Degree polynomial for vertices in a graph and its behavior under graph operations

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Abstract. We introduce a new concept namely the degree polynomial for the vertices of a simple graph. This notion leads to a concept, namely, the degree polynomial sequence which is stronger than the concept of degree sequence. After obtaining the degree polynomial sequence for some well-known graphs, we prove a theorem which gives a necessary condition for the realizability of a sequence of polynomials with positive integer coefficients. Also we calculate the degree polynomial for the vertices of the join, Cartesian product, tensor product, and lexicographic product of two simple graphs and for the vertices of the complement of a simple graph. Some examples, counterexamples, and open problems concerning these subjects is given as well.

Keywords: degree polynomial; degree polynomial sequence; degree sequence; graph operation

Classification: 05C07, 05C31, 05C76

1. Introduction

The degree sequence of a graph is one of the interesting invariants of a graph. In recent decades, many mathematicians have investigated the various aspects and applications of this invariant. Especially, several studies have been done for testing whether a non-increasing sequence of nonnegative integers is a degree sequence or not (the realizability of the sequence) and some interesting criterions have been obtained, see [1], [3], [5], [6], [7], [8], [10], [11], [12].

The degree sequence of a graph, is not the only descriptive parameter on the degrees of the vertices of that graph. A. N. Patrinos and S. L. Hakimi in [9] introduced another parameter named the integer-pair sequence for a simple graph and studied some aspect concerning this parameter. Y. Amanatidis, B. Green, and M. Mihail have introduced and studied a reformulation of integer-pair sequences named the join degree matrix.

The integer-pair sequence and also the join degree matrix give more information about a graph than a degree sequence does. More recently, M. D. Barrus and

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E. A. Donovan in [2] have introduced another degree-related parameter known as the neighborhood degree list (NDL) that yields still more. They studied several various aspects about the neighborhood degree list. Specially, they prove a theorem which gives a necessary and sufficient condition for a given feasible tableau to be the neighborhood degree list of a simple graph [2, Theorem 2.1].

In this paper we introduce a concept, called the "degree polynomial" for the vertices of a simple graph. This notion leads to the concept of degree polynomial sequence which contains precisely the same information as the neighborhood degree list.

Although the degree polynomial sequence is only a reformulation of the neighborhood degree list, there are many advantages arising from recording the informations in polynomial form. Some of these advantages will be revealed in the following.

After obtaining the degree polynomial sequence for some well-known graphs as the cycles, the complete graphs, the complete bipartite graphs, etc., we prove a theorem which gives a necessary condition for the realizability of a sequence of polynomials with positive integer coefficients. Also we study the behavior of the degree polynomial, under several graph operations. More precisely, we calculate the degree polynomial for the vertices of the join, Cartesian product, tensor product, and lexicographic product of two simple graphs and also for the vertices of the complement of a simple graph. Some important examples, counterexamples, and open problems are presented, as well.

2. Preliminaries

In the following, we use [4] for the basic terminologies and notation in graph theory.

Let G be a simple graph. For two vertices $u, v \in V(G)$, if u is adjacent to v, we write $u \sim v$.

Let G be a simple graph of order n. A non-increasing sequence of nonnegative integers $q = (d_1, \ldots, d_n)$ is said to be the degree sequence of G, whenever there exists an ordering $V_1, \ldots, V_n$ of the vertices of G, such that d_i is the degree of v_i for $1 \leq i \leq n$. A sequence $q = (d_1, \ldots, d_n)$ of integers is realizable, if there exists a simple graph G, such that q is the degree sequence of G. Since adding a finite number of isolated vertices to a graph, and deleting a finite number of such vertices from a nonempty graph makes no change in the degree of the other vertices, we can consider only the case in which each d_i, $1 \leq i \leq n$, is positive.

Let G and H be simple graphs with disjoint vertex sets. The join of G and H, denoted by $G \vee H$, is a simple graph with vertex set $V(G) \cup V(H)$, in which for two vertices u and v, $u \sim v$ if and only if

(1) $u, v \in V(G)$ and $u \sim v$ (in G), or
(2) $u, v \in V(H)$ and $u \sim v$ (in H), or
(3) one of the vertices u and v is in $V(G)$, and the other is in $V(H)$.

Let G and H be two simple graphs. The Cartesian product of G and H, denoted by $G \times H$, is a simple graph with vertex set $V(G) \times V(H)$, in which for two vertices (u_1, v_1) and (u_2, v_2), $(u_1, v_1) \sim (u_2, v_2)$ if and only if

(1) $u_1 = u_2$ and $v_1 \sim v_2$ (in H), or
(2) $v_1 = v_2$ and $u_1 \sim u_2$ (in G).

