On the class of order almost L-weakly compact operators

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Abstract. We introduce a new class of operators that generalizes L-weakly compact operators, which we call order almost L-weakly compact. We give some characterizations of this class and we show that this class of operators satisfies the domination problem.

Keywords: order bounded weakly convergent sequence; L-weakly compact set; order almost L-weakly compact operator; L-weakly compact operator

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1. Introduction and notation

Along this paper the term operator means a bounded linear mapping, E, F mean Banach lattices, X means a Banach space and G, H mean Riesz spaces.

Recently, K. Bouras et al. in [5] introduced and studied the class of almost L-weakly compact operators. Namely, an operator $T\colon X \longrightarrow F$ is called almost L-weakly compact if T carries relatively weakly compact subsets of X into L-weakly compact subsets of F. Alternatively, $T\colon X \longrightarrow F$ is almost L-weakly compact if and only if $f_n(T(x_n)) \to 0$ for every weakly convergent sequence (x_n) of X and every disjoint sequence (f_n) of $B_{F'}$, where $B_{F'}$ is the closed unit ball of F'. Also, in another work [10] they considered operators which send order bounded subsets to L-weakly compact subsets. Indeed, an operator $T\colon E \longrightarrow F$ is said to be order L-weakly compact if $T([0,x])$ is an L-weakly compact subset of F for every $x \in E^+$. Alternatively, $T\colon E \longrightarrow F$ is order L-weakly compact if and only if $f_n(T(x_n)) \to 0$ for every order bounded sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'}$. After that, A. Elbour et al. in [7] gave some characterizations of almost L-weakly compact operators and they study the connections between this class of operators and other classes (as Dunford–Pettis operators, compact operators and L-weakly compact operators). Note that order bounded subset is not in general relatively weakly compact and conversely a relatively weakly compact subset is not in general order bounded.

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The main purpose of this paper is to consider operators which send order bounded relatively weakly compact subsets to L-weakly compact subsets. We give some important characterizations of this class of operators and we show that this class satisfies the domination problem.

To state our results, we need to fix some notation and recall some definitions. In what follows, E^+ denotes the positive cone of E, $\mathrm{sol}(A)$ denotes the solid hull of the subset A, $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$. For every norm bounded subset $A \subset E$, $\varrho_A\colon E' \to \mathbb{R}^+$ is the lattice semi norm defined by $\varrho_A(f) = \sup\{|f||x|\colon x \in A\} = \sup\{|g(x)|\colon x \in A \text{ and } |g| \leq |f|\}$. Then E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A nonzero element x of G is discrete if the order ideal generated by x equals the vector subspace generated by x. A space G is discrete, if it admits a complete disjoint system of discrete elements. A space G is σ-Dedekind complete if every majorized countable nonempty subset of G has a supremum. A space E has the positive Schur property whenever $0 \leq x_n \xrightarrow{w} 0$ implies $\lim_{n\to\infty} \|x_n\| = 0$. A nonempty bounded subset A of E is said to be L-weakly compact if $\lim_{n\to\infty} \|x_n\| = 0$ for every disjoint sequence $(x_n) \subset \mathrm{sol}(A)$. A net (x_α) of E is unbounded absolutely weakly convergent (uaw-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges weakly to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{\text{uaw}} x$.

Let E^a denote the maximal ideal in E on which the induced norm is order continuous. From Proposition 2.4.10 in [11], we note that E^a is closed and that

$$E^a = \{x \in E\colon \text{ each monotone sequence in } [0, |x|] \text{ is convergent}\}.$$

A linear mapping between G and H is positive if $T(x) \geq 0$ in H, whenever $x \geq 0$ in G. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T\colon E \longrightarrow F$ is positive then, its adjoint $T'\colon F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

We need to recall definitions of the following operators:

- An operator $T\colon E \longrightarrow F$ is said to be regular if it can be written as a difference of two positive operators.
- An operator $T\colon E \longrightarrow F$ is said to be order bounded if $T(A)$ is an order bounded subset of F for every order bounded subset of E.
- An operator $T\colon X \longrightarrow F$ is said to be norm-order bounded if $T(A)$ is an order bounded subset of F for every norm bounded subset of X.
- An operator $T\colon X \longrightarrow F$ is said to be L-weakly compact if $T(B_X)$ is an L-weakly compact subset of F.

