On Szymański theorem on hereditary normality of $\beta\omega$

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Abstract. We discuss the following result of A. Szymański in "Retracts and non-normality points" (2012), Corollary 3.5.: If F is a closed subspace of ω^* and the π-weight of F is countable, then every nonisolated point of F is a non-normality point of ω^*.

We obtain stronger results for all types of points, excluding the limits of countable discrete sets considered in "Some non-normal subspaces of the Čech–Stone compactification of a discrete space" (1980) by A. Błaszczyk and A. Szymański. Perhaps our proofs look "more natural in this area".

Keywords: Čech–Stone compactification; non-normality point; butterfly-point; countable π-weight

Classification: 54D15, 54D35, 54D40, 54D80, 54E35, 54G20

1. Introduction

We investigate hereditary normality of Čech–Stone compactification βX of a completely regular space X.

Is $X^* \setminus \{p\}$ non-normal for any point p of the remainder $X^* = \beta X \setminus X$?

If so, then p is called a *non-normality point of* X^*. Usually, in order to answer this question positively, we have to show that p is a *butterfly-point* or a *b-point of* βX, see [4], i.e. to construct sets $F, G \subset X^* \setminus \{p\}$, which are closed in $\beta X \setminus \{p\}$, so that $\{p\} = [F] \cap [G]$, see also [6]. A. Szymański in [7] gave a different approach.

Particularly this question is intriguing for countable discrete space $\omega = \{0, 1, 2, \dots\}$.

A. Błaszczyk and A. Szymański in [2] proved in 1980 that p is a *non-normality point of* ω^*, if p is a limit point of some countable discrete set $P \subset \omega^*$.

A point p is called a *Kunen point*, if there exists a discrete set $P \subset \omega^*$ of cardinality ω_1, that is, no more than countable outside any neighbourhood of p. Every Kunen point is a non-normality point of ω^* (E. K. van Douwen, unpublished).

Some other more technical results were obtained in [3].

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The answer is known and positive under CH (continuum hypothesis), see N. Warren [8] and M. Rajagopalan, [5] 1972, or even MA (Martin's axiom), see A. Bešlagić and E. van Douwen, [1] 1990.

In 2012 A. Szymański in [7] obtained the following result:

Corollary 3.5. If F is a closed subspace of ω^* and the π-weight of F is countable, then every nonisolated point of F is a non-normality point of ω^*.

Let D be all isolated points of F. If $p \in [D]$, then Corollary 3.5. reduces to the well known result of A. Błaszczyk and A. Szymański in [2]. Otherwise, we can assume F to be crowded.

Theorem 1. *If F is a closed crowded subspace of ω^* and the π-weight of F is countable, then every point of F is a non-normality point of F.*

We show that F has a π-base $\mathcal{B}$ with the following property:

(*) If $\mathcal{D}, \mathcal{C} \subset \mathcal{B}$ and $\left(\bigcup \mathcal{D}\right) \cap \left(\bigcup \mathcal{C}\right) = \emptyset$, then $\left[\bigcup \mathcal{D}\right] \cap \left[\bigcup \mathcal{C}\right] = \emptyset$.

Then we obtain Theorem 1 as a corollary of the next

Theorem 2. *Let a normal realcompact crowded space X have a weakly embedded σ-cellular π-base $\mathcal{B}$ with the property (*). Then every point $p \in X^*$ is a b-point of βX. Hence $\beta X \setminus \{p\}$ is not normal.*

2. Preliminaries

A space X is crowded, if X has no isolated points, $3 = \{0, 1, 2\}$. By $[\ ]$ we always denote the closure operator in βX. Let $\mathcal{B}$ be a family of nonempty open sets. Then $\mathcal{B}$ is weakly embedded, if any two sets of $\mathcal{B}$ are either disjoint or one of them contains the other and σ-cellular, if $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ and every $\mathcal{B}_n$ is cellular. A set $U \in \mathcal{B}$ is a maximal set of $\mathcal{B}$, if U is a proper subset of V for no $V \in \mathcal{B}$. Moreover, $\mathcal{B}$ is a π-base of X, if any nonempty open set O contains some $U \in \mathcal{B}$, $\mathcal{B}(O) = \{U \in \mathcal{B} \colon U \cap O \neq \emptyset\}$.

