Mal'tsev–Neumann products of semi-simple classes of rings

Barry James Gardner

Abstract. Malt'tsev–Neumann products of semi-simple classes of associative rings are studied and some conditions which ensure that such a product is again a semi-simple class are obtained. It is shown that both products, $\mathcal{S}_1 \circ \mathcal{S}_2$ and $\mathcal{S}_2 \circ \mathcal{S}_1$ of semi-simple classes $\mathcal{S}_1$ and $\mathcal{S}_2$ are semi-simple classes if and only if they are equal.

Keywords: radical class; semi-simple class; Mal'tsev–Neumann product

Classification: 16N80, 08C99

1. Introduction

A *Mal'tsev–Neumann product* $\mathcal{X} \circ \mathcal{Y}$ of classes $\mathcal{X}, \mathcal{Y}$ of rings is the class of rings A which have an ideal $I \in \mathcal{X}$ with $A/I \in \mathcal{Y}$. The concept was introduced about the same time by A. I. Mal'tsev in [4] for general algebras, where the role of ideals is played by congruence classes which are subalgebras, and in H. Neumann's book [5] for groups. In [4] there was some concentration on varieties and quasivarieties, while [5] was concerned with varieties of groups. Subsequently products have been studied in various contexts; a good survey is given in the introduction to [6].

It seems natural to ask when a Mal'tsev–Neumann product of radical classes (or semi-simple classes) is again a radical class (semi-simple class, respectively). In [2] we examined the former question with some emphasis on elementary radical classes. One quite general result was that for radical classes $\mathcal{R}_1$ and $\mathcal{R}_2$, $\mathcal{R}_1 \circ \mathcal{R}_2$ and $\mathcal{R}_2 \circ \mathcal{R}_1$ are *both* radical classes if and only if $\mathcal{R}_1 \circ \mathcal{R}_2 = \mathcal{R}_2 \circ \mathcal{R}_1$.

Here we investigate the question for semi-simple classes. We fare a bit better than in [2] with examples. Thus if $\mathcal{S}_2$ is the semi-simple class corresponding to a hereditary radical class, then $\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class for every semi-simple class $\mathcal{S}_1$, though the hereditary condition is not necessary. Hence if $\mathcal{S}_1$ and $\mathcal{S}_2$ both correspond to hereditary radicals, then both their products are semi-simple classes. As with radical classes this means that the products must coincide. The

DOI 10.14712/1213-7243.2023.004

behaviour relating to products of the semi-simple classes corresponding to special radicals is completely described.

All rings throughout are associative, and we are dealing with rings, not rings with identity. Two results from radical theory for associative rings that are crucially involved in our proofs are the following.

(1) A nonempty class of rings is a semi-simple class if and only if it is isomorphically closed, hereditary and closed under subdirect products and extensions.

(2) Absolute direct summand (ADS) condition (Anderson, Divinsky and Suliński condition). For every radical class $\mathcal{R}$, if $I \triangleleft A$ then $\mathcal{R}(I) \triangleleft A$.

In some contexts, e.g., alternative rings, groups and modules, (1) and (2) are valid and so many of our results carry over with only cosmetic changes. In other settings, though, (1) and/or (2) may fail, notoriously in the class of all (not necessarily associative) rings.

We use the terms "radical class" and "radical" as synonyms, but mostly use the former. For unexplained notation and terminology pertaining to radical theory see [3]. For terms from abelian group theory see [1].

2. Results

Throughout this paper, $\mathcal{S}_1$ and $\mathcal{S}_2$ always denote semi-simple classes, $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, the corresponding radical classes, though distinctive notation will be used for specific radical and semi-simple classes. The first, simple result will be indispensible for some of our proofs.

Proposition 2.1. *For semi-simple classes $\mathcal{S}_1$ and $\mathcal{S}_2$, the following conditions are equivalent for a ring A*

(i) $A \in \mathcal{S}_1 \circ \mathcal{S}_2$.

(ii) $\mathcal{R}_2(A) \in \mathcal{S}_1$.

