Quasicontinuous spaces

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Abstract. We lift the notion of quasicontinuous posets to the topology context, called quasicontinuous spaces, and further study such spaces. The main results are:

(1) A T_0 space (X,τ) is a quasicontinuous space if and only if $SI(X)$ is locally hypercompact if and only if $(\tau_{SI},\subseteq)$ is a hypercontinuous lattice;

(2) a T_0 space X is an SI-continuous space if and only if X is a meet continuous and quasicontinuous space;

(3) if a C-space X is a well-filtered poset under its specialization order, then X is a quasicontinuous space if and only if it is a quasicontinuous domain under the specialization order;

(4) there exists an adjunction between the category of quasicontinuous domains and the category of quasicontinuous spaces which are well-filtered posets under their specialization orders.

Keywords: quasicontinuous space; hypercontinuous lattice; SI-continuous space; locally hypercompact space; meet continuous space

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1. Introduction

In their study of the spectral theory of primally generated lattices, G. Gierz, J. D. Lawson and A. Stralka, see [12], introduced quasicontinuous domains. They proved that quasicontinuous domains equipped with their Scott topologies are precisely the spectra of hypercontinuous distributive lattices. Later, P. Venugopalan, see [20], also uncovered some algebraic properties of quasicontinuous domains.

In [14], R. Heckmann and K. Keimel proved further connections between quasicontinuous domains and the powerdomains of finitely generated compact saturated subsets. It is well known that Rudin's lemma, see Lemma III–3.3 of [10], has played a crucial role in the theory of quasicontinuous domains. In [14], the authors established a topological variant of Rudin's lemma, where directed sets are replaced by irreducible sets. Based on the topological Rudin's lemma, they

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showed that a dcpo is quasicontinuous if and only if the poset of finitely generated upper sets ordered by reverse inclusion is a continuous poset. In particular, a well-known result is that a dcpo P is a quasicontinuous domain if and only if for any $x \in P$ and any Scott open set U, $x \in U$ implies the existence of a finite subset $F \subseteq P$ satisfying $x \in \mathrm{int}_{\sigma(P)}(\uparrow F) \subseteq \uparrow F \subseteq U$, where $\mathrm{int}_{\sigma(P)}(\uparrow F)$ is the interior of $\uparrow F$ with respect to the Scott topology. X. Mao and L. Xu first introduced quasicontinuous posets in terms of the Scott topology, and proved the major properties of such posets similar to that of quasicontinuous domains.

Recently, motivated by the definition of the Scott topology on posets, D. Zhao and W. K. Ho, see [22], introduced a method for deriving a new topology τ_{SI} from a given one τ, in a similar way as one derives the Scott topology on a poset from the Alexandroff topology. They called this topology the irreducibly-derived topology (or simply, SI-topology). In addition, they introduced and studied SI-continuous spaces, which generalize the notion of continuous posets. The main objective of this paper is to lift the notion of quasicontinuous posets to the topology context. More precisely, we introduce the notion of quasicontinuous space, using the SI-topology. The main results obtained in this paper include: (1) A T_0 space (X, τ) is a quasicontinuous space if and only if $SI(X)$ is locally hypercontinuous, if and only if $(\tau_{SI}, \subseteq)$ is a hypercontinuous lattice; (2) a T_0 space X is an SI-continuous space if and only if X is a meet continuous and quasicontinuous space; (3) if a C-space X is a well-filtered poset under its specialization order, then X is a quasicontinuous space if and only if it is a quasicontinuous domain under the specialization order. Based on these results, we then construct an adjunction between the category of quasicontinuous domains and the category of quasicontinuous spaces which are well-filtered posets under their specialization orders. The work carried out here is another response to the call by J. D. Lawson to develop domain theory in the context of T_0 spaces.

2. Preliminaries

Throughout the paper, we refer the reader to [10] for domain theory, to [6] for general topology, and to [1] for category theory.

Let P be a poset. A nonempty subset D of P is directed if every finite subset of D has an upper bound in D. A subset A of P is upper if $A = \uparrow A = \{x \in P\colon x \geq y \text{ for some } y \in A\}$. The Alexandroff topology $\alpha(P)$ on P is the topology consisting of all its upper subsets. A subset U of P is called Scott open if (i) $U = \uparrow U$ and (ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets on P form the Scott topology $\sigma(P)$. The space $(P, \sigma(P))$ is denoted by ΣP, called the Scott space of P. The topology generated by the collection of sets $P \setminus \downarrow x$ (as a subbase) is called the upper topology and denoted by $\vartheta(P)$.

A map $f\colon P \longrightarrow Q$ between two posets is Scott continuous if it is continuous with respect to the Scott topologies on P and Q. It is well known that f is Scott continuous if and only if it preserves all existing directed suprema.

Let P be a poset. For any $x, y \in P$, define $x \ll y$ if for any directed set D, $y \leq \bigvee D$ implies $x \in \downarrow D$ whenever $\bigvee D$ exists. We denote the set $\{x \in P\colon x \ll y\}$ by $\Downarrow y$, and the set $\{x \in P\colon y \ll x\}$ by $\Uparrow y$. Then P is called continuous if for each $y \in P$, the set $\Downarrow y$ is directed and $\bigvee \Downarrow y = y$. A dcpo which is continuous as a poset will be called a domain, see [10]. We order the collection $\mathcal{P}(L)$ of all nonempty subsets of a dcpo L by $G \leq H$ if $\uparrow H \subseteq \uparrow G$. Then $(\mathcal{P}(L), \leq)$ is a quasi-ordered set. Define $G \ll H$ if for any directed set $D \subseteq L$, $\bigvee D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. One uses $G \ll x$ for $G \ll \{x\}$ and lets $\Uparrow F = \{x \in L\colon F \ll x\}$. Let $\mathcal{F}L$ denote the set of all nonempty finite subsets of L. A dcpo L is called a quasicontinuous domain if for any $x \in L$ the family $\mathrm{fin}(x) = \{F \in \mathcal{F}L\colon\ F \ll x\}$ is a directed subset of $\mathcal{P}(L)$, and whenever $x \not\leq y$, then there exists $F \in \mathrm{fin}(x)$ such that $y \notin \uparrow F$. Let **QDOM** denote the category of quasicontinuous domains and Scott continuous maps.