- An operator $T\colon E \longrightarrow Y$ is said to be M-weakly compact if $\lim_{n\longrightarrow\infty}\|T(x_n)\| = 0$ for every disjoint sequence of B_E.
- An operator $T\colon E \longrightarrow X$ is said to be order weakly compact if $T([0,x])$ is a relatively weakly compact subset of X for every x in E^+.
- An operator $T\colon E \longrightarrow X$ is said to be AM-compact if $T([0,x])$ is a relatively compact subset of X for every x in E^+.
- An operator $T\colon E \longrightarrow F$ is said to be a lattice homomorphism whenever it preserves the lattice operations. That is, whenever $T(x\vee y) = T(x)\vee T(y)$ holds for all $x, y \in E$.
- A positive operator $T\colon E \longrightarrow F$ is said to be almost interval preserving, if $T[0,x]$ is dense in $[0,T(x)]$ for every $x \in E^+$.

2. Main results

Definition 2.1. An operator $T\colon E \longrightarrow F$ is called order almost L-weakly compact if T maps order bounded relatively weakly compact subsets of E to L-weakly compact subsets of F. The class of all almost L-weakly compact operators from E to F will be denoted by o-ALWC(E,F).

Let us denote by LWC(E,F), ALWC(E,F) and o-LWC(E,F) the class of L-weakly compact operators, the class of almost L-weakly compact operators and the class of order L-weakly compact operators, respectively. We have the following inclusions:
LWC$(E,F) \subset$ ALWC$(E,F) \subset$ o-ALWC(E,F); and
LWC$(E,F) \subset$ o-LWC$(E,F) \subset$ o-ALWC(E,F).

The following lemmas will be used throughout this paper.

Lemma 2.2 ([5, Lemma 2.4])**.** *For every nonempty bounded subset $A \subset E$, the following assertions are equivalent:*

(1) *A is L-weakly compact;*
(2) *$f_n(x_n) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{F'}$.*

Lemma 2.3. *If E^a is discrete, then every L-weakly compact subset of E is relatively compact.*

Proof: Let A be a L-weakly compact subset of E. By [11, Proposition 3.6.2] there exists $x \in (E^a)^+$ such that $A \subset [-x,x]+\varepsilon B_E$. Since E^a is discrete, it follows from [12, Theorem 6.1] and [1, Theorem 3.1] that A is a relatively compact subsets of E. $\square$

Lemma 2.4. *If $T\colon E \longrightarrow F$ is an almost interval preserving operator then for every L-weakly compact subset A of E we have $T(A)$ is an L-weakly compact subset of F.*

Proof: Let $T\colon E \longrightarrow F$ be an almost interval preserving operator, A be an L-weakly compact subset of E, (x_n) be a sequence of A and (f_n) be a norm bounded disjoint sequence of F'. Since T is almost interval preserving, then by [11, Theorem 1.4.19] T' is a lattice homomorphism and hence $(T'(f_n))$ is a norm bounded disjoint sequence of E'. By Lemma 2.2, we have $T'(f_n)(x_n) \longrightarrow 0$ which means that $f_n(T(x_n)) \longrightarrow 0$ and hence by Lemma 2.2 we infer that $T(A)$ is an L-weakly compact subset of F. □

The converse of the inclusions cited below are not true in general as we show in the following examples;

Examples. (1) The identity operator of an infinite dimensional Banach lattice with the positive Schur property ($L^1([0,1])$ for example) is an almost L-weakly compact operator, see Proposition 2.2 in [5], which is not L-weakly compact, see Theorem 3.1 in [3].

