On the recognizability of some projective general linear groups by the prime graph

Masoumeh Sajjadi

Abstract. Let G be a finite group. The prime graph of G is a simple graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq. A group G is called k-recognizable by prime graph if there exist exactly k nonisomorphic groups H satisfying the condition $\Gamma(G) = \Gamma(H)$. A 1-recognizable group is usually called a recognizable group. In this problem, it was proved that $\mathrm{PGL}(2, p^{\alpha})$ is recognizable, if p is an odd prime and $\alpha > 1$ is odd. But for even α, only the recognizability of the groups $\mathrm{PGL}(2, 5^2)$, $\mathrm{PGL}(2, 3^2)$ and $\mathrm{PGL}(2, 3^4)$ was investigated. In this paper, we put $\alpha = 2$ and we classify the finite groups G that have the same prime graph as $\Gamma(\mathrm{PGL}(2, p^2))$ for $p = 7, 11, 13$ and 17. As a result, we show that $\mathrm{PGL}(2, 7^2)$ is unrecognizable; and $\mathrm{PGL}(2, 13^2)$ and $\mathrm{PGL}(2, 17^2)$ are recognizable by prime graph.

Keywords: projective general linear group; prime graph; recognition

Classification: 20D05, 20D60, 20D08

1. Introduction

Let G be a finite group. We denote by $\omega(G)$ the set of orders of elements of G. This set is closed under divisibility; hence is uniquely determined by a set $\mu(G)$ of elements in $\omega(G)$ which are maximal under divisibility relation. All the prime divisors of $|G|$ is denoted by $\pi(G)$. The prime graph of G is a simple graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq, and in this case we will write $p \sim q$. Symbol $t(G)$, is the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. The number of connected components of $\Gamma(G)$ is denoted by $s(G)$ and the set of $\pi_1(G), \pi_2(G), \ldots, \pi_{s(G)}(G)$, the connected components of $\Gamma(G)$, is denoted by $S(G)$. If $2 \in \pi(G)$, we assume $\pi_1(G)$ is the connected component containing 2.

Definition 1.1. A finite group G is called k-recognizable by prime graph if there exist exactly k nonisomorphic groups H satisfying condition $\Gamma(G) = \Gamma(H)$. A 1-recognizable group is usually called a recognizable group.

DOI 10.14712/1213-7243.2023.009

A group is said to be an almost simple group if there is a nonabelian simple group such that the given group can be embedded between the simple group and its automorphism group. Many articles are devoted to the recognition of almost simple groups by prime graph. As the projective general linear groups are under study in the present paper, we only review the results obtained up to now for these groups.

1. Let G be a finite group, and let p be a prime number such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,p))$, where $p \neq 11, 19$ and p is not a Mersenne or Fermat prime. If $p \neq 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,13)$ or $\mathrm{PSL}(2,27)$, see [8].

2. If $q = p^\alpha$, where p is an odd prime and $\alpha > 1$ is odd, then $\mathrm{PGL}(2,q)$ is uniquely determined by its prime graph, see [1].

3. If $q = p^\alpha$, where p is an odd prime and α is an even number, then $\mathrm{PGL}(2,q)$ for $q = 5^2, 3^4$ is uniquely determined by its prime graph, see [13], [11]. Also, $\mathrm{PGL}(2,3^2)$ is unrecognizable by prime graph, see [12].

In this paper, we investigate the recognizability of almost simple groups $\mathrm{PGL}(2,p^2)$ for $p = 7, 11, 13$ and 17. As a result, the group $\mathrm{PGL}(2,7^2)$, as the unrecognizable group, and $\mathrm{PGL}(2,13^2)$ and $\mathrm{PGL}(2,17^2)$, as the recognizable groups by prime graph, are added to the third part of the above list.

Assume that π is a set of prime numbers. A positive integer n is said to be a π-number if every prime divisor of n belongs to π. By convention 1 is a π-number for every set π of primes, and if $\pi = \emptyset$, 1 is the only π-number. We say that G is a π-group if $|G|$ is a π-number. Let G be a finite group. Then G has a unique largest normal π-subgroup, which is denoted by $O_\pi(G)$ and called the π-radical subgroup of G.