For any T_0 space (X, τ), the specialization order "$\leq$" on X is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned in the context of a T_0 space X, it will be interpreted with respect to the specialization order on X.

A subset A of a topological space X is saturated if A equals the intersection of all open sets containing A. A topological space is well-filtered, see [10], [13], if for every filtered family $\mathcal{F}$ of compact saturated sets with intersection $\bigcap \mathcal{F}$ contained in some open set U, it follows that $F \subseteq U$ for some $F \in \mathcal{F}$. A poset P is called a well-filtered poset if the space ΣP is well-filtered. A T_0 space (X, τ) is called a C-space, see [7], also see [13], if for all $U \in \tau$ and $a \in U$, there is $b \in U$ such that $a \in \mathrm{int}_\tau(\uparrow b)$. Given a topological space (X, τ), a nonempty subset F of X is called a τ-irreducible set (or simply, irreducible) if for any closed subsets A and $B \subseteq X$, $F \subseteq A \cup B$ implies $F \subseteq A$ or $F \subseteq B$. The set of all τ-irreducible sets of X is denoted by $\mathrm{Irr}_\tau(X)$. A space (X, τ) is sober, see [10], if for any irreducible closed set F, there is a unique point $x \in X$ such that $F = \mathrm{cl}(\{x\})$. A space (X, τ) is said to be k-bounded sober, see [22], if for any irreducible closed set F with $\bigvee F$ existing, there is a unique point $x \in X$ such that $F = \mathrm{cl}(\{x\})$. Notice that every sober space is k-bounded sober and every k-bounded sober space is T_0.

Proposition 2.1 ([13])**.** *Let (X, τ) be a T_0 space. Then the following statements hold:*

(1) *For all $a \in X$, $\downarrow a = \{x \in X\colon x \leq a\} = \mathrm{cl}(\{a\})$.*

(2) *If $U \subseteq X$ is an open subset, then $\uparrow U = U$.*
(3) *If $U \subseteq X$ is a closed subset, then $\downarrow U = U$.*
(4) *If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible.*

Definition 2.2 ([22])**.** Let (X, τ) be a T_0 space. A subset U of X is called SI-open if the following conditions are satisfied:

(1) $U \in \tau$;
(2) for any $F \in \mathrm{Irr}_\tau(X)$, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.

The set of all SI-open sets of (X, τ) is denoted by τ_{SI}, which is indeed a topology on X. We call τ_{SI} the irreducibly-derived topology of τ. The space (X, τ_{SI}) will also be simply written as $SI(X)$. Moreover, complements of SI-open sets are called SI-closed sets. The set of all SI-closed sets of X will be denoted by $\Gamma_{SI}(X)$.

Let P be a poset. A nonempty subset $F \subseteq P$ is irreducible with respect to the Alexandroff topology $\alpha(P)$ if and only if it is a directed set. So $SI(P, \alpha(P)) = (P, \sigma(P)) = \Sigma P$. Since a T_0 space (X, τ) is k-bounded sober if and only if $\tau = \tau_{SI}$, we have that the irreducibly-derived topology of a sober space is the original topology. This means that the irreducibly-derived topology of the Scott topology on a quasicontinuous domain is again the Scott topology.

3. Quasicontinuous spaces

A T_0 space (X, τ) is called a *quasicontinuous space* if for any $U \in \tau_{SI}$ and $x \in U$, there exists a nonempty finite subset $F \subseteq X$ such that

$$x \in \mathrm{int}_{\tau_{SI}}(\uparrow F) \subseteq \uparrow F \subseteq U.$$

Remark 3.1. Recall that a poset P is called a quasicontinuous poset, see [18], if for any $U \in \sigma(P)$ and $x \in U$, there exists a nonempty finite subset $F \subseteq X$ such that $x \in \mathrm{int}_{\sigma(P)}(\uparrow F) \subseteq \uparrow F \subseteq U$. Thus P is a quasicontinuous poset if and only if $(P, \alpha(P))$ is a quasicontinuous space.

Example 3.2. (1) Let $X = \{a_i \colon i \in \mathbb{N}\} \cup \{\top\}$, where $\mathbb{N}$ denotes the set of all positive integers. The partial order "$\leq$" on X is defined by

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \top.$$

Then $(X, \leq)$ is a quasicontinuous domain, and thus the irreducibly-derived topology of the Scott topology on X is again the Scott topology. Hence, $(X, \sigma(X))$ is a quasicontinuous space.

(2) Let $X = \{a_i : i \in \mathbb{N}\} \cup \{b_i : i \in \mathbb{N}\} \cup \{\top\}$, where $\mathbb{N}$ denotes the set of all positive integers. The order "$\leq$" on X is defined by

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \top; \qquad b_1 \leq b_2 \leq \cdots \leq b_n \leq \cdots \leq \top.$$

Then $(X, \leq)$ is a quasicontinuous domain. We consider the Scott space $(X, \sigma(X))$. Then the irreducibly-derived topology of $\sigma(X)$ on X is again the Scott topology $\sigma(X)$. Let U be an SI-open set and $x \in U$. Then U is a Scott open set and $x \in U$, there exists a nonempty finite subset F of X such that $x \in \mathrm{int}_{\sigma(X)}(\uparrow F) \subseteq \uparrow F \subseteq U$. Then $(X, \sigma(X))$ is a quasicontinuous space.