(2) The identity operator of an infinite dimensional order continuous Banach lattice (for example, the Banach lattice c_0) is an order L-weakly compact operator, see Corollary 2.1 in [10], which is not L-weakly compact.

(3) The identity operator of an order continuous Banach lattice which does not have the positive Schur property (for example, the Banach lattice c_0) is an order almost L-weakly compact operator, see Theorem 2.5, which is not almost L-weakly compact.

(4) It follows from [13, Proposition 1] that each operator $T\colon l^\infty \longrightarrow c_0$ is Dunford–Pettis, then for every relatively weakly compact subset A of l^∞ we have $T(A)$ is a relatively compact subset of c_0. As c_0 is order continuous, then $T(A)$ is L-weakly compact, see [11, page 212], and hence T is almost L-weakly compact, so the operator T is order almost L-weakly compact. On the other hand, since l^∞ is σ-Dedekind complete and is not a discrete order continuous Banach lattice then it follows from [2, Corollary 1] that there exists an operator $T\colon l^\infty \longrightarrow c_0$ which is not AM-compact and hence is not order L-weakly compact. Otherwise, for every order bounded subset A of l^∞, $T(A)$ should be a L-weakly compact subset of c_0. Since c_0 is a discrete Banach lattice, it follows from Lemma 2.3 that $T(A)$ is a relatively compact subset of c_0. Hence, T is an AM-compact operator and this is a contradiction.

In the following result, we give sequential characterizations of order almost L-weakly compact operators.

Theorem 2.5. *For an operator $T\colon E \longrightarrow F$, the following statements are equivalent:*

(1) *T is order almost L-weakly compact.*
(2) *For every order bounded weakly convergent sequence (x_n) of E and every uaw-null sequence (f_n) of $B_{F'}$, we have $f_n(T(x_n)) \to 0$.*
(3) *For every order bounded weakly convergent sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'}$, we have $f_n(T(x_n)) \to 0$.*
(4) *The following conditions hold simultaneously:*
   (i) *$f_n(T(x_n)) \to 0$ for every order bounded weakly null sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'}$;*
   (ii) *$T(E) \subset F^a$.*

Proof: (1) $\Longrightarrow$ (2) Let (x_n) be an order bounded weakly convergent sequence of E and let (f_n) be a uaw-null sequence of F'. The set $K = \{x_0, x_1, \dots\}$ is order bounded and relatively weakly compact, then by our hypothesis $T(K)$ is L-weakly compact. As $|f_n| \xrightarrow{\text{uaw}} 0$ and $|T(x_n)| \in F^+ \cap \operatorname{sol}(T(K))$, it follows from [8, Proposition 3.2] that $|f_n|(|T(x_n)|) \to 0$ and hence $f_n(T(x_n)) \to 0$, as desired.

(2) $\Longrightarrow$ (3) This follows from the fact that every disjoint sequence is uaw-null, see [14, Lemma 2].

(3) $\Longrightarrow$ (4) (i) Obvious.

(ii) Let $x \in E$. Since the constant sequence (x_n) given by $x_n = x$ is an order bounded weakly convergent sequence of E, then $f_n(T(x)) = f_n(T(x_n)) \to 0$ for every disjoint sequence (f_n) of $B_{F'}$. Hence, it follows from Lemma 2.2 that the singleton $\{T(x)\}$ is L-weakly compact. So, $T(x) \in F^a$, see [11, page 212], as desired.