Also, the tensor product of G and H, denoted by $G \otimes H$, is a simple graph with vertex set $V(G) \times V(H)$, in which for two vertices (u_1, v_1) and (u_2, v_2), $(u_1, v_1) \sim (u_2, v_2)$ if and only if $u_1 \sim u_2$ (in G) and $v_1 \sim v_2$ (in H). Finally, the lexicographic product of G and H, denoted by $G[H]$, is a simple graph with vertex set $V(G)\times V(H)$, in which for two vertices (u_1, v_1) and (u_2, v_2), $(u_1, v_1) \sim (u_2, v_2)$, if and only if

(1) $u_1 \sim u_2$ (in G), or
(2) $u_1 = u_2$ and $v_1 \sim v_2$ (in H).

For a simple graph G, the complement of G, denoted by G^c, is a simple graph with vertex set $V(G)$, in which for two vertices u and v, $u \sim v$ if and only if u is not adjacent with v in G.

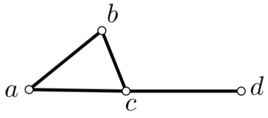
3. Neighborhood degree list (NDL)

Definition 3.1. Let G be a simple graph without any isolated vertex. The neighborhood degree list (NDL) of G is a list

$$\tau(G) = ((\tau_1^1, \dots, \tau_{d_1}^1), (\tau_1^2, \dots, \tau_{d_2}^2), \dots, (\tau_1^n, \dots, \tau_{d_n}^n))$$

for which there exists an ordering $V_1, \dots, V_n$ of the vertices of G, such that $(\tau_1^i, \dots, \tau_{d_i}^i)$ is the list of degrees of the neighbors of v_i for $1 \le i \le n$, and $d_1 \ge d_2 \ge \cdots \ge d_n$ while $\tau_1 \ge \cdots \ge \tau_{d_i}$ for $1 \le i \le n$, see [2].

Example 3.2. Consider the graph G with the following representation.



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The NDL of G is

$$((2, 2, 1), (3, 2), (3, 2), (3)).$$

Definition 3.3. A tableau is a list

$$T = ((\tau_1^1, \dots, \tau_{d_1}^1), \dots, (\tau_1^n, \dots, \tau_{d_n}^n))$$

of n lists of nonnegative integers, where $d_1 \geq \cdots \geq d_n$ and $\tau_1 \geq \cdots \geq \tau_{d_i}$ for $1 \leq i \leq n$, see [2].

A tableau T is called feasible whenever each integer in any list of T is equal to one of the lengths of the lists of T, see [2].

For example, the list

$$((3, 2, 1), (2, 1, 1), (2, 1), (1))$$

is a feasible tableau.

It is clear that for a simple graph G, if

$$\tau(G) = ((\tau_1^1, \dots, \tau_{d_1}^1), (\tau_1^2, \dots, \tau_{d_2}^2), \dots, (\tau_1^n, \dots, \tau_{d_n}^n)),$$

then the degree sequence of G is $d_1, d_2, \dots, d_n$.

4. Degree polynomial

Definition 4.1. For a simple graph G, the degree polynomial of G, denoted by $\mathrm{dp}(G)$, is the polynomial $\sum_i t_i x^i$ in $\mathbb{R}[x]$, in which t_i is the number of vertices of G, each of degree i (specially, t_0 is the number of isolated vertices of G). If Δ is the maximum degree of G, $\mathrm{dp}(G)$ is of degree Δ.

Remark 4.2. It is obvious that if n is the order of G, then the sum of all coefficients of $\mathrm{dp}(G)$ (which is $\mathrm{dp}(G)(1)$) equals n. Also, if m is the size of G, then the sum of all coefficients of the derivative of f with respect to x (which is $(\mathrm{dp}(G))'(1)$) equals $2m$. Therefore some important parameters of a graph G can be achieved by the evaluation process on $\mathrm{dp}(G)$.

Remark 4.3. For two simple graphs G and H, the degree polynomials of the graphs $G \vee H$, $G \times H$, $G \otimes H$, and G^c can be obtained by clear formulas only from $\mathrm{dp}(G)$ and $\mathrm{dp}(H)$. Therefore working with polynomials is certainly convenient in illustrating the effect of graph operations on degree information.

Remarks 4.2 and 4.3 provided motivation for reformulating the concept of neighborhood degree list, with use of polynomials.

Before this, we introduce some notations for convenience. For a polynomial $f(x) = \sum_{i=1}^n a_i x^i \in \mathbb{R}[x]$ with $a_n \neq 0$, we denote the sum of a_i's for $1 \leq i \leq n$, by $\mathrm{sc}(f)$. Also $\mathrm{sec}(f)$ and $\mathrm{soc}(f)$, are used for the sum of a_i's for even i, and sum of a_i's for odd i, respectively. We define $\mathrm{sc}(0) = 0$, as well.

We introduce a total order "$<_{\rm pol}$" on the set of all nonzero polynomials with coefficients in nonnegative integers such that "$<_{\rm pol}$" compares two distinct polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$ with nonnegative integer coefficients, and with $a_n, b_m \neq 0$, as follows:

If $\mathrm{sc}(f) \neq \mathrm{sc}(g)$, then which one of f and g has the sum of coefficients greater (as an integer), will be greater.