(3) The Sorgenfrey line $\mathbb{R}_l$ is the set $\mathbb{R}$ equipped with the topology induced by the quasi-metric $d_l : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty]$ defined as:

$$d_l(x, y) = \begin{cases} y - x, & x \leq y, \\ \infty, & x > y. \end{cases}$$

The set $\mathbf{B}(\mathbb{R}, d_l)$ of formal balls is equipped with a hemi-metric d_l^+, defined

$$d_l^+((x, r), (y, s)) = \max(d_l(x, y) - r + s, 0),$$

and is also a quasi-metric space. The specialization order "$\leq^{d_l^+}$" on $\mathbf{B}(\mathbb{R}, d_l)$ is defined by $(x, r) \leq^{d_l^+} (y, s)$ if and only if $d_l(x, y) \leq r - s$. By Exercise 7.3.12 of [13], $\mathbf{B}(\mathbb{R}, d_l)$ is a domain under the specialization order "$\leq^{d_l^+}$", but $(\mathbb{R}, d_l)$ is not Smyth-complete. Thus the open ball topology τ on $\mathbf{B}(\mathbb{R}, d_l)$ is different from the Scott topology. Next, we shall prove that $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology τ is a quasicontinuous space. Let $U \in \tau_{SI}$, and $(x, r) \in U$. Since $\Downarrow(x, r)$ is a directed set and $(x, r) = \bigvee \Downarrow(x, r)$, we have that $\Downarrow(x, r) \cap U \neq \emptyset$. Then there exists $(y, s) \in U$ such that $(y, s) \ll (x, r)$, implying $(x, r) \in \Uparrow(y, s)$. Since $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology τ is a C-space, $\Uparrow(y, s) = \bigcup_{(z,t) \in \Uparrow(y,s)} \mathrm{int}_\tau(\uparrow(z, t))$ is an SI-open set. Then $(x, r) \in \Uparrow(y, s) \subseteq \mathrm{int}_{\tau_{SI}}(\uparrow(y, s)) \subseteq \uparrow(y, s) \subseteq U$. Therefore, $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology of τ is a quasicontinuous space.

Let (X, τ) be a T_0 space. For all $x, y \in X$, define $x \ll_{SI} y$ if for any irreducible set F, $y \leq \bigvee F$ implies $x \in \downarrow F$ whenever $\bigvee F$ exists. We denote the set $\{x \in X : x \ll_{SI} a\}$ by $\Downarrow_{SI} a$, and the set $\{x \in X : a \ll_{SI} x\}$ by $\Uparrow_{SI} a$.

Definition 3.3 ([22]). A T_0 space X is called SI-continuous if for any $a \in X$, the following conditions are satified:

(SI1) $\Uparrow_{SI} a$ is open in X;

(SI2) $\Downarrow_{SI} a$ is a directed set and $\bigvee \Downarrow_{SI} a = a$.

Remark 3.4. (1) In [22], D. Zhao and W. K. Ho defined the SI-topology on T_0 spaces using irreducible sets as the topological counterparts of directed sets. Since condition (SI2) in Definition 3.3 asks $\Downarrow_{SI} a$ to be a directed set, one may see that the definition of SI-continuity is somewhat not satisfying. Recently, H. J. Andradi et al., see [3], defined the notion of SI^*-continuity by changing the condition (SI2) in Definition 3.3 to the following condition (SI2*).

(SI2*) $\Downarrow_{SI} a$ is an irreducible subset and $\bigvee \Downarrow_{SI} a = a$.

It was proved in [3] that SI-continuity and SI^*-continuity are just the same notions.

(2) In Definition 3.3, condition (SI1) is crucial. If we drop it and change (SI2) to (SI2*), then SI-continuous spaces will coincide with Irr-continuous spaces defined in [2]. It should be pointed out that this idea is similar to a special case of subset systems on the category of posets developed in [4], [5], [8], [21]. It looks being investigated in the context of T_0 spaces, and the essential results, however, are almost the same as those in the context of posets. In fact, the main results in [2] are established on k-bounded sober Irr-continuous spaces. Note that for any continuous poset P, the Scott space ΣP is a k-bounded sober space. Thus the condition (SI1) in SI-continuity plays a similar role as k-bounded sobriety. Although $SI(Y) = \Sigma Y$ for each SI-continuous space Y, see [17], it does not affect the study of SI-continuity. This is because there exists an SI-continuous space (X, τ) such that $\tau \neq \sigma(X)$. In Example 3.2 (3), $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology of d_l^+ is a C-space. Since a C-space is an SI-continuous space if and only if it is a continuous poset under the specialization order, see [17], $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology of d_l^+ is an SI-continuous space, but the open ball topology on $\mathbf{B}(\mathbb{R}, d_l)$ is different from the Scott topology.

Let (X, τ) and (Y, δ) be T_0 spaces. A continuous mapping $f\colon X \longrightarrow Y$ is called an SI-continuous mapping, see [22], if f is a continuous mapping between (X, τ_{SI}) and (Y, δ_{SI}). Let **SIC** denote the category of SI-continuous spaces and SI-continuous maps. Let **QSA** denote the category of quasicontinuous spaces and SI-continuous maps. Let **QSW** denote the category of quasicontinuous spaces which also are well-filtered posets under their specialization orders and SI-continuous maps.

Proposition 3.5. *Every SI-continuous space is quasicontinuous.*

Proof: Let (X, τ) be an SI-continuous space. Suppose that $U \in \tau_{SI}$ and $x \in U$. Since X is an SI-continuous space, we have that $\Downarrow_{SI} x \cap U \neq \emptyset$. There exists $y \in U$ such that $y \ll_{SI} x$, implying $x \in \Uparrow_{SI} y = \mathrm{int}_{\tau_{SI}}(\uparrow y) \subseteq \uparrow y \subseteq U$. Therefore, X is a quasicontinuous space. □

Remark 3.6. (1) Since a quasicontinuous poset need not be a continuous poset, a quasicontinuous space need not be an SI-continuous space.