Let π and σ be any maximal cellular families of open sets. We write $\pi \prec \sigma$ if $U \cap V \neq \emptyset$ implies $U \supsetneq V$ for any $U \in \pi$ and $V \in \sigma$. Set $\mathcal{P}(\pi) = \{F \colon F \subseteq \pi\}$. We define a projection $f^\pi_\sigma \colon \mathcal{P}(\pi) \to \mathcal{P}(\sigma)$ by

$$f^\pi_\sigma F = \left\{V \in \sigma \colon \bigcup F \cap V \neq \emptyset\right\}.$$

Let $p \in X^*$. Then $\mathcal{F} \subset \mathcal{P}(\pi)$ is called a p-filter on π, if any finite subcollection $\{F_0, \dots, F_n\} \subset \mathcal{F}$ satisfies $p \in \left[\bigcup \bigcap_{k=0}^n F_k\right]$. We denote $\bigcap \mathcal{F}^* = \bigcap \left\{\left[\bigcup F\right] \colon F \in \mathcal{F}\right\}$ and $\pi \succ_{\mathcal{F}} \sigma$, if there is $F \in \mathcal{F}$ with $F \succ \sigma$. The image

$f^\pi_\sigma(\mathcal{F}) = \{f^\pi_\sigma F\colon F \in \mathcal{F}\}$ is a p-filter on σ. Obviously, the union of every increasing family of p-filters is also a p-filter. So by Zorn's lemma there are maximal p-filters or p-ultrafilters $\mathcal{F}$ on π, that is $\mathcal{F} = \mathcal{G}$ for any p-filter $\mathcal{G}$ with $\mathcal{F} \subset \mathcal{G}$.

3. Proofs

Lemma 1. *Let a closed subspace F of ω^* have a countable π-base $\{V_i\}_{i<\omega}$ and let p be a nonisolated point of F. Then there is a countable family $\{U_i\}_{i<\omega}$ of clopen subsets of ω^* with the following properties for all $i < \omega$:*

1) $p \notin U_i$;
2) $U_i \cap F$ *is a nonempty subset of* V_i;
3) $\{U_i\}_{i<\omega}$ *is weakly embedded.*

Proof: Assume $\{U_0, \dots, U_{n-1}\}$ have been constructed for some $n < \omega$ so that 1)–3) hold. To get U_n we need one more induction on $k \le n-1$.

Let U^k_n be constructed so that $\{U_0, \dots, U_{k-1}, U^k_n\}$ satisfies 1)–3). We put either $U^{k+1}_n = U^k_n \cap U_k$ if $U^k_n \cap U_k \cap F \neq \emptyset$ or $U^{k+1}_n = U^k_n \setminus U_k$ otherwise. Then $\{U_0, \dots, U_k, U^{k+1}_n\}$ satisfies 1)–3) and, finally, $U_n = U^n_n$. The family $\{U_n\}_{n<\omega}$ is as required. □

Lemma 2. *Theorem* 2 *implies Theorem* 1.

Proof: In the notation of Lemma 1 we put $X = \bigcup_{i<\omega}(U_i \cap F)$ and $\mathcal{B} = \{U_i \cap X\}_{i<\omega}$. If the conditions of Theorem 1 hold, then X and $\mathcal{B}$ satisfy the conditions of Theorem 2. Indeed, if $\mathcal{D}, \mathcal{C} \subset \mathcal{B}$ and $\left(\bigcup \mathcal{D}\right) \cap \left(\bigcup \mathcal{C}\right) = \emptyset$, then $\mathcal{D}' = \{U_i\colon U_i \cap X \in \mathcal{D}\}$ and $\mathcal{C}' = \{U_i\colon U_i \cap X \in \mathcal{C}\}$ satisfy $\left(\bigcup \mathcal{D}'\right) \cap \left(\bigcup \mathcal{C}'\right) = \emptyset$ by our construction. Since $\bigcup \mathcal{D}'$ and $\bigcup \mathcal{C}'$ are open in ω^* and σ-compact, then $\left[\bigcup \mathcal{D}'\right] \cap \left[\bigcup \mathcal{C}'\right] = \emptyset$. Since X is σ-compact and everywhere dense in F, then $F = \beta X$ is a Čech–Stone compactification of X and $p \in X^*$. □