(4) $\Longrightarrow$ (1) Let A be an order bounded relatively weakly compact subsets of E. If $T(A)$ is not an L-weakly compact set, then according to Lemma 2.2 there exist a disjoint sequence $(f_n) \subset B_{F'}$ and a sequence $(x_n) \subset A$ such that $|f_n(T(x_n))| > \varepsilon$ for some $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Pick a subsequence (x_{n_k}) of (x_n) and some $x \in E$ such that $x_{n_k} \xrightarrow{w} x$. Since $(x_{n_k} - x)$ is an order bounded weakly null sequence of E, it follows from our hypothesis that $f_{n_k}(T(x_{n_k} - x)) \to 0$. On the other hand, since $T(E) \subset F^a$ it follows that the singleton $\{T(x)\}$ is an L-weakly compact subset of F. In particular, by Lemma 2.2 we infer that $f_{n_k}(T(x)) \to 0$. Now, from $f_{n_k}(T(x_{n_k})) = f_{n_k}(T(x_{n_k} - x)) + f_{n_k}(T(x))$ we see that $f_{n_k}(T(x_{n_k})) \to 0$, which is impossible. Thus, $T(A)$ is an L-weakly compact subset of F and so T is order almost L-weakly compact. □

In a similar way, we have a dual version.

Theorem 2.6. *For an operator $T\colon E \longrightarrow F$, the following statements are equivalent:*

1. *T' is order almost L-weakly compact.*
2. *For every order bounded weakly convergent sequence (f_n) of F' and every uaw-null sequence (x_n) of B_E, we have $f_n(T(x_n)) \to 0$.*
3. *For every order bounded weakly convergent sequence (f_n) of F' and every disjoint sequence (x_n) of B_E, we have $f_n(T(x_n)) \to 0$.*
4. *The following conditions hold simultaneously:*
   - (i) *$f_n(T(x_n)) \to 0$ for every order bounded weakly null sequence (f_n) of F' and every disjoint sequence (x_n) of B_E.*
   - (ii) *$T'(F') \subset (E')^a$.*

Proof: (1) $\Longrightarrow$ (2) Let $(f_n) \subset F'$, $(x_n) \subset B_E$ be respectively an order bounded weakly convergent and a uaw-null sequences. The set $K = \{f_0, f_1, \dots\}$ is order bounded and relatively weakly compact, then by our hypothesis $A = T'(K)$ is L-weakly compact. We consider the operator $S\colon E \to l^\infty$ defined by

$$S(x) = \big(f_k(T(x))\big)_{k\geq 0} \qquad \text{for each } x \in E.$$

Firstly, we show that S is an M-weakly compact operator. To this end, let (y_n) be a disjoint sequence of B_E and let $\varepsilon > 0$ be fixed. By [11, Proposition 3.6.2], there exists some $g \in ((E')^a)^+$ satisfying

$$A \subset [-g, g] + \varepsilon B_{E'}.$$

This implies that

$$(*) \qquad \begin{aligned} \|S(y_n)\|_\infty &= \sup_k |f_k(T(y_n))| = \sup_k |(T'(f_k))(y_n)| \\ &\leq \sup_{h\in A} |h(y_n)| \leq g(|y_n|) + \varepsilon, \end{aligned}$$

holds for each n. On the other hand, in view of $g \in ((E')^a)^+$, it follows that the singleton $\{g\}$ is an L-weakly compact subset of E', see [11, page 212], and so by Lemma 2.5 in [5] we have

$$(**) \qquad g(|y_n|) \to 0.$$

Therefore, from $(*)$ and $(**)$ we infer that $\limsup \|S(y_n)\|_\infty \leq \varepsilon$ holds. Since $\varepsilon > 0$ is arbitrary, we see that $\|S(y_n)\|_\infty \to 0$ and so S is an M-weakly compact operator. In particular, since the sequence (x_n) is uaw-null, then it follows from [15, Theorem 19] that $\|S(x_n)\|_\infty \to 0$. Now, a glance at the inequality $|f_n(T(x_n))| \leq \|S(x_n)\|_\infty$ shows that $f_n(T(x_n)) \to 0$, as desired.

(2) $\Longrightarrow$ (3) This follows from the fact that every disjoint sequence is uaw-null, see [14, Lemma 2].

(3) $\Longrightarrow$ (4) (i) Obvious.