(2) By Proposition 3.5, we have that **SIC** is a full subcategory of **QSA**. H. Kou, Y. M. Liu and M. K. Luo, see Theorem 4.1 of [15], proved that the category of

domains is not a full reflective subcategory of **QDOM**. Similarly, we have that **SIC** is not a full reflective subcategory of **QSA**.

Proposition 3.7. *Let (X,τ) be a quasicontinuous space. Then $\{\mathrm{int}_{\tau_{SI}}(\uparrow F)\colon F \in \mathcal{F}X\}$ forms a base for the SI-topology.*

PROOF: Straightforward. □

A space is locally hypercompact, see [9], [16], if for any $x \in X$ and every open set U containing x, there exists an open set V and a finite set F such that $x \in V \subseteq \uparrow F \subseteq U$.

Corollary 3.8. *Let (X,τ) be a T_0 space. Then X is a quasicontinuous space if and only if $SI(X)$ is locally hypercompact.*

Let P be a poset. Define a binary relation "$\prec_{\vartheta(P)}$" on P by $x \prec_{\vartheta(P)} y$ if and only if $y \in \mathrm{int}_{\vartheta(P)}(\uparrow x)$, where the interior is taken in the upper topology $\vartheta(P)$. A poset P is called a hypercontinuous poset if for all $x \in P$, the set $\{y \in L\colon y \prec_{\vartheta(P)} x\}$ is directed and $x = \bigvee\{y \in L\colon y \prec_{\vartheta(P)} x\}$, see [18]. A complete lattice L is called a hypercontinuous lattice, see [10], [11], if L is a hypercontinuous poset. A complete lattice L is called a generalized continuous lattice if for any $x, y \in L$ such that $x \not\leq y$, there exists a finite set F such that $F \ll x$ and $\downarrow y \cap F = \emptyset$. If the complete lattice $(L,\leq)$ is a hypercontinuous lattice, then $(L,\leq)$ is a continuous lattice and $(L,\geq)$ is a generalized continuous lattice, see [11].

Proposition 3.9 ([18])**.** *Let P be a poset. If P is hypercontinuous, then for all $x, y \in P$, $x \not\leq y$ implies that there are $u \in P$ and finite set $F = \{v_1,\dots,v_k\} \subseteq P$ such that*

(1) *$u \not\leq y$, $x \not\leq v_i$, $i = 1,2,\dots,k$;*
(2) *for any $z \in P$, $u \leq z$, or there is $j \in \{1,2,\dots,k\}$ such that $z \leq v_j$.*

Theorem 3.10. *A T_0 space (X,τ) is quasicontinuous if and only if the lattice $(\tau_{SI},\subseteq)$ is a hypercontinuous lattice.*

PROOF: Since X is a quasicontinuous space, by Corollary 3.8, we have that $SI(X)$ is locally hypercompact. By Theorem 7 of [9], the lattice $(\tau_{SI},\subseteq)$ is a hypercontinuous lattice.

Conversely, let U be an SI-open set and $x \in U$. Trivially $U \not\subseteq X \setminus \downarrow x$. Since $(\tau_{SI},\subseteq)$ is a hypercontinuous lattice, it follows from Proposition 3.9 that there exist SI-open sets $U_0, V_1,\dots,V_k$ such that

(i) $U_0 \not\subseteq X \setminus \downarrow x$ and $U \not\subseteq V_i$ for all $i \in \{1,\dots,k\}$;
(ii) for any $W \in \tau_{SI}$, $U_0 \subseteq W$ or there exists $j \in \{1,\dots,k\}$ such that $W \subseteq V_j$.

Since $U \not\subseteq V_i$ for all $i \in \{1,\dots,k\}$, there exists $x_i \in U \setminus V_i$ for each $i \in \{1,\dots,k\}$. Let $F = \{x_1,\dots,x_k\}$. Then $\uparrow F \subseteq U$. Assume that $U_0 \not\subseteq \uparrow F$. Then there exists

$u_0 \in U_0$ such that $u_0 \notin \uparrow F$. Let $W_0 = X \setminus \downarrow u_0$. Then $F \subseteq W_0$, and thus $W_0 \nsubseteq V_i$ holds for every $i \in \{1, \ldots, k\}$. By (ii), we have that $U_0 \subseteq W_0$. But this contradicts $u_0 \notin W_0$. Thus $U_0 \subseteq \uparrow F$, and hence $x \in \mathrm{int}_{\tau_{SI}}(\uparrow F) \subseteq \uparrow F \subseteq U$. Therefore, X is a quasicontinuous space. □

If (X, τ) is a sober space, then $X = SI(X)$. Thus by Corollary 3.8 and Theorem 3.10 we deduce the following corollary.

Corollary 3.11. *For any sober space (X, τ), the following statements are equivalent:*

(1) *X is a quasicontinuous space;*
(2) *X is locally hypercompact;*
(3) *the lattice $(\tau, \subseteq)$ is hypercontinuous.*

4. Meet continuous spaces

A poset P is said to be meet continuous, see [19], if for any $x \in P$ and any directed subset D, $\bigvee D$ exists and $x \leq \bigvee D$, then $x \in \mathrm{cl}_\sigma(\downarrow D \cap \downarrow x)$, where $\mathrm{cl}_\sigma(\downarrow D \cap \downarrow x)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$. By Theorem 3.4 of [10], a poset P is meet continuous if and only if for any Scott open set U and $x \in P$, $\uparrow(U \cap \downarrow x)$ is a Scott open set.

Definition 4.1. A T_0 space (X, τ) is called meet continuous if for any $x \in X$ and $U \in \tau_{SI}$, one has $\uparrow(U \cap \downarrow x) \in \tau_{SI}$.

Remark 4.2. (1) A poset P is a meet continuous poset if and only if the space $(P, \alpha(P))$ is meet continuous.

