Exponential separability is preserved by some products

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Abstract. We show that exponential separability is an inverse invariant of closed maps with countably compact exponentially separable fibers. This implies that it is preserved by products with a scattered compact factor and in the products of sequential countably compact spaces. We also provide an example of a σ-compact crowded space in which all countable subspaces are scattered. If X is a Lindelöf space and every $Y \subset X$ with $|Y| \leq 2^{\omega_1}$ is scattered, then X is functionally countable; if every $Y \subset X$ with $|Y| \leq 2^{\mathfrak{c}}$ is scattered, then X is exponentially separable. A Lindelöf Σ-space X must be exponentially separable provided that every $Y \subset X$ with $|Y| \leq \mathfrak{c}$ is scattered. Under the Luzin axiom ($2^{\omega_1} > \mathfrak{c}$) we characterize weak exponential separability of $C_p(X, [0,1])$ for any metrizable space X. Our results solve several published open questions.

Keywords: Lindelöf space; scattered space; σ-product; function space; P-space; exponentially separable space; product; functionally countable space; weakly exponentially separable space

Classification: 54G12, 54G10, 54C35, 54D65

1. Introduction

It was known since late fifties of the last century that real-valued continuous images of a compact scattered space are countable. This was a motivation for introducing *functionally countable spaces* as those ones whose all real-valued continuous images are countable, see [8] and [7]. The importance of this class is best illustrated by the fact that ordinals, σ-products of Cantor cubes and Lindelöf scattered spaces are functionally countable; besides, a compact space is functionally countable if and only if it is scattered.

R. Levy and M. Matveev considered in [5] the spaces with a dense functionally countable subspace and proved that such spaces are functionally countable if they have a weaker version of P-property. In [6], J. T. Moore provided a ZFC (Zermelo–Fraenkel set theory) example of an L-space which is functionally countable. F. Galvin asked in 1983 whether the product of two functionally countable spaces is functionally countable; this question was answered negatively in 1985, see [2].

DOI 10.14712/1213-7243.2022.021

The paper [12] features a natural strengthening of functional countability, called *exponential separability.* It was proved in [12] that both Lindelöf P-spaces and Lindelöf scattered spaces are exponentially separable. Besides, exponential separability coincides with functional countability in perfectly normal spaces, in normal countably compact spaces and Lindelöf Σ-spaces.

In the paper [15] V. V. Tkachuk introduced *weak exponential separability* and proved that all separable spaces, all countably compact spaces as well as all Lindelöf Σ-spaces are weakly exponentially separable. Although $\mathrm{ext}(X) \leq \omega$ for every exponentially separable space X, it was shown in [16] that the extent of a weakly exponentially separable space can be arbitrarily large.

The main motivation for this work was a theorem from the paper [15] which says that the product $X \times Y$ of two functionally countable spaces X and Y must be functionally countable if $X \times Y$ is Lindelöf. While $X \times Y$ can fail to be even weakly exponentially separable for some exponentially separable spaces X and Y, see [15, Example 4.10], the above-mentioned result gives hope that, under some restrictions on exponentially separable spaces X and Y, their product might be exponentially separable. This is, indeed, the case: we prove that the product of two exponentially separable spaces is exponentially separable if one of them is compact; besides, any finite product of sequential countably compact exponentially separable spaces is exponentially separable.

It was known for decades that a compact space K is scattered if and only if all countable subsets of K are scattered. I. Juhász and J. van Mill constructed in [4] an example of a countably compact crowded space in which all countable subsets are scattered. Answering a question from [15], we show that for any infinite cardinal κ, there exists a σ-compact crowded space X in which all subspaces of cardinality less than κ are scattered. This makes it natural to try to find out whether a space X is functionally countable if all of its small subspaces are scattered. We prove, among other things, that a Lindelöf space X is functionally countable if every $Y \subset X$ with $|Y| \leq 2^{\omega_1}$ is scattered.

The final part of this paper contains results on weak exponential separability in function spaces. We show that weak exponential separability of $C_p(X, [0,1])$ implies that $\mathrm{ext}(C_p(X, [0,1])) \leq \mathfrak{c}$ for any metrizable space X and characterize, under Luzin's axiom ($2^{\omega_1} > \mathfrak{c}$), weak exponential separability in $C_p(X, [0,1])$ for any metrizable space X. Our results solve several published open questions.

2. Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$; besides, $\tau(A, X) = \{U \in \tau(X) \colon A \subset U\}$ for any set $A \subset X$. The set $\mathbb{R}$ is the real line with its usual topology, $\mathbb{I} = [0, 1] \subset \mathbb{R}$

while $\mathbb{N} = \omega\backslash\{0\}$ and $\mathbb{D} = \{0, 1\} \subset \mathbb{R}$. For any space X, the expression $I(X)$ always stands for the set of isolated points of X. Say that X is a *P-space* if every G_δ-subset of X is open. A space X is *crowded* if it has no isolated points; the space X is *scattered* if every nonempty subspace of X has an isolated point. The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic. If A is a set and κ is a cardinal, then $[A]^{\leq\kappa} = \{B \subset A \colon |B| \leq \kappa\}$.

Given a space X, let $\mathrm{ext}(X) = \sup\{|D| \colon D$ is a closed discrete subspace of $X\}$; the cardinal $\mathrm{ext}(X)$ is the *extent* of X. Say that a family $\mathcal{F}$ of subsets of a space X is *a network with respect to a cover* $\mathcal{C}$ if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is *Lindelöf* Σ (or has *the Lindelöf* Σ*-property*) if there exists a countable family $\mathcal{F}$ of subsets of X such that $\mathcal{F}$ is a network with respect to a compact cover $\mathcal{C}$ of the space X. A space X is *functionally separable* if for any continuous function $f \colon X \to \mathbb{R}$, there exists a countable set $A \subset X$ such that $f(X) = f(\bar{A})$. The cardinal $d(X) = \min\{|A| \colon \bar{A} = X\}$ is called the *density* of X and $w(X) = \min\{|\mathcal{B}| \colon \mathcal{B}$ is a base of $X\}$ is the *weight* of X.

Suppose that X is a space and $\mathcal{F}$ is a family of subsets of X; then $\bigwedge\mathcal{F}$ is the family of all finite intersections of elements of $\mathcal{F}$. Say that a set $A \subset X$ is *strongly dense in* $\mathcal{F}$ if $A \cap \bigcap\mathcal{F}' \neq \emptyset$ for any family $\mathcal{F}' \subset \mathcal{F}$ such that $\bigcap\mathcal{F}' \neq \emptyset$. The family $\mathcal{F}$ is called *strongly separable* if some countable subset of X is strongly dense in $\mathcal{F}$. The space X is called *exponentially separable* if every countable family of closed subsets of X is strongly separable. Furthermore, X is *weakly exponentially separable* if for each countable family $\mathcal{F}$ of closed subsets of X, there exists a countable set $A \subset X$ such that $\bar{A}$ is strongly dense in $\mathcal{F}$.

The expression $C(X, Y)$ denotes the set of all continuous maps from a space X to a space Y. We follow the usual practice to write $C(X)$ instead of $C(X, \mathbb{R})$. The space $C_p(X)$ is the set $C(X)$ endowed with the pointwise convergence topology. The rest of our topological notation is standard and follows the book [1]. The books [9], [10], [11] contain all necessary facts and notions of C_p-theory.

3. Exponential separability and products

It was proved in [15] that the product $X \times Y$ of functionally countable spaces X and Y is functionally countable if it is Lindelöf. The purpose of this section is to give sufficient conditions for preserving exponential separability by finite products. In particular, it will be shown that the product of two exponentially separable spaces must be exponentially separable if one of them is compact.

We will have to make use of the following lemma proved in [15].

Lemma 3.1. *Given a space X, suppose that $\mathcal{F}$ is a countable family of closed subsets of X. If $Y \subset X$ is strongly dense in $\mathcal{F}$ and a set $A \subset Y$ is strongly dense in the family $\mathcal{F}|Y = \{F \cap Y : F \in \mathcal{F}\}$, then A is strongly dense in $\mathcal{F}$. In particular, if the family $\mathcal{F}|Y$ is strongly separable in Y, then $\mathcal{F}$ is strongly separable in X.*

The following theorem is our main tool for proving that some nice products preserve exponential separability.