Now we only have to prove Theorem 2. To a certain extent, we follow the notation and proof scheme of [4].

Lemma 3. *Under the conditions of Theorem* 2 *the π-base $\mathcal{B}$ satisfying* (*) *can be represented as $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ so that:*

(1) *every $\mathcal{B}_n$ is maximal and cellular in X;*
(2) $\mathcal{B}_{n+1} \succ \mathcal{B}_n$;
(3) *for every $U \in \mathcal{B}_n$ there is $\{U(\nu)\colon \nu < 3\} \subset \mathcal{B}_{n+1}$ with $\bigcup_{\nu<3} U(\nu) \subset U$.*

Proof: Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{D}_n$ be weakly embedded and every $\mathcal{D}_n$ be cellular.

We can choose maximal cellular $\mathcal{B}_0 \subset \mathcal{B}$ so that $\mathcal{D}_0 \subset \mathcal{B}_0$.

Assume $\mathcal{B}_n \subset \mathcal{B}$ has been constructed for some $n < \omega$. We can choose maximal cellular family $\mathcal{B}_{n+1} \subset \mathcal{B}$ so that $\mathcal{B}_{n+1} \succ \mathcal{B}_n$, $\mathcal{B}_{n+1} \succ \mathcal{D}_{n+1}$ and for every $U \in \mathcal{B}_n$ there is $\{U(\nu)\colon \nu < 3\} \subset \mathcal{B}_{n+1}$ with $\bigcup_{\nu<3} U(\nu) \subset U$.

Finally, $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is as required. □

In what follows the π-base $\mathcal{B}$ satisfies the conditions of Lemma 3,

$$\Sigma = \{\sigma \subset \mathcal{B}\colon\ \sigma \text{ maximal cellular in } X\}$$

and $\sigma(\nu) = \{U(\nu)\colon\ U \in \sigma\}$ for every $\sigma \in \Sigma$ and $\nu < 3$.

Lemma 4. *There is $\sigma \in \Sigma$ with the following property: If $\mathcal{F}$ is a p-filter on σ, then $\bigcap \mathcal{F}^* \subset X^*$.*

Proof: We have $p \in \bigcap_{i<\omega} O_i \subset X^*$ for some open $O_i \subset \beta X$. If $O_1 = X$ and $[O_{i+1}] \subset O_i$ for every $i < \omega$, then $\bigcup_{i<\omega}(O_i \setminus [O_{i+2}]) = X$. Denote by σ all maximal sets of the family

$$\{U \in \mathcal{B}\colon\ U \subset O_i \setminus O_{i+2} \text{ for some } i < \omega\}.$$

If $x \in X$ and $x \notin [O_i]$, then $F = \{U \in \sigma\colon\ U \cap [O_{i+2}] \neq \emptyset\}$ satisfies both $\bigcup F \subset O_i$ and $F \in \mathcal{F}$ for any p-filter $\mathcal{F}$. □

Lemma 5. *There are both a well-ordered chain $\{\sigma_\alpha\colon\ \alpha < \lambda\} \subset \Sigma$ and a p-ultrafilter $\mathcal{F}_\alpha$ on every σ_α with the following properties for all $\alpha < \beta < \lambda$:*

(1) $\bigcap \mathcal{F}_0^* \subset X^*$;
(2) $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma_\beta$;
(3) $f^{\sigma_\alpha}_{\sigma_\beta} \mathcal{F}_\alpha \subset \mathcal{F}_\beta$;
(4) *for any* $\sigma \in \Sigma \setminus \{\sigma_\alpha\colon\ \alpha < \lambda\}$ *there is* $\alpha_0 < \lambda$ *with* $\neg(\sigma_{\alpha_0} \prec_{\mathcal{F}_{\alpha_0}} \sigma)$.