Extensions of groups are written in one of the following ways: $A \times B$ denotes a direct product, with normal subgroups A and B; also $A : B$ denotes a semidirect product (or split extension), with a normal subgroup A and a subgroup B; and $A \cdot B$ denotes a non-split extension, with a normal subgroup A and a quotient B, but no subgroup B; finally $A.B$ or just AB denotes an unspecified extension. In the extensions of groups, if B is cyclic of order n, we denote B by n. Our undefined notations are standard as in [5].

2. Preliminary lemmas

The first and second parts of the following remark are used extensively in Section 3 without mentioning it.

Remark 2.1. Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Then:

(1) If $p \sim q$ in $\Gamma(H)$, then $p \sim q$ in $\Gamma(G)$;
(2) If $p \sim q$ in $\Gamma(G/N)$, then $p \sim q$ in $\Gamma(G)$;
(3) If $p \sim q$ in $\Gamma(G)$ and $\{p,q\} \cap \pi(N) = \emptyset$, then $p \sim q$ in $\Gamma(G/N)$.

PROOF: The proof is straightforward. □

We will use the symbol ε to denote either ± 1 or the sign "+" or "−". We write $L_n^+(q)$ for the group $L_n(q) = \mathrm{PSL}(n,q)$ and $L_n^-(q)$ for the group $U_n(q) = \mathrm{PSU}(n,q)$.

Lemma 2.2. (a) *Let $S = S_4(q)$. We assume that $q = p^n$, where $p \neq 3$ is an odd prime. Then the set $\omega(S)$ consists of all divisors of numbers $(q^2+1)/2$, $(q^2-1)/2$, $p(q+1)$ and $p(q-1)$, see* [21].
(b) *We have*

$$\mu(L_3^\varepsilon(q)) = \begin{cases} \{q-\varepsilon 1, p(q-\varepsilon 1)/3, (q^2-1)/3, (q^2+\varepsilon q+1)/\} & \text{if } d=3; \\ \{p(q-\varepsilon 1), (q^2-1), (q^2+\varepsilon q+1)\} & \text{if } d=1, \end{cases}$$

where $q = p^\alpha$ is odd and $d = (3, q-\varepsilon 1)$, see [2, Lemma 10] *for $\varepsilon = -$, and see* [20] *for $\varepsilon = +$.*
(c) *We have $\mu(L_4^\varepsilon(q)) = \{(q^2+1)(q+\varepsilon 1), q^3-\varepsilon 1, 2(q^2-1), 4(q-\varepsilon 1)\}$, where $q = 2^m$, see* [4, Corollary 3].

Definition 2.3. A group G is a 2-Frobenius group if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

K. W. Gruenberg and O. Kegel gave the structure of finite groups with disconnected prime graph in an unpublished manuscript in 1975. When a finite group has a disconnected prime graph, we will be able to determine its structure by the following theorem.

Theorem 2.4 (Gruenberg–Kegel, see [23, Theorem A]). *If G is a finite group whose prime graph has more than one component, then one of the following holds:*

(a) *G is a Frobenius or 2-Frobenius group;*
(b) *there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G.*

Here we list some properties of the Frobenius group whose proofs can be found in [15].

Lemma 2.5. *Let G be a Frobenius group with kernel K and complement H. Then the following holds:*

(a) *K is a nilpotent group; in particular, the prime graph of K is complete;*
(b) *$s(G) = 2$ and $S(G) = \{\pi(K), \pi(H)\}$;*
(c) *$|K| \equiv 1 \pmod{|H|}$; and*
(d) *every subgroup of H of order pq, where p and q are not necessarily deferent prime numbers, is cyclic. In particular each Sylow subgroup of H of odd order is cyclic and a Sylow 2-subgroup of H is either cyclic or a generalized quaternion group. If H is nonsolvable, then there is a normal subgroup H_0 of H such that $|H : H_0| \leq 2$ and $H_0 \cong \mathrm{SL}(2, 5) \times Z$, where every Sylow subgroup of Z is cyclic and $|Z|$ is prime to 2, 3 and 5.*

Lemma 2.6 ([14, Lemma 1])**.** *Let G be a finite group, let $K \trianglelefteq G$, and let G/K be a Frobenius group with kernel F and complement C. If $(|F|, |K|) = 1$ and F does not lie in $KC_G(K)/K$, then $r \cdot |C| \in \omega(G)$ for some prime divisor r of $|K|$.*