(2) There exists a T_2 space that is not meet continuous, see Example 4.3 (1).

Example 4.3. (1) Let X be an infinite set, $x_0 \in X$, and

$$\tau = \{U \subseteq X \colon \text{ the complement of } U \text{ is finite}\} \cup \{U \subseteq X \colon x_0 \notin U\}.$$

Then (X, τ) is a T_2 space, and thus $\tau = \tau_{SI}$. Take $x \in X$ and $x \neq x_0$, then $U = X \setminus \{x\}$ is an open set. Since $\uparrow(U \cap \downarrow x_0) = \uparrow x_0 = \{x_0\}$ is not an open set, X is not meet continuous.

(2) Let $(X, \leq)$ be the poset defined in Example 3.2 (1). Consider the Scott space $(X, \sigma(X))$. Obviously, $X = SI(X)$. Let U be an SI-open set. Then U is a Scott open set. For any $x \in X$, if $x \in U$, then $\uparrow(U \cap \downarrow x)$ is a Scott open set; if $x \notin U$, then $U \cap \downarrow x = \emptyset$. Thus $\uparrow(U \cap \downarrow x) = \emptyset$ is a Scott open set. Therefore $(X, \sigma(X))$ is a meet continuous space.

(3) Let $(X, \sigma(X))$ be the Scott space defined in Example 3.2 (2). Obviously, $\uparrow a_5 \cup \uparrow b_5$ is an SI-open set. Since $\uparrow(\downarrow a_6 \cap (\uparrow a_5 \cup \uparrow b_5)) = \uparrow a_5$ is not an SI-open set, $(X, \sigma(X))$ is not a meet continuous space.

(4) Let $\mathbf{B}(\mathbb{R}, d_l)$ be the quasi-metric space defined in Example 3.2 (3). Next, we shall prove that $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology τ is a meet continuous space. Let $U \in \tau_{SI}$ and $(x, r) \in U$. For all $(y, s) \in \uparrow(U \cap \downarrow(x, r))$. Then there exists $(z, t) \in U \cap \downarrow(x, r)$ such that $(z, t) \leq^{d_l^+} (y, s)$. By Remark 3.4 (2), we have that $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology τ is an SI-continuous space. Then $\Downarrow_{SI}(z, t)$ is a directed set and $(z, t) = \bigvee \Downarrow_{SI}(z, t)$. It follows from $(z, t) \in U$ that $(a, q) \in U$ for some $(a, q) \ll_{SI} (z, t)$. Then $(a, q) \in U \cap \downarrow(x, r) \subseteq \uparrow(U \cap \downarrow(x, r))$, and thus $(y, s) \in \bigcup\{\Uparrow_{SI}(a, q) \colon (a, q) \in \uparrow(U \cap \downarrow(x, r))\}$. This shows that $\uparrow(U \cap \downarrow(x, r)) = \bigcup\{\Uparrow_{SI}(a, q) \colon (a, q) \in \uparrow(U \cap \downarrow(x, r))\}$ is an SI-open set. Therefore, $\mathbf{B}(\mathbb{R}, d_l)$ with the open ball topology τ is a meet continuous space.

Proposition 4.4. *Let X be a meet continuous space. Then $(\Gamma_{SI}(X), \subseteq)$ is a frame.*

Proof: Let $F \in \Gamma_{SI}(X)$ and $\{F_i \colon i \in I\} \subseteq \Gamma_{SI}(X)$. We shall prove that $F \wedge \big(\bigvee_{i\in I} F_i\big) = \bigvee_{i\in I}(F \wedge F_i)$. Obviously, $\bigvee_{i\in I}(F \wedge F_i) \subseteq F \wedge \big(\bigvee_{i\in I} F_i\big)$. Let $x \in F \wedge \big(\bigvee_{i\in I} F_i\big) = F \cap \big(\bigvee_{i\in I} F_i\big)$, and U be an SI-open set containing x. Then $x \in U \cap F$. Since $\uparrow(U \cap F) = \bigcup_{x\in F} \uparrow(U \cap \downarrow x) \in \tau_{SI}$, it follows from $x \in \mathrm{cl}_{SI(X)}\big(\bigcup_{i\in I} F_i\big)$ that $\uparrow(U \cap F) \cap \big(\bigcup_{i\in I} F_i\big) \neq \emptyset$. Then there exists $i_0 \in I$ such that $\uparrow(U \cap F) \cap F_{i_0} \neq \emptyset$, and thus $(U \cap F) \cap \downarrow F_{i_0} = (U \cap F) \cap F_{i_0} \neq \emptyset$. Hence $U \cap \big(\bigcup_{i\in I}(F \cap F_i)\big) \neq \emptyset$. This shows $x \in \mathrm{cl}_{SI(X)}\big(\bigcup_{i\in I}(F \cap F_i)\big) = \bigvee_{i\in I}(F \wedge F_i)$. Therefore, $F \wedge \big(\bigvee_{i\in I} F_i\big) = \bigvee_{i\in I}(F \wedge F_i)$, and so $(\Gamma_{SI}(X), \subseteq)$ is a frame. □

Theorem 4.5. *For any T_0 space (X, τ), the following conditions are equivalent:*

(1) *X is a meet continuous and quasicontinuous space;*

(2) *X is an SI-continuous space.*

Proof: (1) $\Longrightarrow$ (2) Since X is a quasicontinuous space, it follows from Theorem 3.10 that $(\tau_{SI}(X), \subseteq)$ is a hypercontinuous lattice. Then $(\tau_{SI}(X), \subseteq)$ is a continuous lattice and $(\tau_{SI}(X), \supseteq)$ is a generalized continuous lattice, and thus $(\tau_{SI}(X), \supseteq)$ is a quasicontinuous domain. Since X is a meet continuous space, by Proposition 4.4, we have that $(\Gamma_{SI}(X), \subseteq)$ is a frame. Then $(\Gamma_{SI}(X), \subseteq)$ is a meet continuous lattice, and thus $(\tau_{SI}(X), \supseteq)$ is a meet continuous lattice. Hence $(\tau_{SI}(X), \supseteq)$ is a continuous lattice. This shows that $(\tau_{SI}(X), \subseteq)$ is a complete distributive lattice, so we conclude that $SI(X)$ is a C-space, and hence X is an SI-continuous space.