Proof: Let $\mathcal{F}_0$ be any p-ultrafilter on σ_0, constructed in Lemma 4.

For some ordinal β assume σ_α and $\mathcal{F}_\alpha$ have been constructed for all $\alpha < \beta$. If there is $\sigma \in \Sigma$ with $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma$ for every σ_α, then we put $\sigma_\beta = \sigma$ and embed the p-filter $\bigcup_{\alpha<\beta} f^{\sigma_\alpha}_{\sigma_\beta} \mathcal{F}_\alpha$ into some p-ultrafilter $\mathcal{F}_\beta$ on σ_β. Otherwise $\lambda = \beta$ and the proof is complete. □

Denote $f^\alpha_\beta = f^{\sigma_\alpha}_{\sigma_\beta}$ from now on.

Lemma 6. *If $\alpha < \beta < \lambda$, then $\bigcap \mathcal{F}^*_\beta \subset \bigcap \mathcal{F}^*_\alpha$.*

Proof: There is $F \in \mathcal{F}_\alpha$ with $F \prec \sigma_\beta$ by (2). For any $G \in \mathcal{F}_\alpha$ we have $G \cap F \in \mathcal{F}_\alpha$ and $G \cap F \prec \sigma_\beta$. But then $\bigcup f^\alpha_\beta(G \cap F) \in \mathcal{F}_\beta$ implies

$$\bigcap \mathcal{F}^*_\beta \subset \Big[\bigcup f^\alpha_\beta(G \cap F)\Big] \subset \Big[\bigcup (G \cap F)\Big] \subset \Big[\bigcup G\Big].$$

□

Lemma 7. *For any neighbourhood O of p there is $\alpha < \lambda$ with $\bigcap \mathcal{F}^*_\alpha \subset O$.*

Proof: Let σ be all maximal members of the family $\{U \in \mathcal{B}\colon U \subset O$ or $U \cap O = \emptyset\}$. Then $\sigma \in \Sigma$. For any σ_α with $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$ we get $\sigma_\alpha(O) \in \mathcal{F}_\alpha$. Denote $\pi = \{U \in \sigma_\alpha(O)\colon V \subsetneq U$ for some $V \in \sigma\}$ and $\delta = \{U \in \sigma_\alpha(O)\colon U \subset V$ for some $V \in \sigma\}$. Since $\mathcal{B}$ is weakly embedded, $\sigma_\alpha(O) = \pi \cup \delta$. Since $\mathcal{F}_\alpha$ is maximal, then either $\pi \in \mathcal{F}_\alpha$ or $\delta \in \mathcal{F}_\alpha$. But if $\pi \in \mathcal{F}_\alpha$, then $\pi \prec \sigma$ implies $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma$. Hence $\delta \in \mathcal{F}_\alpha$ and

$$\bigcap \mathcal{F}^*_\alpha \subset \Big[\bigcup \delta\Big] \subset \Big[\bigcup \sigma(O)\Big] \subset [O]_{\beta X}.$$

□

Lemma 8. *The set $B_\alpha(\nu) = \bigcap \mathcal{F}^*_\alpha \cap \big(\bigcap_{\beta\in\lambda\setminus\alpha} \big[\bigcup \sigma_\beta(\nu)\big]\big)$ is not empty for any $\alpha < \lambda$ and $\nu < 3$.*

Proof: Let $F \in \mathcal{F}_\alpha$ and let $\alpha < \beta_0 < \dots < \beta_i < \dots < \beta_n < \lambda$ be any finite sequence of indexes. Our goal is to find by induction $U \in \mathcal{B}$ so that $U \subset \bigcup F$ and $U \subset \bigcup \sigma_{\beta_i}(\nu)$ and every $i \le n$.