(iii) *A has an ideal J such that $J \in \mathcal{S}_1$, $A/J \in \mathcal{S}_2$ and $J \triangleleft R$ for every ring R for which $A \triangleleft R$.*

Proof: (i) $\Rightarrow$ (ii) If $I \triangleleft A$, $I \in \mathcal{S}_1$ and $A/I \in \mathcal{S}_2$, then $\mathcal{R}_2(A) \subseteq I$ as A/I is $\mathcal{R}_2$-semi-simple. Thus $\mathcal{R}_2(A) \in \mathcal{S}_1$.

(ii) $\Rightarrow$ (iii) $A/\mathcal{R}_2(A) \in \mathcal{S}_2$ and the other requirement follows from the (ADS) condition.

Clearly (iii) $\Rightarrow$ (i). □

A Mal'tsev–Neumann product of semi-simple classes always satisfies some of the requirements for a semi-simple class.

Proposition 2.2. *For all semi-simple classes $\mathcal{S}_1$ and $\mathcal{S}_2$, the product $\mathcal{S}_1 \circ \mathcal{S}_2$ is*

(i) *hereditary and*

(ii) *closed under subdirect products.*

PROOF: (i) If $A \triangleleft B \in \mathcal{S}_1 \circ \mathcal{S}_2$ let I be an ideal of B with $I \in \mathcal{S}_1$ and $B/I \in \mathcal{S}_2$. Then $I \cap A \triangleleft A$ and $I \cap A \triangleleft I \in \mathcal{S}_1$ so $I \cap A \in \mathcal{S}_1$. Also $A/I \cap A \cong (A+I)/I \triangleleft B/I \in \mathcal{S}_2$, so $A/I \cap A \in \mathcal{S}_2$.

(ii) If A is a subdirect product of rings $A/K_\lambda \in \mathcal{S}_1 \circ \mathcal{S}_2$ with $\bigcap K_\lambda = 0$ for each λ let I_λ be an ideal of A containing K_λ such that $I_\lambda/K_\lambda \in \mathcal{S}_1$ and $A/I_\lambda \cong (A/K_\lambda)/(I_\lambda/K_\lambda) \in \mathcal{S}_2$. Let $I = \bigcap I_\lambda$. Then

$$I/I \cap K_\lambda \cong (I + K_\lambda)/K_\lambda \triangleleft I_\lambda/K_\lambda \in \mathcal{S}_1.$$

Now $\bigcap(I \cap K_\lambda) \subseteq \bigcap K_\lambda = 0$, so I is a subdirect product of the rings $I/I \cap K_\lambda \in \mathcal{S}_1$ whence $I \in \mathcal{S}_1$.

Also $A/I = A/\bigcap I_\lambda$ and $\bigcap(I_\lambda/I) = 0$ so A/I is a subdirect product of the rings $(A/I)/(I_\lambda/I) \cong A/I_\lambda \in \mathcal{S}_2$. Thus $A \in \mathcal{S}_1 \circ \mathcal{S}_2$. □

Corollary 2.3. *$\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class if and only if it is closed under extensions.*

Proposition 2.4. *If $\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class, then $\mathcal{S}_2 \circ \mathcal{S}_1 \subseteq \mathcal{S}_1 \circ \mathcal{S}_2$.*

PROOF: $\mathcal{S}_2$ and $\mathcal{S}_1 \subseteq \mathcal{S}_1 \circ \mathcal{S}_2$ and the latter is closed under extensions. □

The following result mirrors Theorem 2.3 of [2].

Theorem 2.5. *Let $\mathcal{S}_1, \mathcal{S}_2$ be semi-simple classes. Then $\mathcal{S}_1 \circ \mathcal{S}_2$ and $\mathcal{S}_2 \circ \mathcal{S}_1$ are both semi-simple classes if and only if they are equal.*

PROOF: First suppose that $\mathcal{S}_1 \circ \mathcal{S}_2 = \mathcal{S}_2 \circ \mathcal{S}_1$. If $A \triangleleft B$ with $A, B/A \in \mathcal{S}_1 \circ \mathcal{S}_2$, then A has an ideal $I \in \mathcal{S}_1$ with $A/I \in \mathcal{S}_2$ and B has an ideal J containing A with $J/A \in \mathcal{S}_1$ and $B/J \cong (B/A)/(J/A) \in \mathcal{S}_2$. By Proposition 2.1 it can be assumed that $I \triangleleft J$ and $I \triangleleft B$. Since $A/I \in \mathcal{S}_2$ and $J/A \in \mathcal{S}_1$ the exact sequence

$$0 \to A/I \to J/I \to J/A \to 0$$

shows that $J/I \in \mathcal{S}_2 \circ \mathcal{S}_1 = \mathcal{S}_1 \circ \mathcal{S}_2$. Hence J has an ideal K containing I such that $K/I \in \mathcal{S}_1$ and $J/K \cong (J/I)/(K/I) \in \mathcal{S}_2$.