If $\mathrm{sc}(f) = \mathrm{sc}(g)$, $i_1 = \max\{i\colon\ a_i \neq 0 \text{ or } b_i \neq 0\}$ and $a_{i_1} \neq b_{i_1}$, then whichever of f and g has greater coefficient in x^{i_1}, will be greater.

If $\mathrm{sc}(f) = \mathrm{sc}(g)$, $a_{i_1} = b_{i_1}$, $i_2 = \max\{i\colon\ i < i_1,\ a_i \neq 0 \text{ or } b_i \neq 0\}$, and $a_{i_2} \neq b_{i_2}$, then whichever of f and g has greater coefficient in x^{i_2}, will be greater; and so on.

For example,

$$\begin{aligned} 2x^4 + 12x^3 &>_{\rm pol} 3x^5 + x^2, \\ 2x^4 + 12x^2 &<_{\rm pol} 2x^5 + 12x^2,\ x^5 + 13x^2, \\ 2x^4 + 12x^2 &>_{\rm pol} 2x^4 + 11x^2 + x. \end{aligned}$$

Let $f = \sum_{a_i \neq 0} a_i x^i$ be a nonzero polynomial in $\mathbb{R}[x]$ with nonnegative integer coefficients where $a_i x^i$'s are the nonzero terms of f. For $n \in \mathbb{N}$, we denote the polynomial $\sum_{a_i \neq 0} a_i x^{in}$ by $f^{\curlywedge \times n}$. Also we set $0^{\curlywedge \times n} = 0$. If $\deg f \leq n$, we denote the polynomial $\sum_{a_i \neq 0} a_i x^{n-i}$ by $f^{\curlywedge n-}$. Also we set $0^{\curlywedge n-} = 0$.

Definition 4.4. Let $f = \sum_{a_i \neq 0} a_i x^i$, $g = \sum_{b_j \neq 0} b_j x^j$ be two nonzero polynomials in $\mathbb{R}[x]$ with nonnegative integer coefficients where $a_i x^i$'s and $b_j x^j$'s are the nonzero terms of f and g, respectively. The tensor product of f and g, denoted by $f \otimes g$, is the polynomial $\sum c_t x^t$ in which t's are the distinct products of i's and j's, and

$$c_t = \sum_{i.j=t} a_i b_j.$$

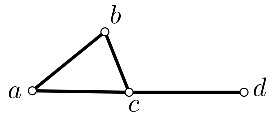
Also we set $0 \otimes 0 = 0$, $0 \otimes f = 0$, where 0 is the zero polynomial.

Under the conditions of Definition 4.4., it is observed simply that, first, $f \otimes g$ can be achieved by tensor-multiplying the nonzero terms of f by the nonzero terms of g, one by one, and secondly for each f and g with variable x and nonnegative integer coefficients, $f \otimes g = g \otimes f$.

Now we introduce a new concept, that is the concept of degree polynomial.

Definition 4.5. Let G be a simple graph. For a vertex v of G, the degree polynomial of v, denoted by $\mathrm{dp}(v)$, is a polynomial with nonnegative integer coefficients, in which the coefficient of x^i is the number of neighbors of v, each of degree i. Especially, for an isolated vertex v, $\mathrm{dp}(v) = 0$.

Example 4.6. For the graph G with representation



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we have

$$\mathrm{dp}(a) = x^2 + x^3,$$
$$\mathrm{dp}(b) = x^2 + x^3,$$
$$\mathrm{dp}(c) = 2x^2 + x,$$
$$\mathrm{dp}(d) = x^3.$$

Since adding a finite number of isolated vertices to a simple graph, and deleting a finite number of such vertices from a nonempty simple graph makes no change in the degree polynomials of the other vertices, we will consider only the graphs which has no isolated vertices.

Definition 4.7. For a simple graph G of order n without any isolated vertex, a sequence $q = (f_1, f_2, \ldots, f_n)$ of polynomials is said to be the degree polynomial sequence of G, if

(a) $f_1 \geq_{\text{pol}} \cdots \geq_{\text{pol}} f_n$,
(b) There exists an ordering $V_1, \ldots, V_n$ of the vertices of G, such that f_i be the degree polynomial of v_i for $1 \leq i \leq n$.

Example 4.8. For the graph G in Example 4.6, the degree polynomial sequence is

$$2x^2 + x, x^2 + x^3, x^2 + x^3, x^3.$$

Proposition 4.9. *Let G be a nonempty simple graph. Graph G is r-regular if and only if each term of the degree polynomial sequence of G be in the form rx^r.*

If G is a nontrivial complete graph, K_n, it is obvious that the degree polynomial sequence of G is

$$(n-1)x^{n-1}, \ldots, (n-1)x^{n-1}$$

where the number of terms is n. If G is a path with n vertices, P_n, then if $n = 2$, the degree polynomial sequence of G is

$$x, x,$$

if $n = 3$, that will be

$$2x, x^2, x^2,$$

if $n = 4$, that will be

$$x + x^2, x + x^2, x^2, x^2,$$

and finally, if $n \geq 5$, that will be

$$2x^2, \ldots, 2x^2, x + x^2, x + x^2, x^2, x^2$$

where the number of terms $2x^2$ is $n - 4$.

