \neq \tau_{c,x}^{x,1}(Y_0, \ldots, Y_{2v})$, but the only way $\gamma(X_0, \ldots, X_{2v})$ can even be in the image of $\tau_{c,x}^{x,1}$ is if $X_0 \subseteq \{(x,1),(x,2)\}$ as $\gamma(X_0, \ldots, X_{2v}) = x$. By definition of adjacency in $\Omega_{2w-1}(K_{2c-2})$ we then have $\{(x,1),(x,2)\} \cap Y_1 \neq \emptyset$, so that $\tau_{c,x}^{x,1}(Y_0, \ldots, Y_{2v}) = c$. Thus, γ is adjacent to $\tau_{c,x}^{x,1}$.

Moreover, $\tau_{i,j}^{x,l}$ is adjacent to $\tau_{k,i}^{x,l+1}$ for any three distinct values $i, j, k \in \{1, \ldots, c\}$. Indeed i is the only common value in the image of these two functions. If $(Y_0, \ldots, Y_{2v})$ is adjacent to $(X_0, \ldots, X_{2v})$ with $\tau_{i,j}^{x,l}(X_0, \ldots, X_{2v}) = i$, then

$$\{(x,1),(x,2)\} \cap X_l \subseteq \{(x,1),(x,2)\} \cap Y_{l+1} \neq \emptyset,$$

therefore $\tau_{k,i}^{x,l+1}(Y_0, \ldots, Y_{2v}) = k$.

For $x \in \{1, \ldots, c-1\}$ and $i \in \{3, \ldots, c\}$, the *clique-member* κ_i^x is the element of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ defined by

$$\kappa_i^x(X_0, \ldots, X_{2v}) = \begin{cases} 1 & \text{if } (x,1) \in X_{2v}, \\ 2 & \text{if } (x,1) \notin X_{2v} \text{ and } (x,2) \in X_{2v}, \\ i & \text{otherwise.} \end{cases}$$

Then $\{\kappa_3^x, \ldots, \kappa_c^x\}$ induces a complete subgraph of $K_c^{\Omega_{2w-1}(K_{2c-2})}$. Indeed for $i \neq j$, the intersection of the images of κ_i^x and κ_j^x is $\{1, 2\}$. If $(X_0, \ldots, X_{2v})$ and $(Y_0, \ldots, Y_{2v})$ are neighbours in $\Omega_{2w-1}(K_{2c-2})$, then X_{2v} and Y_{2v} are disjoint, hence they cannot contain the same element (x, q). Therefore $\kappa_i^x(X_0, \ldots, X_{2v}) \neq \kappa_j^x(Y_0, \ldots, Y_{2v})$. Moreover, for any $j \notin \{1, 2, i\}$, $\tau_{i,j}^{x,2v-1}$ is adjacent to κ_i^x for the same reason that $\tau_{i,j}^{x,l}$ is adjacent to $\tau_{k,i}^{x,l+1}$.

In summary, the various γ, $\tau_{i,j}^{x,l}$ and κ_i^x defined here are connected in the following ways:

- γ is adjacent to $\tau_{c,x}^{x,1}$ for $x \in \{1, \ldots, c-1\}$;
- $\tau_{i,j}^{x,l}$ is adjacent to $\tau_{k,i}^{x,l+1}$ for $x \in \{1, \ldots, c-1\}$, i, j, k distinct in $\{1, \ldots, c\}$, and $l \in \{1, \ldots, 2v-1\}$;

- $\tau_{i,j}^{x,2v-1}$ is adjacent to κ_i^x for $x \in \{1,\dots,c-1\}$, $i \in \{3,\dots,c\}$ and $j \in \{1,\dots,c\} \setminus \{i\}$;
- κ_i^x is adjacent to κ_j^x for $x \in \{1,\dots,c-1\}$ and i, j distinct in $\{3,\dots,c\}$.

5.3 Subsets of H_{id}. For $x \in \{1,\dots,c-1\}$, we now consider the subsets N_0^x, N_1^x,…,N_{2v}^x of elements of H_{id} defined recursively as follows. Let N_0^x be the set of functions f such that $f(\gamma) = x$, that is, the set $\Phi_0^{-1}(x)$ according to the notation introduced in Section 5.2. Then for $l \in \{1,\dots,2v\}$, N_l^x is defined as the set containing all f in $V(H_{\mathrm{id}})$ that have a neighbour g in N_{l-1}^x. The properties of the various elements γ, $\tau_{i,j}^{x,l}$, κ_i^x of $K_c^{\Omega_{2w-1}(K_{2c-2})}$ discussed above have the following consequences:

(i) If $f \in N_0^x$, then $f(\gamma) = x$ by definition.
(ii) If $f \in N_1^x$ is adjacent to $g \in N_0^x$, then $f(\tau_{c,x}^{x,1}) \neq g(\gamma) = x$. Therefore $f(\tau_{c,x}^{x,1}) = c$.
(iii) For $l \geq 2$, if $f \in N_l^x$ is adjacent to $g \in N_{l-1}^x$, then $f(\tau_{k,i}^{x,l}) \neq g(\tau_{i,j}^{x,l-1})$ when i, j, k are distinct. Thus if $g(\tau_{i,j}^{x,l-1}) = i$, then $f(\tau_{k,i}^{x,l}) = k$. As l increases, the consequences are as follows:
$l = 2$: $f(\tau_{k,c}^{x,2}) = k$ for $k \notin \{x, c\}$, since $g(\tau_{c,x}^{x,1}) = c$;
$l = 3$: $f(\tau_{j,k}^{x,3}) = j$ for $j \neq c$ and $k \notin \{j, x, c\}$, since $g(\tau_{k,c}^{x,2}) = k$;
$l = 4$: $f(\tau_{i,j}^{x,4}) = i$ for $i \neq j \neq c$, since for $k \notin \{i, j, x, c\}$, $g(\tau_{j,k}^{x,3}) = j$;
$l \geq 5$: $f(\tau_{k,i}^{x,l}) = k$ for $k \neq i$, since for $j \notin \{i, k, c\}$, $g(\tau_{i,j}^{x,l-1}) = i$.
(iv) Finally if $f \in N_{2v}^x$ is adjacent to $g \in N_{2v-1}^x$, then for $k \in \{3,\dots,c\}$ we have $f(\kappa_k^x) \neq g(\tau_{k,i}^{x,2v-1}) = k$ for $i \neq k$, thus $f(\kappa_k^x) \in \{1, 2\}$.

Note that N_l^x contains N_{l-2}^x for $l \in \{2, 3, \dots, 2v\}$. We define $\Psi_x \colon N_{2v}^x \to \{1, 2\}$ by letting $\Psi_x(f)$ be a colour used at least twice as $f(\kappa_k^x)$, $k \in \{3,\dots,c\}$. The map $\Phi_1 \colon H_{\mathrm{id}} \to \{1, 2\}$ is $\bigcup_{x \in \{1,\dots,c-1\}} \Psi_x|_{N_0^x}$, that is, $\Phi_1(f) = \Psi_x(f)$ when $x = \Phi_0(f)$.

It remains to show that $\Phi = (\Phi_0, \Phi_1)$ is a $(v+1)$-wide colouring of H_{id}. Let $f_0, \dots, f_{2v+1}$ be a walk in H_{id}. If $\Phi_0(f_0) \neq \Phi_0(f_{2v+1})$, then $\Phi(f_0) \neq \Phi(f_{2v+1})$. Suppose that $\Phi_0(f_0) = \Phi_0(f_{2v+1}) = x$. Then each f_n is in N_{2v}^x, since it is joined to either f_0 or f_{2v+1} by a $2v$-walk. Thus Ψ_x is defined on the whole walk $f_0, \dots, f_{2v+1}$, and it must take values that alternate between 1 and 2 as n goes from 0 to $2v+1$. Indeed for $n \in \{0,\dots,2v\}$ there exist distinct values $i, j \in \{3,\dots,c\}$ such that $\Psi_x(f_n) = f_n(\kappa_i^x) \neq f_{n+1}(\kappa_j^x) = \Psi_x(f_{n+1})$. Therefore $\Phi_1(f_0) = \Psi_x(f_0) \neq \Psi_x(f_{2v+1}) = \Phi_1(f_{2v+1})$. This shows that $\Phi = (\Phi_0, \Phi_1)$ is a wide colouring of H_{id}.

Recall that the $(v+1)$-wide colouring $\Phi \colon H_{\mathrm{id}} \to K_{2c-2}$ can be viewed as a colouring $\Phi \colon \Gamma_w(H_{\mathrm{id}}) \to K_{2c-2}$. By Lemma 2, there exists a homomorphism $\Phi' \colon H_{\mathrm{id}} \to \Omega_w(K_{2c-2})$.

5.4 Conclusion. We have outlined the construction of a homomorphism

$$\Upsilon\colon\ K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}} \to K_c + \Omega_w(K_{2c-2})$$

as follows:

- ◦ If f is nonbijective on C, then $\Upsilon(f) = \Xi(f)$ in K_c;
- ◦ otherwise if f is isolated, then $\Upsilon(f)$ can be any vertex of $K_c + \Omega_w(K_{2c-2})$;
- ◦ otherwise, $\Upsilon(f) = \Phi'(\Pi(f))$ in $\Omega_w(K_{2c-2})$.

Therefore, $K_c^{K_c^{\Omega_{2w-1}(K_{2c-2})}} \to K_c + \Omega_w(K_{2c-2})$. The second statement $K_c + \Omega_w(K_{2c-2}) \to K_c^{K_c^{\Omega_w(K_{2c-2})}}$ is just a restatement of the basic property $(K_c + \Omega_w(K_{2c-2})) \times K_c^{\Omega_w(K_{2c-2})} \to K_c$. This completes the proof of Lemma 7. □

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C. Tardif:
Department of Mathematics and Computer Science,
Royal Military College of Canada, 13 General Crerar Crescent, Kingston, ON K7K 7B4, Ontario, Canada

E-mail: Claude.Tardif@rmc.ca

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