(2) $\Longrightarrow$ (1) By Proposition 3.5, we have that X is a quasicontinuous space. Let U be an SI-open set and $x \in X$. For all $y \in \uparrow(U \cap \downarrow x)$, there exists $a \in U \cap \downarrow x$ such that $a \leq y$. Since X is an SI-continuous space, we have that

$\Downarrow_{SI} a$ is a directed set and $a = \bigvee \Downarrow_{SI} a$. It follows from $a \in U$ that $b \in U$ for some $b \ll_{SI} a$. Then $b \in U \cap {\downarrow} x \subseteq {\uparrow}(U \cap {\downarrow} x)$, and thus $y \in \bigcup\{\Uparrow_{SI} b\colon b \in {\uparrow}(U \cap {\downarrow} x)\}$. This shows that ${\uparrow}(U \cap {\downarrow} x) = \bigcup\{\Uparrow_{SI} b\colon b \in {\uparrow}(U \cap {\downarrow} x)\}$ is an SI-open set. Therefore, X is a meet continuous space. □

By Remark 4.2 (1) and Theorem 4.5, we have the following corollary.

Corollary 4.6 ([18])**.** *For any poset P, the following conditions are equivalent:*

(1) *P is continuous;*
(2) *P is meet continuous and quasicontinuous;*
(3) *$(P, \alpha(P))$ is a meet continuous and quasicontinuous space.*

5. An adjunction between QDOM and QSW

We now construct an adjunction between the categories **QDOM** and **QSW**. We first prove some results on the links between quasicontinuous spaces and well-filtered posets.

Proposition 5.1. *Let X be a quasicontinuous space. If $(X, \leq)$ is a well-filtered poset, then $\tau_{SI} = \sigma(X)$.*

Proof: Obviously, $\tau_{SI} \subseteq \sigma(X)$. If $x \in U \in \sigma(X)$, then $U \ll x$. Put

$$\mathcal{F}_x = \{F \subseteq X \colon F \text{ is a nonempty finite set and } x \in \mathrm{int}_{\tau_{SI}}({\uparrow} F)\}.$$

Since X is a quasicontinuous space, there exists a nonempty finite set F such that $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} F) \subseteq {\uparrow} F \subseteq U$. Then $F \in \mathcal{F}_x$, and thus $\mathcal{F}_x \neq \emptyset$. Let F_1, $F_2 \in \mathcal{F}_x$. Then $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} F_1) \cap \mathrm{int}_{\tau_{SI}}({\uparrow} F_2)$, and thus $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} G) \subseteq {\uparrow} G \subseteq \mathrm{int}_{\tau_{SI}}({\uparrow} F_1) \cap \mathrm{int}_{\tau_{SI}}({\uparrow} F_2)$ for some nonempty finite set G. It follows from that $\mathcal{F}_x$ is a directed set. Obviously, ${\uparrow} x \subseteq \bigcap_{F \in \mathcal{F}_x} {\uparrow} F$. If $x \not\leq y$, then $x \in X \setminus {\downarrow} y$. Since $X \setminus {\downarrow} y$ is an SI-open set, there exists a nonempty finite set H such that $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} H) \subseteq {\uparrow} H \subseteq X \setminus {\downarrow} y$. Then $y \notin {\uparrow} H$ and $H \in \mathcal{F}_x$. This shows that ${\uparrow} x = \bigcap_{F \in \mathcal{F}_x} {\uparrow} F$. It follows from $x \in U$ that $\bigcap_{F \in \mathcal{F}_x} {\uparrow} F \subseteq U$. Since X is a well-filtered poset, we have that ${\uparrow} F_0 \subseteq U$ for some $F_0 \in \mathcal{F}_x$. Then $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} F_0) \subseteq {\uparrow} F_0 \subseteq U$, and thus $U \in \tau_{SI}$. Therefore $\tau_{SI} = \sigma(X)$. □

Corollary 5.2. *Let X be a quasicontinuous space. Then $(X, \leq)$ is a well-filtered poset if and only if $(X, \leq)$ a quasicontinuous domain.*

Proof: Let $(X, \leq)$ be a well-filtered poset. Then $(X, \leq)$ is a dcpo. Let $x \in U \in \sigma(X)$. By Proposition 5.1, we have that $\tau_{SI} = \sigma(X)$. Then $x \in U \in \tau_{SI}$. Since X is a quasicontinuous space, we have that $x \in \mathrm{int}_{\tau_{SI}}({\uparrow} F) \subseteq {\uparrow} F \subseteq U$ for some nonempty finite subset F of X. Obviously, $\mathrm{int}_{\tau_{SI}}({\uparrow} F)$ is a Scott open set.

Then $x \in \mathrm{int}_{\sigma(X)}(\uparrow F) \subseteq \uparrow F \subseteq U$, and thus $(X, \leq)$ is a quasicontinuous poset. Conversely, if $(X, \leq)$ is a quasicontinuous domain, then $(X, \sigma(X))$ is a sober space, and thus $(X, \leq)$ is a well-filtered poset. □

Remark 5.3. Let X be an infinite set, and $\tau = \{A \subseteq X\colon$ the complement of A is finite$\} \cup \{\emptyset\}$ be the co-finite topology. Then (X, τ) is a T_1 space, and thus the specialization order is the discrete order. Hence, X is a well-filtered poset and a quasi-continuous poset under the specialization order. Next, we shall show that X is not a quasicontinuous space. Suppose that X is a quasicontinuous space. Let $x \in X$. Then there exists a nonempty finite set F such that $x \in \mathrm{int}_{\tau_{SI}}(\uparrow F) \subseteq \uparrow F = F \subseteq X$. This contradicts the fact that $\mathrm{int}_{\tau_{SI}}(\uparrow F) = \emptyset$. Therefore, X is not a quasicontinuous space.