Theorem 3.2. *Suppose that $f : X \to Y$ is a continuous closed onto map and $f^{-1}(y)$ is exponentially separable for every $y \in Y$. If the fiber $f^{-1}(y)$ is countably compact for each $y \in Y$, and the space Y is exponentially separable, then X is also exponentially separable.*

Proof: Let $\mathcal{F} = \{F_n : n \in \omega\}$ be a family of closed subsets of X. If $G_n = f(F_n)$ for every $n \in \omega$, then $\mathcal{G} = \{G_n : n \in \omega\}$ is a family of closed subsets of Y. If $\mathcal{H} = \bigwedge \mathcal{G}$, then $\mathcal{H}$ is also a countable family of closed subsets of Y so there is a countable set $B \subset Y$ which is strongly dense in $\mathcal{H}$. We claim that $f^{-1}(B)$ is strongly dense in $\mathcal{F}$. To see it, take any set $S \subset \omega$ such that $Q = \bigcap_{n\in S} F_n \neq \emptyset$. Take a decreasing sequence $\{Q_n : n \in \omega\} \subset \bigwedge \mathcal{F}$ such that $Q = \bigcap_{n\in\omega} Q_n$. Our choice of the set B shows that there exists $b \in B \cap \bigcap_{n\in\omega} f(Q_n)$ whence $f^{-1}(b) \cap Q_n \neq \emptyset$ for every $n \in \omega$. Since the family $\{f^{-1}(b) \cap Q_n : n \in \omega\}$ is decreasing and consists of nonempty closed subsets of a countably compact space $f^{-1}(b)$, we conclude that $f^{-1}(b) \cap Q \neq \emptyset$ and therefore $f^{-1}(B)$ is strongly dense in $\mathcal{F}$.

By exponential separability of the set $f^{-1}(b)$, there exists a countable set $A_b \subset f^{-1}(b)$ which is strongly dense in the family $\{F_n \cap f^{-1}(b) : n \in \omega\}$ for every $b \in B$. Then $A = \bigcup_{b\in B} A_b$ is easily seen to be a countable set that is strongly dense in $\mathcal{F}$. □

Corollary 3.3. *Assume that $f : X \to Y$ is a perfect map such that Y is an exponentially separable space. If $f^{-1}(y)$ is exponentially separable for any $y \in Y$, then X is exponentially separable.*

Corollary 3.4. *Given an exponentially separable space X, if K is a scattered compact space, then $X \times K$ is exponentially separable.*

Proof: The projection $p : X \times K \to X$ is known to be a perfect map and the set $p^{-1}(x) \simeq K$ is exponentially separable being a compact scattered space, see [12, Corollary 3.14] for any $x \in X$. Corollary 3.3 does the rest. □

Proposition 3.5. *Suppose that X_i is a countably compact sequential space which is exponentially separable for each $i \leq n$. Then the product $X = \prod_{i\leq n} X_i$ is exponentially separable.*

PROOF: We will proceed by induction on n. As there is nothing to prove if $n = 1$, assume that $n \in \mathbb{N}$ and any product of at most n-many sequential countably compact exponentially separable spaces is exponentially separable. Take any collection $\{X_1, \ldots, X_n, X_{n+1}\}$ of sequential countably compact exponentially separable spaces and consider the space $X = \prod_{i \leq n+1} X_i$; we will also need the product $Y = X_1 \times \ldots \times X_n$ which is exponentially separable by the induction hypothesis.

Any countable product of sequential countably compact spaces is countably compact, see Proposition 3.4.B of [17], so both spaces X and Y are countably compact; let $p\colon X \to X_{n+1}$ be the projection. It is straightforward that any continuous map of a countably compact space onto a sequential one is closed so our projection p is a closed map.

Given any $x \in X_{n+1}$, the fiber $p^{-1}(x)$ is homeomorphic to the space Y so it is countably compact and exponentially separable; this makes it possible to apply Theorem 3.2 to convince ourselves that X is exponentially separable. □

Observation 3.6. It was proved in [15, Example 4.10] that the product of the one-point Lindelöfication L of a discrete space of cardinality ω_1 and the ordinal ω_1 with its order topology is not weakly exponentially separable. Therefore the product of a scattered Lindelöf P-space with a first countable countably compact exponentially separable space can fail to be exponentially separable. This observation should be compared with Corollary 3.4 and Proposition 3.5.