If G is a cycle C_n ($n \geq 3$), then the degree polynomial sequence of G is

$$2x^2, \ldots, 2x^2,$$

where the number of terms is n.

If G is a complete bipartite graph, $K_{r,s}$ where $r \geq s$, then the degree polynomial sequence of G is

$$rx^s, \ldots, rx^s, sx^r, \ldots, sx^r,$$

where s terms are rx^s and r terms are sx^r.

Remark 4.10. In the following, we will see that the reformulation of the concept of the neighborhood degree list in polynomial form has benefits similar to those described in Remarks 4.2 and 4.3.

Remark 4.11. Supposing that $q = (f_1, \ldots, f_n)$ is the degree polynomial sequence of a simple graph G, and f_i is the degree polynomial of the vertex v_i, the degree sequence for G is

$$\mathrm{sc}(f_1), \ldots, \mathrm{sc}(f_n),$$

since $\mathrm{sc}(f_i)$ is the degree of v_i. Consequently, if a sequence $q = (f_1, \ldots, f_n)$ of nonzero polynomials is realizable, then the sequence

$$\mathrm{sc}(f_1), \mathrm{sc}(f_2), \ldots, \mathrm{sc}(f_n),$$

that is, the sequence

$$f_1(1), \ldots, f_n(1)$$

of integers is realizable. The following example shows that the inverse case is not true in general.

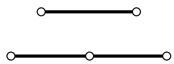
Example 4.12. Consider the sequence

$$2x, x^2, x, x, x$$

of nonzero polynomials with nonnegative integer coefficients. Although the sequence

$$\mathrm{sc}(2x), \mathrm{sc}(x^2), \mathrm{sc}(x), \mathrm{sc}(x), \mathrm{sc}(x)$$

is realized by the simple graph with representation



but the sequence

$$2x, x^2, x, x, x$$

is not realizable, by Theorem 4.14 (part c).

Remark 4.13. Considering that the concept of degree polynomial sequence is a reformulation for the concept of NDL, the following facts can be extracted from [2]:

- Two simple graphs with the same degree sequence, can have different degree polynomial sequences.
- Two non-isomorphic graphs can have the same degree polynomial sequences.

Now we prove a theorem which gives a necessary condition for the realizability of a sequence of polynomials with nonnegative integer coefficients.

Theorem 4.14. *If G is a simple graph without any isolated vertices, and $q = (f_1, \ldots, f_n)$ where $f_1 \geq_{\text{pol}} \cdots \geq_{\text{pol}} f_n$ is the degree polynomial sequence of G, then*

(a) $\sum_{i=1}^{n} \mathrm{sc}(f_i)$ *is even,*

(b) *for each nonzero coefficient k of a term kx^i in the degree polynomial of a vertex v, there are at least k distinct vertices $v_1, \ldots, v_k$, all distinct from v, such that*

$$\mathrm{sc}(\mathrm{dp}(v_1)) = \cdots = \mathrm{sc}(\mathrm{dp}(v_k)) = i,$$

(c) $\sum_{\mathrm{sc}(f_j) \text{ is odd}} \mathrm{sec}(f_j)$ *and* $\sum_{\mathrm{sc}(f_j) \text{ is even}} \mathrm{sec}(f_j)$ *are even.*

Proof: (a) Let $f_i = \mathrm{dp}(v_i)$ for $1 \leq i \leq n$. We have $\sum_{i=1}^{n} \mathrm{sc}(f_i) = \sum_{i=1}^{n} \deg(v_i)$. Thus $\sum_{i=1}^{n} \mathrm{sc}(f_i)$ is even.

(b) Let $k (\neq 0)$ be the coefficient of kx^i in $\mathrm{dp}(v)$. Therefore v has exactly k neighbors $v_1, \ldots, v_k$ of degree i. Now, $v_1, \ldots, v_k$ are distinct from v, and

$$\mathrm{sc}(\mathrm{dp}(v_1)) = \cdots = \mathrm{sc}(\mathrm{dp}(v_k)) = i.$$

(c) Let $\{a_1, \ldots, a_s\}$ be the set of odd vertices of G. Then $\sum_{j=1}^{n} \deg(a_j)$ is even. That is, $\sum_{j=1}^{s} \mathrm{sc}(\mathrm{dp}(a_j))$ is even. Thus $\sum_{j=1}^{s} \mathrm{sec}(\mathrm{dp}(a_j)) + \sum_{j=1}^{s} \mathrm{soc}(\mathrm{dp}(a_j))$ is even. For each a_j, $1 \leq j \leq s$, if $a_{j'}$ is an odd neighbor of a_j, then a_j is an odd neighbor of $a_{j'}$, as well. Therefore the edge between a_j and $a_{j'}$, accurs two times in calculating $\sum_{j=1}^{s} \mathrm{soc}(\mathrm{dp}(a_j))$, once in $\mathrm{soc}(\mathrm{dp}(a_j))$ and again in