We can assume that $K/I \triangleleft B/I$ (by Proposition 2.1) whence $K \triangleleft B$. Since I and $K/I \in \mathcal{S}_1$, so also is $K \in \mathcal{S}_1$. Moreover, $J/K \in \mathcal{S}_2$ and $B/J \in \mathcal{S}_2$, so from the exact sequence

$$0 \to J/K \to B/K \to B/J \to 0$$

we see that $B/K \in \mathcal{S}_2$. But then $B \in \mathcal{S}_1 \circ \mathcal{S}_2$ as required, and $\mathcal{S}_1 \circ \mathcal{S}_2$ is therefore closed under extensions and hence a semi-simple class.

The converse follows from Proposition 2.4. □

This is all well and good, but as yet we have given no examples of products of semi-simple classes which are themselves semi-simple classes. The next result provides a plentiful supply of such examples.

Theorem 2.6. *If the radical class $\mathcal{R}_2$ corresponding to $\mathcal{S}_2$ is hereditary, then $\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class.*

Proof: Let A be an ideal of a ring C with A and $C/A \in \mathcal{S}_1 \circ \mathcal{S}_2$. Then $\mathcal{R}_2(A)$ and $\mathcal{R}_2(C/A)$ are in $\mathcal{S}_1$ by Proposition 2.1. Now $\mathcal{R}_2(A) \triangleleft C$ so $\mathcal{R}_2(a) \triangleleft \mathcal{R}_2(C)$ and

$$\mathcal{R}_2(C)/\mathcal{R}_2(A) = \mathcal{R}_2(C)/(A \cap \mathcal{R}_2(C)) \cong (\mathcal{R}_2(C) + A)/A \triangleleft C/A$$

and $\mathcal{R}_2(C)/\mathcal{R}_2(A) \in \mathcal{R}_2$ so $(\mathcal{R}_2(C) + A)/A \triangleleft \mathcal{R}_2(C/A) \in \mathcal{S}_1$, consequently $\mathcal{R}_2(C)/\mathcal{R}_2(A) \in \mathcal{S}_1$. But also $\mathcal{R}_2(A) \in \mathcal{S}_1$ and thus $C \in \mathcal{S}_1 \circ \mathcal{S}_2$. By Corollary 2.3, $\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class. □

Corollary 2.7. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are both hereditary then $\mathcal{S}_1 \circ \mathcal{S}_2$ and $\mathcal{S}_2 \circ \mathcal{S}_1$ are both semi-simple classes and hence equal.*

Example 2.8. If $\mathcal{R}_2$ is not hereditary then $\mathcal{S}_1 \circ \mathcal{S}_2$ need not be a semi-simple class.

Let $\mathcal{B}$ denote the prime (Baer) radical class and $\mathcal{E}$ the radical class of all idempotent rings, and let $S(\mathcal{B}), S(\mathcal{E})$, respectively, denote their corresponding semi-simple classes. Then $\mathcal{B}$ is hereditary, so $S(\mathcal{E}) \circ S(\mathcal{B})$ is a semi-simple class. If $S(\mathcal{B}) \circ S(\mathcal{E})$ were a semi-simple class, we should have $S(\mathcal{B}) \circ S(\mathcal{E}) = S(\mathcal{E}) \circ S(\mathcal{B})$. For a prime p, let $\mathbb{Z}(p)^0 * \mathbb{Z}(p)$ denote the standard unital extension obtained by the adjunction to the zeroring $\mathbb{Z}(p)^0$ on the cyclic group of order p of the identity of the field $\mathbb{Z}(p)$. Then $\mathbb{Z}(p)^0 * \mathbb{Z}(p)$ is in the semi-simple class $S(\mathcal{E}) \circ S(\mathcal{B})$. But $\mathbb{Z}(p)^0 * \mathbb{Z}(p)$ has no nonzero semiprime ideals and is idempotent, so it is not in $S(\mathcal{B}) \circ S(\mathcal{E})$.