4. Small scattered subspaces of Lindelöf spaces

It is a well-known folklore result that a compact space X is scattered if and only if all countable subsets of X are scattered. In [4], I. Juhász and J. van Mill gave an example of a countably compact crowded space in which all countable subsets are scattered. It was asked in [15, Question 5.5] whether a Lindelöf Σ-space X must be scattered provided that all countable subsets of X are scattered. The theorem below shows that the answer is negative even for σ-compact spaces. This makes it natural to study what happens in the spaces in which all small subspaces are scattered.

Theorem 4.1. *For any uncountable cardinal κ, there exists a space M_κ with the following properties:*

(a) *M_κ is σ-compact and dense in $\mathbb{D}^\kappa$;*
(b) *if $Y \subset M_k$ and $|Y| < \kappa$, then Y is scattered.*

Proof: Fix a disjoint family $\mathcal{S} = \{S_n : n \in \omega\}$ of subsets of κ such that $|S_n| = \kappa$ for any $n \in \omega$. Let $\xi_n(\alpha) = 1$ if $\alpha \in S_n$ and $\xi_n(\alpha) = 0$ for all $\alpha \in \kappa \backslash S_n$; then ξ_n is the characteristic function of S_n for each $n \in \omega$.

The set $\sigma_n = \{x \in \mathbb{D}^\kappa : |x^{-1}(1)| \leq n\}$ is well known to be a scattered compact subspace of $\mathbb{D}^\kappa$ and therefore $Q_n = \{\xi_n + x : x \in \sigma_n\}$ is also a scattered compact subspace of $\mathbb{D}^\kappa$. We claim that $M_\kappa = \bigcup_{n\in\omega} Q_n$ is as promised. It is immediate that M_k is σ-compact. Given any $U \in \tau^*(\mathbb{D}^\kappa)$, there is a finite set $A \subset \kappa$ and a function $\varphi : A \to \mathbb{D}$ such that $[\varphi, A] = \{x \in \mathbb{D}^\kappa : x|A = \varphi\} \subset U$. If $|A| = n$, then define a function $x : \kappa \to \mathbb{D}$ by the equalities $x|A = \varphi$ and $x|(\kappa\backslash A) = \xi_n|(\kappa\backslash A)$. Then $x \in Q_n \cap [\varphi, A] \subset M_\kappa \cap U$ and therefore the set M_κ is dense in $\mathbb{D}^\kappa$.

Now, if $Y \subset M_\kappa$ and $|Y| < \kappa$, then let $Y_n = Y \cap Q_n$ for every $n \in \omega$; we will show that every Y_n is a clopen subset of Y. Since $\{Y_n : n \in \omega\}$ is a disjoint partition of Y, it suffices to show that Y_n is open in Y for every $n \in \omega$. Take any $x \in Q_n$ and let $B = \bigcup\{y^{-1}(1) \cap S_n : y \in Y\backslash Y_n\}$; then $|B| < \kappa$ and hence we can pick $\alpha \in S_n\backslash(B \cup x^{-1}(0))$. Then $V = \{y \in \mathbb{D}^\kappa : y(\alpha) = 1\}$ is an open set such that $x \in V$ and $V \cap (Y\backslash Y_n) = \emptyset$. Thus $\{Y_n : n \in \omega\}$ is a clopen partition of Y; since every Y_n is scattered, the space Y is scattered as well. □

Corollary 4.2. *There is a σ-compact dense subspace M of the Cantor cube $\mathbb{D}^{\omega_1}$ in which every countable subset is scattered.*

Lemma 4.3. *Given a space X and an infinite cardinal κ, assume that every $Y \in [X]^{\leq\kappa}$ is scattered. Then every subspace $Y \subset X$ such that $hd(Y) \leq \kappa$ is scattered. In particular, if $Y \subset X$ and $w(Y) \leq \kappa$, then Y is scattered.*

Proof: Take any nonempty set $F \subset Y$ and observe that we have the inequalities $d(F) \leq hd(Y) \leq \kappa$ so there exists a set $D \subset F$ of cardinality less than or equal to κ which is dense in F. The set D being scattered, it has an isolated point x; it is standard to see that x is also isolated in F so Y is scattered. □

Theorem 4.4. *Suppose that X is a Lindelöf space. If every $Y \subset X$ with $|Y| \leq 2^{\omega_1}$ is scattered, then X is functionally countable.*