$\mathrm{soc}(\mathrm{dp}(a_{j'}))$. Thus $\sum_{j=1}^{s} \mathrm{soc}(\mathrm{dp}(a_j))$ is even and so is $\sum_{j=1}^{s} \mathrm{sec}(\mathrm{dp}(a_j))$. Hence $\sum_{\mathrm{sc}(f_j) \text{ is odd}} \mathrm{sec}(f_j)$, is an even integer.

The argument for second part is similar, with the difference that we should start with the set of all even vertices, $\{b_1, \ldots, b_t\}$. □

Example 4.15. The sequence

$$s_1 = (2x, x^2, x, x, x)$$

of polynomials satisfies (a) and (b), but not (c); the sequence

$$s_2 = (2x, x^2, x^2, x, x, x)$$

satisfies (b) and (c), but not (a). Finally the sequence

$$s_3 = (2x^2, x, x, x, x)$$

satisfies (a) and (c), but not (b). Therefore by Theorem 4.14, the sequences s_1, s_2 and s_3 are not realizable. Meanwhile it is possible that a non-increasing sequence $q = (f_1, f_2, \ldots, f_n)$ of nonzero polynomials with nonnegative integer coefficients satisfies (a), (b) and (c), but it is not realizable yet. Consider for example, the sequence

$$2x^2, 2x, 2x, x, x.$$

Note that if the sequence was realizable, then any vertex v for which $\mathrm{dp}(v) = 2x$, should be adjacent only with two vertices with degree polynomials x and x (name (a) and (b)). But in this case, the degree polynomial of (a) and (b) will not be x.

Remark 4.16. Of course, one can examine the non-realizability of the sequences in Example 4.15 by Theorem 2.1 in [2]. But note that the use of this theorem practically requires a long process but Theorem 4.14 uses only polynomial invariants.

Now we study the behavior of the degree polynomial under graph operations.

Theorem 4.17. *Let G and H be two simple graphs with disjoint vertex sets, and u be a vertex in G. Then*

$$\mathrm{dp}_{G \vee H}(u) = x^{n_2} \mathrm{dp}_G(u) + x^{n_1} \mathrm{dp}(H),$$

where n_1 and n_2 are the orders of G and H, respectively.

Proof: If u is an isolated vertex in G, then $\mathrm{dp}_G(u) = 0$. In this case, u is adjacent to all vertices of H in $G \vee H$ (by definition of $G \vee H$) and u is not adjacent to any vertex of G. If H has t_0 vertices of degree 0, t_1 vertices of degree $1, \ldots, t_\Delta$ vertices of degree Δ (Δ is the maximum degree of H), then the neighbors of u in $G \vee H$ are restricted to

$$\begin{gathered} t_0 \text{ vertices of degree } n_1 + 0, \\ t_1 \text{ vertices of degree } n_1 + 1, \\ \vdots \\ t_\Delta \text{ vertices of degree } n_1 + \Delta, \end{gathered}$$

and therefore the degree polynomial of u in $G \vee H$ is

$$t_0 x^{n_1} + t_1 x^{n_1+1} + \cdots + t_\Delta x^{n_1+\Delta} = x^{n_1} \mathrm{dp}(H),$$

and the conclusion holds.

Now let u be non-isolated vertex in G. Suppose that $\mathrm{dp}_G(u) = \sum_{s=1}^{k} c_{i_s} x^{i_s}$ where c_{i_s}'s are positive integers and i_s's are the distinct degrees of neighbors of u in G. It means that the neighbors of u in G are restricted to

$$\begin{gathered} c_{i_1} \text{ vertices of degree } i_1, \\ \vdots \\ c_{i_k} \text{ vertices of degree } i_k. \end{gathered}$$

Now by definition of $G \vee H$, u will be adjacent in $G \vee H$ to all of the above vertices, and also with any vertex in H. Thus the neighbors of u in $G \vee H$ are restricted to

$$\begin{gathered} c_{i_1} \text{ vertices of degree } n_2 + i_1, \\ \vdots \\ c_{i_k} \text{ vertices of degree } n_2 + i_k, \\ t_0 \text{ vertices of degree } n_1 + 0, \\ t_1 \text{ vertices of degree } n_1 + 1, \\ \vdots \\ t_\Delta \text{ vertices of degree } n_1 + \Delta, \end{gathered}$$

where $\mathrm{dp}(H) = \sum_{i=0}^{\Delta} t_i x^i$. Therefore

$$\begin{aligned} \mathrm{dp}(G \vee H) &= c_{i_1} x^{n_2+i_1} + \cdots + c_{i_k} x^{n_2+i_k} + t_0 x^{n_1} + t_1 x^{n_1+1} + \cdots + t_\Delta x^{n_1+\Delta} \\ &= x^{n_2} \mathrm{dp}_G(u) + x^{n_1} \mathrm{dp}(H). \end{aligned}$$

□

Remark 4.18. Since for every two simple graphs G and H, $G \vee H = H \vee G$, the above theorem, in practice, provides a tool for calculating the degree polynomial of any vertex of $G \vee H$.