Example 2.9. Nevertheless, $\mathcal{R}_2$ need not be hereditary for $\mathcal{S}_1 \circ \mathcal{S}_2$ to be a semi-simple class.

Let $\beth$ be the radical class of boolean rings and $\mathcal{D}$ the A-radical class of *divisible* rings, i.e. rings with divisible additive groups, $S(\beth)$ and $S(\mathcal{D})$, respectively, their associated semi-simple classes. (It would be unwise to call the rings in $S(\mathcal{D})$ *reduced* because of an established meaning of that word in ring theory.) Since $\beth$ is hereditary, $S(\mathcal{D}) \circ S(\beth)$ is a semi-simple class. In fact for *every* ring A we have $2\beth(A) = 0$, so $\beth(A) \in S(\mathcal{D})$, whence $S(\mathcal{D}) \circ S(\beth)$ is the class of all rings. We shall show that $S(\beth) \circ S(\mathcal{D})$ is also the class of all rings. Since $S(\beth)$ and $S(\mathcal{D}) \subseteq S(\beth) \circ S(\mathcal{D})$ we consider an arbitrary ring A with $\beth(A) \neq 0$. For convenience let $I = \beth(A)$. Likewise we can assume $\mathcal{D}(A/I) \neq 0$ and let $D/I = \mathcal{D}(A/I)$.

We need to consider additive groups for a moment: let R^+ denote the additive group of a ring R. Now

$$\mathrm{Ext}((D/I)^+, I^+) = \mathrm{Ext}(2(D/I)^+, I^+) = 2\mathrm{Ext}((D/I)^+, I^+) = 0,$$

as $2I = 0$, see [1], pages 266–267, Lemma 3.1 and (C), so that $D^+ = I^+ \oplus E$, where E is a group isomorphic to $(D/I)^+$. But then E is the maximum divisible subgroup of D^+, so $E = \mathcal{D}(D)^+$. Moreover $I = \beth(D)$, so the group direct sum $D^+ = I^+ \oplus E$ is "really" a ring direct sum $D = \beth(D) \oplus \mathcal{D}(D)$.

Now $I \in S(\mathcal{D})$,

$$I \cong I/I \cap \mathcal{D}(D) \cong (I + \mathcal{D}(D))/\mathcal{D}(D)$$

and

$$A/I \oplus \mathcal{D}(D) \cong (A/I)/((I + \mathcal{D}(D)/I) = (A/I)/(D/I) = (A/I)/\mathcal{D}(A/I) \in S(\mathcal{D}).$$

Hence from the exact sequence

$$0 \to (I + \mathcal{D}(D))/\mathcal{D}(D) \to A/\mathcal{D}(D) \to A/(I + \mathcal{D}(D)) \to 0$$

we deduce that $A/\mathcal{D}(D) \in S(\mathcal{D})$. (Note that $A/\mathcal{D}(D)$ exists by (ADS).)

Since $\mathcal{D}(D) \in S(\beth)$ we see that A is in $S(\beth) \circ S(\mathcal{D})$.

We can completely describe the behaviour of semi-simple classes of *special* radicals with respect to Mal'tsev–Neumann products. For this we make use of the following result which generalizes a theorem of R. L. Snider in [7]. Recall that an ideal I of a ring A is *essential* if it has nonzero intersection with every nonzero ideal of A, and A is then said to be an *essential extension* of I. We write $I \triangleleft^{\bullet} A$ to denote this situation.

Theorem 2.10. *Let $\mathcal{X}$ be a class of semiprime rings closed under essential extensions, $\mathcal{Y}$ a class of rings. If A is an ideal of a ring B, A is a subdirect product of rings in $\mathcal{X}$ and B/A is a subdirect product of rings in $\mathcal{Y}$, then B is a subdirect product of rings in $\mathcal{X} \cup \mathcal{Y}$.*

Proof: Let A have ideals I_λ such that each $A/I_\lambda \in \mathcal{X}$ and $\bigcap I_\lambda = 0$, and let B/A have ideals J_μ/A such that each $(B/A)/(J_\mu/A) \in \mathcal{Y}$ and $\bigcap J_\mu/A = 0$ (so that each $J_\mu \triangleleft B$, each $B/J_\mu \in \mathcal{Y}$ and $\bigcap J_\mu = A$).