Proof: Suppose that the set $f(X)$ is uncountable for some continuous function $f : X \to \mathbb{R}$. Choose a set $A \subset X$ such that $|A| = \omega_1$ and $f|A$ is injective. Observe that the subspace $Y = \bar{A}$ is Lindelöf and $w(Y) \leq 2^{\omega_1}$ which, together with Lemma 4.3 shows that Y is scattered. Since $f(A) \subset f(Y)$, we conclude that $f|Y$ is a continuous function on Y with an uncountable image. However, Lindelöf scattered spaces are functionally countable; this contradiction proves that X is functionally countable. □

Corollary 4.5. *Under $2^{\omega_1} = \mathfrak{c}$, if X is a Lindelöf space and every set $Y \in [X]^{\leq\mathfrak{c}}$ is scattered, then X is functionally countable.*

Proposition 4.6. *Assume that X is a functionally separable Lindelöf space in which every subset $Y \in [X]^{\leq \mathfrak{c}}$ is scattered. Then X is functionally countable.*

PROOF: Take any continuous function $f: X \to \mathbb{R}$; there exists a countable set $A \subset X$ such that $f(X) = f(\bar{A})$. The subspace $Y = \bar{A}$ has weight not exceeding $\mathfrak{c}$ so it is scattered by Lemma 4.3. Since any Lindelöf scattered space must be functionally countable, the space Y is functionally countable and therefore $f(X) = f(Y)$ is countable; this proves that X is functionally countable. □

Theorem 4.7. *Suppose that X is a Lindelöf space. If every $Y \subset X$ with $|Y| \leq 2^{\mathfrak{c}}$ is scattered, then X is exponentially separable.*

PROOF: Take a countable family $\mathcal{F}$ of closed subsets of X. Since there are at most $\mathfrak{c}$-many nonempty intersections of subfamilies of $\mathcal{F}$, we can choose a set $A \subset X$ such that $|A| \leq \mathfrak{c}$ and A is strongly dense in $\mathcal{F}$. The set $Y = \bar{A}$ is Lindelöf and $w(Y) \leq 2^{\mathfrak{c}}$ which, together with Lemma 4.3, shows that Y is scattered.

Since scattered Lindelöf spaces are exponentially separable, there exists a countable set $B \subset Y$ which is strongly dense in the family $\{F \cap Y : F \in \mathcal{F}\}$. Finally, apply Lemma 3.1 to see that B is also strongly dense in $\mathcal{F}$ so X is exponentially separable. □

Proposition 4.8. *Assume that X is a weakly exponentially separable Lindelöf space in which every subset $Y \in [X]^{\leq \mathfrak{c}}$ is scattered. Then X is exponentially separable.*

PROOF: Take any countable family $\mathcal{F}$ of closed subsets of X. There exists a countable set $A \subset X$ such that $Y = \bar{A}$ is strongly dense in $\mathcal{F}$. The subspace Y has weight not exceeding $\mathfrak{c}$ so it is scattered by Lemma 4.3. Since any Lindelöf scattered space must be exponentially separable by Proposition 3.4 of [12], we can find a countable set $B \subset Y$ which is strongly dense in $\{F \cap Y : F \in \mathcal{F}\}$. It follows from Lemma 3.1 that B is strongly dense in $\mathcal{F}$ so X is, indeed, exponentially separable. □

Corollary 4.9. *If X is a Lindelöf Σ-space and every subset $Y \in [X]^{\leq \mathfrak{c}}$ is scattered, then X is exponentially separable.*

PROOF: Any Lindelöf Σ-space must be weakly exponentially separable according to [15, Theorem 4.8] so we can apply Proposition 4.8 to convince ourselves that the space X is exponentially separable. □

5. Weak exponential separability in $\mathbf{C_p}(\mathbf{X}, \mathbb{I})$

Under Luzin's axiom ($2^{\omega_1} > \mathfrak{c}$), we will give a complete characterization of weak exponential separability in $C_p(X, \mathbb{I})$ for metrizable spaces X. The following statement generalizes Theorem 3.15 of [16].