Proposition 5.4. *Let (X, τ) be a C-space. If $(X, \leq)$ is a well-filtered poset, then X is a quasicontinuous space if and only if X is a quasicontinuous poset under the specialization order.*

Proof: The necessity follows directly from Corollary 5.2. We now verify the sufficiency. Let $x \in U \in \tau_{SI}$. Then U is a Scott open set, and thus $x \in \mathrm{int}_{\sigma(X)}(\uparrow F) \subseteq \uparrow F \subseteq U$ for some nonempty finite subset F of X. Let $y \in \mathrm{int}_{\sigma(X)}(\uparrow F)$. Since X is a C-space, we have that $A_y = \{z \in X\colon y \in \mathrm{int}_\tau(\uparrow z)\}$ is a directed set and $y = \bigvee A_y$. Then there exists $z \in \mathrm{int}_{\sigma(X)}(\uparrow F)$ such that $y \in \mathrm{int}_\tau(\uparrow z)$, and thus $y \in \mathrm{int}_\tau(\uparrow z) \subseteq \uparrow z \subseteq \mathrm{int}_{\sigma(X)}(\uparrow F)$. So we conclude that $\mathrm{int}_{\sigma(X)}(\uparrow F) = \bigcup\{\mathrm{int}_\tau(\uparrow z)\colon z \in \mathrm{int}_{\sigma(X)}(\uparrow F)\}$, and hence $\mathrm{int}_{\sigma(X)}(\uparrow F)$ is an open set. Let G be an irreducible set with $\bigvee G$ existing, and $\bigvee G \in \mathrm{int}_{\sigma(X)}(\uparrow F)$. Since X is a C-space, there exists a directed set $D \subseteq \downarrow G$ such that $\bigvee D = \bigvee G$. It follows from $\bigvee G \in \mathrm{int}_{\sigma(X)}(\uparrow F)$ that $\bigvee D \in \mathrm{int}_{\sigma(X)}(\uparrow F)$. Then $D \cap \mathrm{int}_{\sigma(X)}(\uparrow F) \neq \emptyset$, and thus $G \cap \mathrm{int}_{\sigma(X)}(\uparrow F) \neq \emptyset$. Hence, $\mathrm{int}_{\sigma(X)}(\uparrow F)$ is an SI-open set. So we conclude that $x \in \mathrm{int}_{\sigma(X)}(\uparrow F) = \mathrm{int}_{\tau_{SI}}(\uparrow F) \subseteq \uparrow F \subseteq U$. Therefore, X is a quasicontinuous space. □

Lemma 5.5 ([22]). *Let (X, τ) and (Y, δ) be T_0 spaces. Then a continuous mapping $f\colon X \longrightarrow Y$ is an SI-continuous mapping if and only if f preserves all existing irreducible suprema.*

Proposition 5.6. *Define $F\colon$ **QDOM** $\longrightarrow$ **QSW** as follows:*

$$(f\colon X \longrightarrow Y) \longmapsto \big(f\colon (X, \alpha(X)) \longrightarrow (Y, \alpha(Y))\big)$$

Then F is a full and faithful functor.

Proof: By Remark 3.1, we have that $(X, \alpha(X))$ is a quasicontinuous space for every quasicontinuous domain X. Since X is a quasicontinuous domain, we have that ΣX is a sober space. Then $(X, \leq)$ is a well-filtered poset. Let X, Y be two quasicontinuous domains and $f\colon X \longrightarrow Y$ be a Scott continuous map. Then $F(f)$ is an SI-continuous map. It is clear that F preserves composition and identity morphisms. Then F is a functor. Obviously, F is a full and faithful functor. □

Definition 5.7 ([1])**.** A functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is called isomorphism-dense if there exists some $\mathcal{A}$-object A such that $F(A)$ is isomorphic to B for any $\mathcal{B}$-object B.

Proposition 5.8. *Define* $G\colon \mathbf{QSW} \longrightarrow \mathbf{QDOM}$ *as follows:*

$$(g\colon X \longrightarrow Y) \longmapsto (g\colon (X, \leq) \longrightarrow (Y, \leq)).$$

Then G *is a faithful and isomorphism-dense functor.*

Proof: Let (X, τ) be a quasicontinuous space which is also a well-filtered poset under its specialization order. By Corollary 5.2, we have that $(X, \leq)$ is a quasicontinuous domain. Let X, Y be two quasicontinuous spaces and $g\colon X \longrightarrow Y$ be an SI-continuous map. It follows from Lemma 5.5 that g preserves all existing irreducible suprema. Then $G(g)$ preserves all existing directed suprema, and thus $G(g)$ is a Scott continuous map. It is clear that G preserves composition and identity morphisms. Then G is a functor. Let L be a quasicontinuous domain. Then $(L, \alpha(L))$ is a quasicontinuous space and $(X, \leq)$ is a well-filtered poset, and thus $G\big((L, \alpha(L))\big) = L$. It follows from that G is an isomorphism-dense functor. □

Definition 5.9 ([1])**.** Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ be functors. The functor F is called a left adjoint of G (or G is a right adjoint of F) or (F, G) is an adjunction between $\mathcal{A}$ and $\mathcal{B}$, in symbols $F \dashv G\colon \mathcal{A} \rightharpoonup \mathcal{B}$, if for each $\mathcal{A}$-object A, there exists a universal pair $(\varepsilon_A, F(A))$ (or equivalently, for each $\mathcal{B}$-object B, there exists a co-universal pair $(G(B), \eta_B)$).