Then $I_\lambda \triangleleft A \triangleleft B$ and A/I_λ is semiprime, so $I_\lambda \triangleleft B$ (e.g. by the Andrunakievich lemma) for each λ. For each λ let $N_\lambda/I_\lambda \triangleleft B/I_\lambda$ be maximal with respect to having zero intersection with A/I_λ. Then $N_\lambda \cap A = I_\lambda$ for each λ. Now

$$\begin{aligned} A/I_\lambda &\cong (A/I_\lambda)/(A/I_\lambda) \cap (N/I_\lambda) \cong (A/I_\lambda + N_\lambda/I_\lambda)/(N_\lambda/I_\lambda) \\ &\triangleleft^\bullet (B/I_\lambda)/(N_\lambda/I_\lambda) \cong B/N_\lambda. \end{aligned}$$

Hence $B/N_\lambda \in \mathcal{X}$ for all λ. Now

$$\bigcap N_\lambda \cap \bigcap J_\mu = \bigcap N_\lambda \cap A = \bigcap (N_\lambda \cap A) = \bigcap I_\lambda = 0.$$

This shows that B is a subdirect product of the $B/N_\lambda \in \mathcal{X}$ and the $B/J_\mu \in \mathcal{Y}$. □

In Snider's theorem, $\mathcal{X}$ is a class of prime rings with identity. This is a special case of Theorem 2.10 since rings with identity have no proper essential extensions. Now for special radicals.

Theorem 2.11. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be special radical classes with semi-simple classes $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and for $i = 1, 2$ let $\mathcal{C}_i$ be a special class of prime rings with upper radical class $\mathcal{R}_i$. Then $\mathcal{S}_1 \circ \mathcal{S}_2$ is the semi-simple class corresponding to the upper radical class defined by $\mathcal{C}_1 \cup \mathcal{C}_2$, which is special.*

Proof: Special radicals are hereditary, so by Theorems 2.5 and 2.6, $\mathcal{S}_1 \circ \mathcal{S}_2$ $(= \mathcal{S}_2 \circ \mathcal{S}_1)$ is a semi-simple class. Since special classes are closed under essential extensions, Theorem 2.10 says that every ring in $\mathcal{S}_1 \circ \mathcal{S}_2$ is a subdirect product of rings in the special class $\mathcal{C}_1 \cup \mathcal{C}_2$. But $\mathcal{C}_1$ and $\mathcal{C}_2 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S}_1 \circ \mathcal{S}_2$, so all subdirect products of rings from $\mathcal{C}_1 \cup \mathcal{C}_2$ are in $\mathcal{S}_1 \circ \mathcal{S}_2$. □

Theorem 2.10 might suggest that if $\mathcal{S}_1$ is the semi-simple class of a special radical, then $\mathcal{S}_1 \circ \mathcal{S}_2$ is a semi-simple class for every semi-simple class $\mathcal{S}_2$. This is not so. See Example 2.8: $\mathcal{B}$ is special.

References

[1] Fuchs L., *Abelian Groups*, Springer Monographs in Mathematics, Springer, Cham, 2015.

[2] Gardner B. J., *A note on Mal'tsev–Neumann products of radical classes*, Int. Electron. J. Algebra **24** (2018), 1–11.

[3] Gardner B. J., Wiegandt R., *Radical Theory of Rings*, Monographs and Textbooks in Pure and Applied Mathematics, 261, Marcel Dekker, New York, 2004.

[4] Mal'tsev A. I., *Ob umnozhenii klassov algebraicheskikh sistem*, Sibirskii Mat. Zh. **8** (1967), 346–365 (Russian).

[5] Neumann H., *Varieties of Groups*, Springer, New York, 1967.

[6] Penza T., Romanowska A. B., *Mal'tsev products of varieties, I.*, Algebra Universalis **82** (2021), no. 2, Paper No. 33, 19 pages.
[7] Snider R. L., *Subdirect decompositions of extension rings*, Michigan Math. J. **16** (1969), 225–226.

B. J. Gardner:
Discipline of Mathematics, University of Tasmania, PB 37, Hobart TAS 7001, Australia

E-mail: Barry.Gardner@utas.edu.au

(*Received* July 27, 2021, *revised* January 17, 2022)