Proposition 5.1. *Suppose that X is a space that has a closed discrete subset D such that $|D| = \kappa \leq \mathfrak{c}$. If Y is a space such that $X \times Y$ is weakly exponentially separable, then any set $E \subset Y$ of cardinality not exceeding κ is dominated by a countable set, i.e., there exists a countable set $A \subset Y$ such that $E \subset \bar{A}$.*

Proof: Let $\pi \colon X \times Y \to Y$ be the projection and choose a faithful enumeration $\{d_\alpha \colon \alpha < \kappa\}$ of the set D. Take any enumeration $\{e_\alpha \colon \alpha < \kappa\}$ of the set E. It is easy to see that the set $F = \{(d_\alpha, e_\alpha) \colon \alpha < \kappa\}$ is closed and discrete in $X \times Y$; since also $|F| \leq \mathfrak{c}$, we can apply Theorem 3.15 of [16] to find a countable set $B \subset X \times Y$ such that $F \subset \overline{B}$. Then $A = \pi(B) \subset Y$ is countable and $E \subset \bar{A}$. □

Corollary 5.2. *Suppose that X is a space in which there is a closed discrete subset D such that $|D| = \mathfrak{c}$. If Y has a dense Čech-complete subspace and $X \times Y$ is weakly exponentially separable, then Y is separable.*

Proof: By Proposition 5.1, every $A \in [Y]^{\leq \mathfrak{c}}$ is dominated by a countable set so we can apply Corollary 3.29 of the paper [3] to conclude that Y is separable. □

It was established in the paper [16] that for a metrizable space X, weak exponential separability of $C_p(X)$ is equivalent to its separability. However, this is not true for $C_p(X, \mathbb{I})$ because $C_p(X, \mathbb{I})$ is even compact and hence weakly exponentially separable for any discrete space X. This was a motivation for Question 4.10 in [16] on whether weak exponential separability of the space $C_p(X, \mathbb{I})$ implies that $\mathrm{ext}(C_p(X, \mathbb{I})) \leq \mathfrak{c}$ for a metrizable space X. We will show next that the answer to this question is positive.

Theorem 5.3. *If X is a metrizable space and $C_p(X, \mathbb{I})$ is weakly exponentially separable, then $\mathrm{ext}(C_p(X, \mathbb{I})) \leq \mathfrak{c}$.*

Proof: It was proved in [16, Theorem 3.20] that $|X \backslash I(X)| \leq \mathfrak{c}$. If $|X| \leq \mathfrak{c}$, then $\mathrm{ext}(C_p(X, \mathbb{I})) \leq w(C_p(X, \mathbb{I})) \leq \mathfrak{c}$ so we must only consider the case when $|X| > \mathfrak{c}$ and hence $\mathrm{ext}(X) > \mathfrak{c}$. Then there exists a closed discrete set $D \subset I(X)$ such that $|D| = \mathfrak{c}^+$. It is easy to see that $X \simeq X \oplus D$ and therefore the space $C_p(X, \mathbb{I})$ is homeomorphic to $C_p(X, \mathbb{I}) \times \mathbb{I}^D$. If $\mathrm{ext}(C_p(X, \mathbb{I})) > \mathfrak{c}$, then we can choose a closed discrete set $E \subset C_p(X, \mathbb{I})$ such that $|E| = \mathfrak{c}$. Since $\mathbb{I}^D$ is a nonseparable space, it follows from Corollary 5.2 that $C_p(X, \mathbb{I}) \simeq C_p(X, \mathbb{I}) \times \mathbb{I}^D$ is not weakly exponentially separable; this contradiction shows that $\mathrm{ext}(C_p(X, \mathbb{I})) \leq \mathfrak{c}$. □

Theorem 5.4 ($2^{\omega_1} > \mathfrak{c}$)**.** *Given a metrizable space X, the following conditions are equivalent:*

(1) $C_p(X, \mathbb{I})$ *is weakly exponentially separable;*
(2) $C_p(X, \mathbb{I})$ *is either separable or Lindelöf Σ.*

PROOF: The implication (2) $\Longrightarrow$ (1) is trivial because both Lindelöf Σ-spaces and separable ones are weakly exponentially separable, see Theorem 4.8 of [15]. So, assume that $C_p(X, \mathbb{I})$ is weakly exponentially separable. If it is not Lindelöf Σ, then $\text{ext}(C_p(X, \mathbb{I})) > \omega$, see [14, Theorem 3.11], and hence we can fix a closed discrete set $D \subset C_p(X, \mathbb{I})$ with $|D| = \omega_1$.