(ii) Let $f \in F'$. Since the constant sequence (f_n) given by $f_n = f$ is an order bounded weakly convergent sequence of F', then

$$(T'(f))(x_n) = f(T(x_n)) = f_n(T(x_n)) \to 0$$

for every disjoint sequence (x_n) of B_E. Hence, it follows from [5, Lemma 2.5] that the singleton $\{T'(f)\}$ is L-weakly compact. So, $T'(F') \subset (E')^a$, see [11, page 212], as desired.

(4) $\Longrightarrow$ (1) Let A be an order bounded relatively weakly compact subset of F'. If $T'(A)$ is not an L-weakly compact set, then according to [5, Lemma 2.5] there exist a disjoint sequence $(x_n) \subset B_E$ and a sequence $(f_n) \subset A$ satisfying $|f_n(T(x_n))| > \varepsilon$ for some $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Pick a subsequence (f_{n_k}) of (f_n) and some $f \in F'$ such that $f_{n_k} \xrightarrow{w} f$. Note that since $(f_{n_k} - f)$ is an order bounded weakly null sequence of E, then it follows from our hypothesis that $(f_{n_k} - f)(T(x_{n_k})) \to 0$. On the other hand, since $T'(F') \subset (E')^a$ it follows that the singleton $\{T'(f)\}$ is an L-weakly compact subset of E'. In particular, by [5, Lemma 2.5] we infer that $f(T(x_{n_k})) \to 0$. Now, from the equality

$$f_{n_k}(T(x_{n_k})) = (f_{n_k} - f)(T(x_{n_k})) + f(T(x_{n_k}))$$

we see that $f_{n_k}(T(x_{n_k})) \to 0$, which contradicts the fact that $|f_n(T(x_n))| > \varepsilon$. Thus, $T'(A)$ is an L-weakly compact subset of E' and so T' is order almost L-weakly compact. □

In terms of norm-order bounded weakly compact and L-weakly compact operators, the order almost L-weakly compact operators are characterized as follows.

Theorem 2.7. *For an operator $T\colon E \longrightarrow F$, the following statements are equivalent:*

1. *T is order almost L-weakly compact.*
2. *If $S\colon X \longrightarrow E$ is a norm-order bounded weakly compact operator from an arbitrary Banach space X into E, then the product $T \circ S$ is L-weakly compact.*
3. *If $S\colon l^1 \longrightarrow E$ is a norm-order bounded weakly compact operator, then the product $T \circ S$ is L-weakly compact.*

The proof is virtually identical with that of [5, Theorem 2.4] which we omit.

Remarks. (1) We can check easily that the space o-ALWC(E, F) is a norm closed vector subspace of the space $L(E, F)$ of all operators from E into F.

(2) Consider the schema of operators $E \xrightarrow{T} F \xrightarrow{S} G$.

(a) If S is an order almost L-weakly compact operator, then $S \circ T$ is not necessarily order almost L-weakly compact. In fact, by [4, Lemma 2.4] there exists a non regular operator $T: c \longrightarrow c_0$ which is certainly not Dunford–Pettis. The operator T is not order almost L-weakly compact. Otherwise, since c_0 is discrete then it follows from Lemma 2.3 that $T(A)$ is relatively compact for every order bounded relatively weakly compact subsets A of the Banach lattice c. As c is an AM-space with unit, then T should be a Dunford–Pettis operator and this is a contradiction.
Now, if $S: c_0 \longrightarrow c_0$ is the identity operator on c_0 then S is order almost L-weakly compact, see Theorem 2.5, but $S \circ T = T$ is not order almost L-weakly compact.

(b) However, if S is an order almost L-weakly compact operator and T is an order bounded operator, then the composed operator $S \circ T$ is order almost L-weakly compact. In fact, let A be an order bounded relatively weakly compact subset of E then $T(A)$ is an order bounded relatively weakly compact subset of F. Since S is order almost L-weakly compact, then $S(T(A))$ is a L-weakly compact subset of G. That is, $S \circ T$ is order almost L-weakly compact.