Lemma 2.7 ([6, Lemma 3])**.** *Let G be a 2-Frobenius group. Then G is a solvable group.*

Lemma 2.8 ([7, Theorem 1])**.** *If G is a finite solvable group all of whose elements are of prime power order, then $|\pi(G)| \leq 2$.*

Lemma 2.9 ([9])**.** *Let $n \geq 2$ and $q = p^f$. Then*

(a) *$\mathrm{Out}(\mathrm{PSL}(n, q)) \cong \mathbb{Z}_{(n,q-1)} : \mathbb{Z}_f : \mathbb{Z}_2$, if $n \geq 3$;*
(b) *$\mathrm{Out}(\mathrm{PSL}(2, q)) \cong \mathbb{Z}_{(2,q-1)} \times \mathbb{Z}_f$.*

Notation 2.10. Let G be an almost simple group related to $L = \mathrm{PSL}(2, p^2))$, where p is an odd prime. By Lemma 2.9, $\mathrm{Out}(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{Aut}(L) \cong L \cdot 2^2$ (the Klein's four group is denoted by 2^2). We note that the number of nontrivial proper subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ up to conjugacy is three subgroups of order 2: the field, diagonal and field-diagonal automorphisms of L, which are denoted by 2_1, 2_2 and 2_3, respectively. So if G is an almost simple group related to L, i.e. $L \trianglelefteq G \leq \mathrm{Aut}(L)$, then G is isomorphic to one of these groups: L, $L : 2_1 \cong \mathrm{PGL}(2, p^2)$, $L : 2_2 \cong P\Sigma L(2, p^2)$, $L \cdot 2_3$, $L \cdot 2^2 \cong \mathrm{Aut}(L)$.

Given a prime $p \geqslant 5$, we denote by $\mathfrak{S}_p$ the set of all finite nonabelian simple groups G, such that $p \in \pi(G) \subseteq \{2, 3, \ldots, p\}$.

Lemma 2.11 ([18, Lemma 2.1])**.** *Let P be a nonabelian simple group that belongs to $\mathfrak{S}_p$, where $5 \leqslant p \leqslant 997$. Then $\pi(\mathrm{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$, and 11 divides only the order of the outer automorphism group of $L_2(2^{11})$.*

Lemma 2.12 ([17])**.** *Let G be a finite group and N a nontrivial normal p-subgroup for some prime p and set $K = G/N$. Suppose that K contains an element x of order $m$coprime to p such that $\langle \varphi |_{\langle x \rangle}, 1 |_{\langle x \rangle} \rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) K in characteristic p. Then G contains elements of order pm.*

Let $t > 1$ and n be natural numbers. A primitive prime divisor of $t^n - \varepsilon^n$ is a prime that divides $t^n - \varepsilon^n$ and does not divide $t^i - \varepsilon^i$ for $1 \leqslant i < n$, which is denoted by $t_{[\varepsilon n]}$. The following lemma is taken from [26].

Lemma 2.13 ([26, corollary 9]). *Let $G = L_n^{\varepsilon}(q)$, $q = p^m$, be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Denote $H = W : G$.*

(1) *If $q = p$ and $(n, q - \varepsilon) > 1$, then $pq_{[\varepsilon n]} \in \omega(H)$.*
(2) *If n is odd, then $pq_{[\varepsilon(n-1)]} \in \omega(H)$.*
(3) *If $n = 3$ and $(q - \varepsilon)_3 = 3$, then $3p \in \omega(H)$.*
(4) *If $n = 2$ and q is odd, then $2p \in \omega(H)$.*

Lemma 2.14 ([22, Lemma 5]). *Let L be a finite simple group $L_n(q)$, $d = (n, q - 1)$.*

(1) *If there exists a primitive prime divisor r of $q^n - 1$, then L includes a Frobenius subgroup with kernel of order r and cyclic complement of order n.*
(2) *Group L includes a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/d$.*

Definition 2.15. A finite nonabelian simple group G is called a simple K_n-group, if the order of G has exactly n distinct prime factors.