We may assume $F \prec \sigma_{\beta_0}$, choose $G_i \in \mathcal{F}_{\beta_i}$ so that $G_i \prec \sigma_{\beta_{i+1}}$ for each $i < n$ and put $G_n = \sigma_{\beta_n}$. Then the sets $F_0 = f^\alpha_{\beta_0} F \cap G_0$ and $F_{i+1} = f^{\beta_i}_{\beta_{i+1}} F_i \cap G_{i+1}$ satisfy the following conditions: $F_i \in \mathcal{F}_{\beta_i}$, $F_i \prec F_{i+1}$ and $\bigcup F_{i+1} \subset \bigcup F_i$. For any $U_n \in F_n$ we find $U_i \in F_i$ so that $U_n \subset U_i$ to get the sequence

$$U_n \subsetneq \dots \subsetneq U_i \subsetneq \dots \subsetneq U_1 \subsetneq U_0 \subset \bigcup F$$

and put $\Delta_0 = \{\sigma_{\beta_0}, \dots, \sigma_{\beta_n}\}$, $\Theta_0 = \emptyset$ and $W_0 = U_0$.

Let us construct for some $m \in \omega$ a sequence

$$U_n \subseteq \dots \subseteq U_{i+1} = W_m \subsetneq U_i(\nu) \subsetneq U_i \subsetneq \dots \subsetneq U_0(\nu) \subsetneq U_0 \subset \bigcup F$$

of sets $U_i \in \sigma_{\beta_i}$. Then $\Delta_m = \{\sigma_{\beta_{i+1}}, \dots, \sigma_{\beta_n}\}$ and $\Theta_m = \{\sigma_{\beta_0}, \dots, \sigma_{\beta_i}\}$ satisfy the following conditions:

(1) $\Delta_m \cap \Theta_m = \emptyset$;
(2) $\Delta_m \cup \Theta_m = \Delta_0$;
(3) $W_m \subset \bigcup F$;
(4) $W_m \subseteq \bigcup \sigma(\nu)$ for any $\sigma \in \Theta_m$;
(5) for any $\sigma \in \Delta_m$ there is $U_\sigma \in \sigma$ with $U_\sigma \subseteq W_m$.

Let $\Omega = \{\sigma \in \Delta_m\colon U_\sigma = W_m\}$.

If $\Delta_m \ne \Omega$, then we put $\Delta_{m+1} = \Delta_m \setminus \Omega$ and $\Theta_{m+1} = \Theta_m \cup \Omega$. As $\sigma \in \Delta_{m+1}$ are nice, we can choose $U'_\sigma \in \sigma$ so that $\bigcap\{U'_\sigma\colon \sigma \in \Delta_{m+1}\} \cap W_m(\nu) \ne \emptyset$. Then $U_\sigma \subsetneq W_m$ implies $U'_\sigma \subseteq W_m(\nu)$ by our construction. We define W_{m+1} to be the maximal member of embedded sequence $\{U'_\sigma\colon \sigma \in \Delta_{m+1}\}$.

If, finally, $\Delta_m = \Omega$, then W_m is as required. □

Lemma 9. *The point p is a b-point in βX.*

Proof: Define $F_\nu = \{p_\alpha(\nu)\colon \alpha < \lambda\}$ for all $\nu < 3$, where $p_\alpha(\nu) \in B_\alpha(\nu)$. By our construction, $F_\nu \subset \bigcap \mathcal{F}_0^* \subset X^*$ and for any neighbourhood O of p there is $\alpha < \lambda$ with

$$\{p_\beta(\nu)\colon \beta \in \lambda \setminus \alpha\} \subset \bigcap \mathcal{F}_\alpha^* \subset O.$$

Then the condition $\{p_\beta(\nu)\colon \beta < \alpha\} \subset \left[\bigcup \sigma_\alpha(\nu)\right]$ implies that the sets $[F_\nu] \setminus \{p\}$ are pairwise disjoint and $p \in F_\nu$ for no more then one unique F_ν. The other two ensure that p is a b-point in βX. Our proof is complete. □

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