Theorem 4.19. *If G and H are two simple graphs, and u and v are vertices of G and H, respectively, then*

$$\mathrm{dp}_{G\times H}((u,v)) = x^{\deg u}\mathrm{dp}(v) + x^{\deg v}\mathrm{dp}(u).$$

PROOF: If u in G and v in H are isolated, then by definition of $G \times H$, (u,v) in $G \times H$ is an isolated vertex and therefore $\mathrm{dp}((u,v)) = 0$. On the other hand, $\mathrm{dp}(u) = 0$ and $\mathrm{dp}(v) = 0$. Therefore the conclusion holds.

If u is isolated in G but v is non-isolated in H, supposing that $\mathrm{dp}(v) = \sum_{r_j \neq 0} r_j x^j$ where r_j's are positive integers and j's are the disjoint degrees of the neighbors of v in H, by definition of $G \times H$, each neighbor of (u,v) in $G \times H$ is in the form (u,v') with $v' \sim v$ since u has not any adjacent vertex in G. Meanwhile the degree of such (u,v') in $G \times H$ is $\deg u + \deg v'$. Since for each j, v has r_j neighbors of degree j, the number of neighbors of (u,v) of degree $\deg u + j$ will be r_j. Thus

$$\mathrm{dp}_{G\times H}((u,v)) = \sum_{r_j} r_j x^{\deg u + j} = x^{\deg u}\mathrm{dp}(v).$$

But in this case, $\mathrm{dp}(u) = 0$ and therefore the conclusion holds.

The argument in the case that v is isolated but u is not, is similar to the argument in the previous case.

Now let neither of u and v be isolated. Suppose that $\mathrm{dp}(u) = \sum_{s=1}^{k} c_{i_s} x^{i_s}$, and $\mathrm{dp}(v) = \sum_{t=1}^{k'} r_{j_t} x^{j_t}$, where c_{i_s}'s and r_{j_t}'s are positive integers, and i_s's and j_t's are the disjoint degrees of the neighbors of u and v, respectively. This means that the neighbors of u in G are restricted to

$$c_{i_1} \text{ vertices of degree } i_1,$$
$$\vdots$$
$$c_{i_k} \text{ vertices of degree } i_k,$$

and the neighbors of v in H are restricted to

$$r_{j_1} \text{ vertices of degree } j_1,$$
$$\vdots$$
$$r_{j_{k'}} \text{ vertices of degree } j_{k'}.$$

By definition of $G \times H$, the adjacent vertices of (u, v) in $G \times H$ are of two kinds below:

(i) the vertices in the form (u, b) where b is adjacent to v in H,
(ii) the vertices in the form (a, v) where a is adjacent to u in G.

Since, for all vertices of kind (i), u is fixed, such vertices are restricted to

$$r_{j_1} \text{ vertices of degree } \deg u + j_1,$$
$$\vdots$$
$$r_{j_{k'}} \text{ vertices of degree } \deg u + j_{k'}.$$

Also, since, for the vertices of kind (ii), v is fixed, such vertices are restricted to

$$c_{i_1} \text{ vertices of degree } i_1 + \deg v,$$
$$\vdots$$
$$c_{i_k} \text{ vertices of degree } i_k + \deg v.$$

Note that the degree of each vertex, (x, y), in $G \times H$ is $\deg_G x + \deg_H y$. Therefore

$$\begin{aligned} \mathrm{dp}_{G\times H}((u,v)) &= (r_{j_1} x^{\deg u + j_1} + \cdots + r_{j_{k'}} x^{\deg u + j_{k'}}) \\ &\quad + \ (c_{i_1} x^{i_1 + \deg v} + \cdots + c_{i_k} x^{i_k + \deg v}) \\ &= x^{\deg u} \mathrm{dp}(v) + x^{\deg v} \mathrm{dp}(u). \end{aligned}$$

□

Theorem 4.20. *If G and H are two simple graphs, and u and v are vertices of G and H, respectively, then*

$$\mathrm{dp}_{G\otimes H}((u,v)) = \mathrm{dp}(u) \otimes \mathrm{dp}(v).$$

Proof: If at least one of u and v are isolated, then by definition of $G \otimes H$, (u, v) is an isolated vertex in the graph $G \otimes H$, and therefore $\mathrm{dp}((u, v)) = 0$. On the other hand, in this case, at least one of $\mathrm{dp}(u)$ and $\mathrm{dp}(v)$ is zero. Therefore by definition of the tensor product of polynomials, $\mathrm{dp}(u) \otimes \mathrm{dp}(v)$ is zero as well, and the conclusion holds.