(c) If T is an order almost L-weakly compact operator, then $S \circ T$ is not necessarily order almost L-weakly compact. In fact, consider the operator $S: l^1 \to l^\infty$ defined by

$$S((\lambda_n)_n) = \left(\sum_{n=1}^{\infty} \lambda_n\right)(1, 1, \ldots)$$

for all $(\lambda_n) \in l^1$. Note that S is not order almost L-weakly compact. Indeed, let $e = (1/n^2)_{n \in \mathbb{N}^*}$ and (e_n) be the sequence of the standard unit vectors of l^∞. Since $|e_n| \leq S(e)$, then (e_n) is a disjoint sequence in the solid hull of $\{S(e)\}$ satisfying $\|e_n\|_\infty \nrightarrow 0$ and hence $\{S(e)\}$ is not L-weakly compact. As the singleton $\{e\}$ is an order bounded relatively weakly compact subset of l^1, then S fails to be order almost L-weakly compact. If $T: l^1 \longrightarrow l^1$ is the identity operator on l^1, then T is order almost L-weakly compact but $S \circ T = S$ is not order almost L-weakly compact.

(d) However, if S is an almost interval preserving operator and T is order almost L-weakly compact, then the composed operator $S \circ T$ is order almost L-weakly compact. In fact, let A be an order bounded relatively weakly compact subset of E, then $T(A)$ is an L-weakly compact set in F. As S is an almost interval preserving operator, then by the Lemma 2.4 we have $S(T(A))$ is a L-weakly compact subset of G. Hence, $S \circ T$ is order almost L-weakly compact.

In the next result, we characterize the positive order almost L-weakly compact operators.

Theorem 2.8. *For a positive operator $T\colon E \longrightarrow F$, the following statements are equivalent:*

(1) *T is order almost L-weakly compact.*
(2) *T carries the solid hull of each order bounded relatively weakly compact subset of E to L-weakly compact subset of F.*
(3) *For every order bounded relatively weakly compact set $W \subset E$ and $\varepsilon > 0$, there exists some $g \in (F')^+$ such that*

$$(|f| - g)^+(T|x|) < \varepsilon$$

holds for all $x \in W$ and $f \in B_{F'}$.

Proof: (1) $\Longrightarrow$ (2) Let W be an order bounded relatively weakly compact subset of E. We shall see that $T(\mathrm{sol}(W))$ is an L-weakly compact subset of F. The proof will be based upon two steps.

Step 1: Let (f_n) be a disjoint sequence of $B^+_{F'}$ and $\varepsilon > 0$. We claim that there exist $u \in E^+$ and a natural number k such that

$$(*) \qquad f_n(T((|x| - u)^+)) < \varepsilon$$

holds for all $x \in W$ and all $n > k$. To see this, assume by way of contradiction that $(*)$ is false. That is, assume that for each $u \in E^+$ and each k there exist $x \in W$ and $m > k$ with $f_m(T((|x| - u)^+)) \geq \varepsilon$. An easy inductive argument shows that there exist a sequence $(x_n) \subset W$ and a subsequence (g_n) of (f_n) such that

$$g_n\Bigg(T\bigg(\Big(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i|\Big)^+\bigg)\Bigg) \geq \varepsilon \qquad \text{holds for all } n.$$

Let $x = \sum_{n=1}^{\infty} 2^{-n}|x_n|$, $y_n = \big(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i|\big)^+$ and $z_n = \big(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i| - 2^{-n}x\big)^+$. By [1, Lemma 4.35], the sequence (z_n) is disjoint and lies in the solid hull of W. Thus, by [1, Theorem 4.34] we see that the sequence (z_n) is order bounded and weakly null in E. Hence, it follows from our hypothesis that $g_n(T(z_n)) \to 0$. On the other hand, the inequality $0 \leq y_n \leq z_n + 2^{-n}x$ implies

$$0 < \varepsilon \leq g_n(T(y_n) \leq g_n(T(z_n)) + 2^{-n} g_n(T(x)) \to 0,$$

which is impossible. Therefore, $(*)$ is true.