If $|X| \leq \mathfrak{c}$, then $w(X) \leq \mathfrak{c}$ and hence $iw(X) \leq \omega$, see Problem 102 of [10]; this implies that $C_p(X, \mathbb{I})$ is separable by Problem 174 and Problem 92 of [9]. If $|X| > \mathfrak{c}$, then, recalling that $X \backslash I(X)$ has cardinality not exceeding $\mathfrak{c}$ according to Theorem 3.20 of [16], we conclude that there exists a closed discrete set $D \subset I(X)$ such that $|D| = \mathfrak{c}^+$. It is easy to see that $X \simeq X \oplus D$ and therefore $C_p(X, \mathbb{I})$ is homeomorphic to $C_p(X, \mathbb{I}) \times \mathbb{I}^D$.

By our assumption, $\mathfrak{c}^+ \leq 2^{\omega_1}$ and hence $d(\mathbb{I}^D) = \omega_1$; fix a dense set $A \subset \mathbb{I}^D$ such that $|A| = \omega_1$. Proposition 5.1 implies that there exists a countable set $E \subset \mathbb{I}^D$ such that $A \subset \overline{E}$. As a consequence, E is dense in $\mathbb{I}^D$ which is a contradiction because $d(\mathbb{I}^D) > \omega$. Thus, $|X| \leq \mathfrak{c}$ and therefore $C_p(X, \mathbb{I})$ is separable. □

Not much is known so far about weak exponential separability in $C_p(X)$ for general spaces X. The paper [16] provides an example of a weakly exponentially separable space Y with $\text{ext}(Y) > \mathfrak{c}$ and poses a question on whether such an example is possible for $Y = C_p(X)$ for some space X. Our last result gives an affirmative answer. Recall that X is a *space with exponential ω-domination* if for every set $Y \in [X]^{\leq \mathfrak{c}}$, there exists a countable set $A \subset X$ such that $Y \subset \bar{A}$.

Example 5.5. It was proved in [16, Theorem 3.22] that for any cardinal $\kappa > \mathfrak{c}$, there exists a space S_κ with exponential ω-domination such that $\text{ext}(S_\kappa) = \kappa$. If $X = C_p(S_\kappa)$, then the space S_κ embeds in $C_p(X)$ as a closed subspace, see Problem 167 of [9], and therefore $\text{ext}(C_p(X)) \geq \kappa$. By Corollary 3.17 of [13] the space $C_p(X) = C_p(C_p(S_\kappa))$ also features exponential ω-domination which, together with Proposition 3.8 of [16] shows that $C_p(X)$ is weakly exponentially separable. Therefore the extent of a weakly exponentially separable space $C_p(X)$ can be as large as we wish.

6. Open questions

The results of this paper make it clear that exponential separability displays a very nontrivial behavior in products and function spaces so the purpose of the following list of open questions is to outline some possible lines of research toward better understanding of the respective phenomena.

Question 6.1. Suppose that X is a Lindelöf exponentially separable space and Y is a Lindelöf P-space. Is it true that the product $X \times Y$ is exponentially separable?

Question 6.2. Suppose that X is a countably compact exponentially separable space. Is it true that $X \times X$ is exponentially separable?

Question 6.3. Suppose that X is a countably compact exponentially separable space. Is it true that $X \times X$ is countably compact?

Question 6.4. Suppose that X is a countably compact exponentially separable space of countable tightness. Is it true in ZFC that $X \times X$ is exponentially separable?

Question 6.5. Suppose that X is first countable exponentially separable space. Is it true that $X \times X$ is exponentially separable?

Question 6.6. Suppose that X is first countable pseudocompact exponentially separable space. Is it true that $X \times X$ is exponentially separable?

Question 6.7. Let X be a Lindelöf Σ-space in which every countable set is scattered. Must X be functionally countable?

Question 6.8. Let X be a Lindelöf Σ-space in which every set of cardinality less than or equal to ω_1 is scattered. Is it true in ZFC that X is functionally countable?

Question 6.9. Let X be a Lindelöf space in which every set of cardinality less than or equal to $\mathfrak{c}$ is scattered. Is it true in ZFC that X is functionally countable?

Question 6.10. Assume that X is a metrizable space such that $C_p(X, \mathbb{I})$ is weakly exponentially separable. Is it true in ZFC that the space $C_p(X, \mathbb{I})$ is either separable or has the Lindelöf Σ-property?

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(*Received* May 12, 2021)