In the following, we shall prove that the pair $F \dashv G$ is an adjunction.

Theorem 5.10. *We have* $F \dashv G\colon \mathbf{QDOM} \rightharpoonup \mathbf{QSW}$.

Proof: Let (X, τ) be a quasicontinuous space which is also a well-filtered poset under its specialization order. Define $\eta\colon FG(X) \longrightarrow X$ as follows:

$$\forall\, x \in X \quad \eta(x) = x.$$

Let U be an open set. Then $\eta^{-1}(U) = U$. Obviously, $\eta^{-1}(U)$ is an upper set. Thus η is continuous. Let V be an SI-open set. Then $\eta^{-1}(V) = V$ is a Scott open set. So we conclude that η is an SI-continuous map. Let Y be a quasicontinuous domain, and $f\colon F(Y) \longrightarrow X$ be an SI-continuous map. Now, define a map $g\colon Y \longrightarrow G(X)$ as follows:

$$\forall\, y \in Y \quad g(y) = f(y).$$

Now, we shall prove that g is a Scott continuous map. Obviously, g is a monotone map. Let $\{y_j : j \in J\} \subseteq Y$ be a directed set with $\bigvee_{j\in J} y_j$ existing. Since f is an SI-continuous map, we have that

$$g\Big(\bigvee_{j\in J} y_j\Big) = f\Big(\bigvee_{j\in J} y_j\Big) = \bigvee_{j\in J} f(y_j) = \bigvee_{j\in J} g(y_j).$$

Thus g is a Scott continuous map. It is straightforward to verify that $\eta\circ F(g) = f$. Suppose that there exists a Scott continuous map $h\colon Y \longrightarrow G(X)$ such that $\eta \circ F(h) = f$. Then $\eta \circ F(g) = \eta \circ F(h)$. Since η is an injective map, we have that $g = h$. Thus F is a left adjoint of G. □

Remark 5.11. A natural question arises if there is an adjunction between the category of quasicontinuous spaces and the category of quasicontinuous posets. At present, we do not know the answer yet. To answer it, more connections between quasicontinuous spaces and quasicontinuous posets need to be found.

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References

[1] Adámek J., Herrlich H., Strecker G. E., *Abstract and Concrete Categories*, The Joy of Cats, Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1990.

[2] Andradi H., Ho W. K., *On a new convergence class in sup-sober spaces*, available at arXiv:1709.03269v1 [cs.LO] (2017), 13 pages.

[3] Andradi H., Shen C., Ho W. K., Zhao D., *A new convergence inducing the SI-topology*, Filomat **32** (2018), no. 17, 6017–6029.

[4] Bandelt H.-J., Erné M., *The category of Z-continuous posets*, J. Pure and Appl. Algebra **30** (1983), no. 3, 219–226.

[5] Baranga A., *Z-continuous posets*, Discrete Math. **152** (1996), no. 1–3, 33–45.

[6] Engelking R., *General Topology*, Mathematical Monographs, 60, PWN—Polish Scientific Publishers, Warszawa, 1977.

[7] Erné M., *The ABC of order and topology*, Category Theory at Work, Bremen, 1990, Res. Exp. Math., 18, Heldermann, Berlin, 1991, pages 57–83.

[8] Erné M., *Z-continuous posets and their topological manifestation*, Applications of ordered sets in computer science, Braunschweig, 1996, Appl. Categ. Structures **7** (1999), no. 1–2, 31–70.

[9] Erné M., *Infinite distributive laws versus local connectedness and compactness properties*, Topology Appl. **156** (2009), no. 12, 2054–2069.

[10] Gierz G., Hofmann K. H., Keimel K., Lawson J. D., Mislove M., Scott D. S., *Continuous Lattices and Domains*, Encyclopedia of Mathematics and Its Applications, 93, Cambridge University Press, Cambridge, 2003.

[11] Gierz G., Lawson J. D., *Generalized continuous and hypercontinuous lattices*, Rocky Mountain J. Math. **11** (1981), no. 2, 271–296.

[12] Gierz G., Lawson J. D., Stralka A., *Quasicontinuous posets*, Houston J. Math. **9** (1983), no. 2, 191–208.

[13] Goubault-Larrecq J., *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs, 22, Cambridge University Press, Cambridge, 2013.

[14] Heckmann R., Keimel K., *Quasicontinuous domains and the Smyth powerdomain*, Proc. of the Twenty-Ninth Conf. Mathematical Foundations of Programming Semantics, MFPS XXIX, Electron. Notes Theor. Comput. Sci., 298, Elsevier, Amsterdam, 2013, pages 215–232.

[15] Kou H., Liu Y.-M., Luo M.-K., *On meet-continuous dcpos*, Domain Theory, Logic and Computation, Semant. Struct. Comput., 3, Kluwer Acad. Publ., Dordrecht, 2003, pages 137–149.

[16] Lawson J. D., *T_0-spaces and pointwise convergence*, Topology Appl. **21** (1985), no. 1, 73–76.

[17] Lu J., Zhao B., Wang K., *SI-continuous spaces and continuous posets*, Topology Appl. **264** (2019), 313–321.

[18] Mao X., Xu L., *Quasicontinuity of posets via Scott topology and sobrification*, Order **23** (2006), no. 4, 359–369.

[19] Mao X., Xu L., *Meet continuity properties of posets*, Theoret. Comput. Sci. **410** (2009), no. 42, 4234–4240.

[20] Venugopalan P., *Quasicontinuous posets*, Semigroup Forum **41** (1990), no. 2, 193–200.

[21] Wright J. B., Wagner E. G., Thatcher J. W., *A uniform approach to inductive posets and inductive closure*, Theoret. Comput. Sci. **7** (1978), no. 1, 57–77.

[22] Zhao D., Ho W. K, *On topologies defined by irreducible sets*, J. Log. Algebr. Methods Program. **84** (2015), no. 1, 185–195.

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