Step 2: Now, we claim that $T(\mathrm{sol}(W))$ is an L-weakly compact set in F. To this end, assume by way of contradiction that this is not the case. Then, by applying Lemma 2.2 we can assume that there exist a disjoint sequence $(f_n) \subset B_{F'}$ and a sequence $(y_n) \subset \mathrm{sol}(W)$ such that $f_n(T(y_n)) \nrightarrow 0$ for all $n \in \mathbb{N}$. By passing to a subsequence, we can assume that for some $\varepsilon > 0$ we have $|f_n(T(y_n))| > \varepsilon$ for all $n \in \mathbb{N}$. On the other hand, by the Step 1 we pick $u \in E^+$ and k such that $(*)$ is valid. Note that the singleton $\{u\}$ is an order bounded relatively weakly compact subset of E, hence $\{T(u)\}$ is an L-weakly compact subset of F (because T is order almost L-weakly compact), so it follows from Lemma 2.2 that $f_n(T(u)) \to 0$.

Now, we choose $m > k$ such that $f_n(T(u)) < \varepsilon$ holds for all $n \geq m$ and we pick a sequence $(x_n) \subset W$ with $|y_n| \leq |x_n|$ for all n, and we note that

$$\begin{aligned} |T'(f_n)(y_n)| &\leq T'(f_n)|x_n| \leq T'(f_n)(|x_n| - u)^+ + T'(f_n)(u) \\ &\leq f_n(T(|x_n| - u)^+) + f_n(T(u)) \leq \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

holds for all $n \geq m$. The above inequality contradicts the fact that $|f_n(T(y_n))| > \varepsilon$. Therefore, $T(\mathrm{sol}(W))$ is an L-weakly compact subset of F.

(2) $\Longrightarrow$ (3) Let W be an order bounded relatively weakly compact subset of E and let $\varepsilon > 0$. According to our hypothesis $T(\mathrm{sol}(W))$ is L-weakly compact and hence it follows from [11, Proposition 3.6.3] that $\varrho_{T(\mathrm{sol}(W))}(f_n) \to 0$ for each disjoint sequence $(f_n) \subset B_{F'}$. Let ϱ be the norm continuous semi-norm defined by

$$\varrho(f) = \sup\{|f||x| \colon x \in W\}.$$

Then, the positivity of T implies $\varrho(T'f_n) \leq \varrho_{T(\mathrm{sol}(W))}(f_n)$ for each disjoint sequence $(f_n) \subset B_{F'}$. In particular, $\varrho(T'f_n) \to 0$ for each disjoint sequence $(f_n) \subset B_{F'}$. Therefore, by [1, Theorem 4.36] there exists some $g \in (F')^+$ satisfying

$$(|f| - g)^+(T(|x|)) \leq \varrho(T'[|f_n| - g)^+]) < \varepsilon$$

holds for all $f \in B_{F'}$ and $x \in W$.

(3) $\Longrightarrow$ (1) Let $(f_n) \subset B_{F'}$ be a disjoint sequence and (x_n) be an order bounded weakly convergent sequence of E. According to Theorem 2.5, it suffices to show that $f_n(T(x_n)) \longrightarrow 0$.

The set $W = \{x_1, x_2, \ldots\}$ is an order bounded weakly relatively compact subset of E. Choose $x \in E$ with $|x_n| \leq x$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$, by our hypothesis there exists some $0 \leq g \in (F')^+$ satisfying

$$(|f_n| - g)^+(T(|x_n|)) < \varepsilon$$

for all $n \in \mathbb{N}$. Since $(|f_n| \wedge g)$ is an order bounded disjoint sequence, then we have $|f_n| \wedge g \xrightarrow{w^*} 0$ in F' and so $(|f_n| \wedge g)(T(|x|)) \to 0$. Thus, for every n we have

$$|f_n(T(x_n))| \leq |f_n|(T(|x_n|)) \leq ((|f_n| - g)^+)(T(|x_n|)) + (|f_n| \wedge g)(T(|x|))$$
$$\leq \varepsilon + (|f_n| \wedge g)(T(|x|)).$$

This shows that $f_n(T(x_n)) \to 0$, as desired. □

As consequence of Theorem 2.8, we obtain that the class of order almost L-weakly compact operators satisfies the domination problem.