Now let none of u and v be isolated. Suppose that $\mathrm{dp}(u) = \sum_i c_i x^i$ and $\mathrm{dp}(v) = \sum_j r_j x^j$, where c_i's and r_j's are positive integers, and i's and j's are the disjoint degrees of the neighbors of u and v, respectively. By definition of $G \otimes H$, first, each neighbor of (u, v) in $G \otimes H$ is in the form (u', v') where u' is a neighbor of u, and v' is a neighbor of v, and secondly if u' is a neighbor of u of degree i and v' is a neighbor of v of degree j, then (u', v') is a neighbor of (u, v)

of degree $i \cdot j$. Since for each i, u has exactly c_i neighbors of degree i, and for each j, v has exactly r_j neighbors of degree j, the number $c_i r_j$ is calculated in the coefficient of $x^{i \cdot j}$ in the degree polynomial of (u, v). This implies that

$$\mathrm{dp}_{G \otimes H}((u, v)) = \mathrm{dp}(u) \otimes \mathrm{dp}(v)$$

by definition of $\mathrm{dp}(u) \otimes \mathrm{dp}(v)$. □

Theorem 4.21. *If G and H are two simple graphs and u and v are vertices of G and H, respectively, then*

$$\begin{aligned}\mathrm{dp}_{G[H]}((u, v)) &= (\mathrm{dp}(u))^{\curlywedge \times n_2} \mathrm{dp}(H) + x^{(\deg u) n_2} \mathrm{dp}(v) \\ &(= \mathrm{dp}(u)(x^{n_2}) \mathrm{dp}(H) + x^{(\deg u) n_2} \mathrm{dp}(v)),\end{aligned}$$

in which n_2 is the order of H.

Proof: If u and v are both isolated, by definition of $G[H]$, (u, v) is isolated in $G[H]$, and therefore $\mathrm{dp}_{G[H]}((u, v)) = 0$. But in this case, both $\mathrm{dp}(u)$ and $\mathrm{dp}(v)$ are zero polynomials, and therefore the conclusion holds.

If u is isolated in G, but v is not isolated in H, supposing that $\mathrm{dp}(v) = \sum_{t=1}^{k'} r_{j_t} x^{j_t}$, in which r_{j_t}'s are positive integers and j_t's are the disjoint degrees of the neighbors of v in H, the neighbors of v in H are restricted to

$$\begin{gathered} r_{j_1} \text{ vertices of degree } j_1, \\ \vdots \\ r_{j_{k'}} \text{ vertices of degree } j_{k'}. \end{gathered}$$

Since u has no any neighbor in G, by definition of $G[H]$, every neighbors of (u, v) is of the form (u, b) with degree $(\deg u) n_2 + \deg b$, where b is a neighbor of v in H. Hence since u is fixed, the neighbors of (u, v) are restricted to

$$\begin{gathered} r_{j_1} \text{ vertices of degree } (\deg u) n_2 + j_1, \\ \vdots \\ r_{j_{k'}} \text{ vertices of degree } (\deg u) n_2 + j_{k'}. \end{gathered}$$

Thus

$$\mathrm{dp}_{G[H]}((u, v)) = r_{j_1} x^{(\deg u) n_2 + j_1} + \cdots + r_{j_{k'}} x^{(\deg u) n_2 + j_{k'}} = x^{(\deg u) n_2} \mathrm{dp}(v),$$

and since $\mathrm{dp}(u) = 0$, the conclusion in this case holds.

If v is isolated in H but u is not isolated in G, since v has not any neighbor in H, each neighbor of (u, v) is in the form (a, b) such that a is a neighbor of u, and b is a vertex of H, and the degree of (a, b) in $G[H]$ is $(\deg a) n_2 +$

$\deg b$. Suppose that $\mathrm{dp}(u) = \sum_{s=1}^{k} c_{i_s} x^{i_s}$ where c_{i_s}'s are positive integers and i_s's are the distinct degrees of the neighbors of u in G. Supposing that $\mathrm{dp}(H) = \sum_{p=0}^{\Delta} l_p x^p$ where l_p's are the number of vertices of H, each one of degree p, and Δ is the maximum degree of H, for each p, the neighbors of (u, v) whose second components are of degree p, are restricted to

$$l_p c_{i_1} \text{ vertices of degree } i_1 n_2 + p,$$
$$\vdots$$
$$l_p c_{i_k} \text{ vertices of degree } i_k n_2 + p.$$

Therefore each $l_p c_{i_s}$ is calculated in the coefficient of $x^{i_s n_2 + p}$ in $\mathrm{dp}((u, v))$. Thus

$$\begin{aligned}\mathrm{dp}_{G[H]}((u,v)) &= \sum_{p=0}^{\Delta}\sum_{s=1}^{k} l_p c_{i_s} x^{i_s n_2 + p} = \sum_{s=1}^{k} c_{i_s} x^{i_s n_2} \sum_{p=0}^{\Delta} l_p x^p \\ &= \sum_{s=1}^{k} c_{i_s} x^{i_s n_2} \mathrm{dp}(H) = (\mathrm{dp}(u))^{\curlywedge n_2} \mathrm{dp}(H),\end{aligned}$$

and since $\mathrm{dp}(v) = 0$, the conclusion holds.