Corollary 2.9. *If a positive operator $S\colon E \longrightarrow F$ is dominated by an order almost L-weakly compact operator $T\colon E \longrightarrow F$, then S is an order almost L-weakly compact operator.*

Proof: Let $S, T\colon E \longrightarrow F$ be two positive operators such that $0 \leq S \leq T$ holds and T is order almost L-weakly compact. We have to show that S is order almost L-weakly compact. Indeed, let W be an order bounded relatively weakly compact subset of E and let $\varepsilon > 0$. By Theorem 2.8, there exists some $g \in (F')^+$ such that

$$(|f| - g)^+(T|x|) < \varepsilon$$

holds for all $x \in W$ and $f \in B_{F'}$. Hence,

$$(|f| - g)^+(S|x|) \leq (|f| - g)^+(T|x|) < \varepsilon$$

holds for all $x \in W$ and $f \in B_{F'}$. So, it follows from Theorem 2.8 that S is an order almost L-weakly compact operator. □

An easy application of Theorem 2.5 shows that the identity operator on E is order almost L-weakly compact if and only if E is order continuous. On the other hand, it is well known that E is order continuous if and only if the identity operator on E is an order weakly compact operator. Motivated by the last results, it is natural to ask the following question: What is the relationship that combines order almost L-weakly compact operators and order weakly compact operators. A direct application of Theorem 2.5 and [6, Corollary 2.6] shows that each positive order almost L-weakly compact operator is order weakly compact but the converse is false in general. Indeed, in the previous example mentioned in Remarks (Remark (2) (c)), the positive operator $S\colon l^1 \longrightarrow l^\infty$ defined by

$$S((\lambda_n)_n) = \left(\sum_{n=1}^{\infty} \lambda_n\right)(1, 1, \ldots)$$

for all $(\lambda_n) \in l^1$ is an order weakly compact operator but it is not order almost L-weakly compact.

For the almost interval preserving operators, the situation is quite different. More precisely, we have the following proposition.

Proposition 2.10. *Each almost interval preserving order weakly compact operator $T\colon E \longrightarrow F$ is order almost L-weakly compact.*

Proof: Let (x_n) be an order bounded weakly convergent sequence of E and let (f_n) be a disjoint sequence of $B_{F'}$. We have to show that $f_n(T(x_n)) \to 0$. To this end, let $\varepsilon > 0$ and pick some $0 \leq x \in E$ such that $|x_n| \leq x$ for all $n \in \mathbb{N}$. As T is order weakly compact, then by [9, Corollary 3.5] there exists some $0 \leq g \in E'$ such that

$$(|T'(f_n)| - g)^+(x)$$

holds for all $n \in \mathbb{N}$. Since T' is a lattice homomorphism, then $(|T'(f_n)| \wedge g)$ is an order bounded disjoint sequence of E' and this implies that $(|T'(f_n)| \wedge g)$ is weakly null. Hence, we have $(|T'(f_n)| \wedge g)(x) \to 0$. Thus, for every $n \in \mathbb{N}$ we get

$$\begin{aligned}|f_n(T(x_n))| \leq |T'(f_n)||x| &= (|T'(f_n)| - g)^+(x) + (|T'(f_n)| \wedge g)(x)\\ &\leq \varepsilon + (|T'(f_n)| \wedge g)(x).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the latter inequalities imply that $f_n(T(x_n)) \to 0$ and this proves that T is an order almost L-weakly compact operator. □

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