Now let none of u and v be isolated. Suppose that $\mathrm{dp}(u) = \sum_{s=1}^{k} x_{i_s} x^{i_s}$ and $\mathrm{dp}(v) = \sum_{t=1}^{k'} r_{j_t} x^{j_t}$ where c_{i_s}'s and r_{j_t}'s are positive integers and i_s's and j_t's are the distinct degrees of the neighbors of u and v, respectively. The adjacent vertices of (u, v) in $G[H]$ are of two kinds below:

(i) the vertices in the form (a, b) where a is a neighbor of u in G, and b is a vertex of H,

(ii) the vertices in the form (u, b) where b is a neighbor of v in H.

Let $\mathrm{dp}(H) = \sum_{p=0}^{\Delta} l_p x^p$ where Δ be the maximum degree of H. For each p, the neighbors of (u, v) of kind (i) whose second components are of degree p, are restricted to

$$l_p c_{i_1} \text{ vertices of degree } i_1 n_2 + p,$$
$$\vdots$$
$$l_p c_{i_k} \text{ vertices of degree } i_k n_2 + p.$$

Therefore each $l_p c_{i_s}$ is calculated in the coefficient of $x^{i_s n_2 + p}$. On the other hand, since in all neighbors of kind (ii), u is fixed, such vertices are restricted to

$$r_{j_1} \text{ vertices of degree } (\deg u)n_2 + j_1,$$

$$\vdots$$

$$r_{j_{k'}} \text{ vertices of degree } (\deg u)n_2 + j_{k'}.$$

Thus

$$\begin{aligned}
\mathrm{dp}_{G[H]}((u,v)) &= \sum_{p=0}^{\Delta}\sum_{s=1}^{k} l_p c_{i_s} x^{i_s n_2 + p} + \sum_{t=0}^{k'} r_{j_t} x^{(\deg u)n_2 + j_t} \\
&= (\mathrm{dp}(u))^{\curlywedge n_2}\mathrm{dp}(H) + x^{(\deg u)n_2}\mathrm{dp}(v).
\end{aligned}$$

□

As we saw above, having the degree polynomial sequence of graphs G and H without access to G and H, the degree polynomial sequences of $G \vee H$, $G \times H$, $G \otimes H$, and $G[H]$ are calculatable. The following theorem shows that the degree polynomial sequence of the complement of a graph can be calculated having the degree polynomial sequence of that graph without access to the graph itself.

Theorem 4.22. *Let G be a simple graph and u be a vertex of G. Then*

$$\mathrm{dp}_{G^c}(u) = (\mathrm{dp}(G) - \mathrm{dp}_G(u) - x^{\deg_u G})^{\curlywedge (n-1)-}$$

where n is the order of G.

PROOF: For each integer $i \geq 0$, the coefficient of x^i in $\mathrm{dp}(G)$ is the total number of the vertices of G, each one of degree i, and the coefficient of the same x^i in $\mathrm{dp}(u)$ is exactly the number of the vertices of G, each one of degree i, which are adjacent to u in G. Therefore the coefficient of each x^i in the polynomial

$$\mathrm{dp}(G) - \mathrm{dp}_G(u)$$

is the number of the vertices of degree i (in G) which are non-adjacent to u. Since u itself is non-adjacent to u, the coefficient of x^i in the polynomial

$$\mathrm{dp}(G) - \mathrm{dp}_G(u) - x^{\deg_G u}$$

is exactly the number of the vertices of G, other than u, which are of degree i and non-adjacent to u (in G), and by definition of G^c, this number is exactly the number of the vertices of G^c which are of degree $(n-1) - i$ and adjacent to u (in G^c). Therefore for each i, the coefficient of $x^{(n-1)-i}$ in the polynomial $\mathrm{dp}(G) - \mathrm{dp}_G(u) - x^{\deg_G u}$ equals exactly the number of the vertices of degree i in G^c, which are adjacent to u (in G^c). Thus (based on the meaning of the notation $(\mathrm{dp}(G)-\mathrm{dp}_G(u)-x^{\deg_G u})^{\curlywedge(n-1)-})$ the coefficient of x^i in

$$(\mathrm{dp}(G)-\mathrm{dp}_G(u)-x^{\deg_G u})^{\curlywedge(n-1)-}$$

is exactly the number of the neighbors of u in G^c whose degree is i. □

5. Some open problems

Many new questions and open problems can arise from the above topics. Some of them are:

(1) Classify all degree polynomial sequences of connected graphs and trees.
(2) Characterize all graphs whose degree polynomial sequences are formed by polynomials with only one term.

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