Lemma 2.16 ([19, Theorem 2]). *Let G be a simple K_4-group. Then, G is isomorphic to one of the following simple groups:*

(a) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.
(b) $L_2(r)$, *where r is a prime satisfying the equation $r^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some $a, b, c \geq 1$ and a prime $u > 3$.*
(c) $L_2(2^m)$, *where $m \geq 1$ satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t.*
(d) $L_2(3^m)$, *where $m \geq 1$ satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$, or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u, t are odd primes and $b \geq 1$.*

GAP code enabled us to find the spectrum of the following almost simple groups, as used in the main results.

Lemma 2.17. *We have:*

$$\omega(U_4(5)) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 20, 21, 24, 26, 30, 52, 60, 63\}$$
$$\omega\big(\mathrm{Aut}(L_2(3^5))\big) = \{1, 2, 3, 4, 5, 10, 11, 15, 20, 22, 61, 121, 122, 242, 244\}$$

$$\omega(P\Sigma L(2,3^5)) = \{1,2,3,5,10,11,15,61,121,122\}$$
$$\omega(S_4(8)) = \{1,2,3,4,5,6,7,8,9,13,14,18,21,63,65\}$$
$$\omega(L_3(16)) = \{1,2,3,4,5,7,10,13,15,17,85,91\}$$
$$\omega(S_6(4)) = \{1,2,3,4,5,6,7,8,9,10,12,13,15,17,20,21,30,34,51,63,65,85\}$$
$$\omega(O_8^+(4)) = \{1,2,3,4,5,6,7,8,9,10,12,13,15,17,20,21,30,34,51,63,65,85, 255\}.$$

3. Main results

Lemma 3.1. *Let G be a finite group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,p^2))$, where $p \geq 5$ is prime. Then, either G is a nonsolvable Frobenius group and $p = 7$; or there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G.*

Proof: We know that $\mu(\mathrm{PGL}(2,p^2)) = \{p^2-1, p, p^2+1\}$. Then $\Gamma(G) = \Gamma(\mathrm{PGL}(2,p^2))$ implies that $S(G) = \{\pi_1 = \pi(p^2-1) \cup \pi(p^2+1), \pi_2 = \{p\}\}$.

Clearly 3 does not divide n^2+1 for every natural number n. Therefore if $\nu \in \pi((p^2+1)/2)$, $\nu \neq 3$. Now, let G be a solvable group. Then, G has a solvable Hall $\{3,\nu,p\}$-subgroup T. As there is no edge between 3, ν and p in $\Gamma(G)$, it follows that each element of T has prime power order. Hence $|\pi(T)| \leq 2$ by Lemma 2.8, which is a contradiction. Thus G is not solvable; so by using Lemma 2.7, we conclude that G is not a 2-Frobenius. So by Theorem 2.4, either G is a nonsolvable Frobenius group or there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G.

If G is a Frobenius group with kernel K and complement H, H is nonsolvable, because G is a nonsolvable group. Then $\pi(H) \neq \{p\}$, which implies that $\pi(H) = \pi(p^2-1) \cup \pi(p^2+1)$ and $\pi(K) = \{p\}$. Also by Lemma 2.5, there is a normal subgroup H_0 of H such that $|H : H_0| \leq 2$ and $H_0 \cong \mathrm{SL}(2,5) \times Z$, where $|Z|$ is prime to 2, 3 and 5. Suppose that $Z \neq 1$ and ν is a prime divisor of $|Z|$. Then $\nu \sim 3$ and $\nu \sim 5$ in $\Gamma(H_0) \subseteq \Gamma(G)$, which implies that $\{\nu, 3, 5\}$ is a subset of $\pi(p^2-1)$, because $\nu \neq 2$ and 3 divides p^2-1. Therefore $\pi(H) \subseteq \pi(p^2-1)$, concluding $\pi(p^2+1) = \{2\}$, because $\pi(p^2-1) \cap \pi(p^2+1) = \{2\}$, but it is evident that p^2+1 has at least one odd divisor, which is a contradiction. Then $Z = 1$. So $H_0 = \mathrm{SL}(2,5)$ and then $\pi(H) = \{2,3,5\}$. Therefore $\pi(G) = \{2,3,5\} \cup \{p\}$, where $p > 5$. Since $\pi(\mathrm{PSL}(2,p^2)) = \pi(\mathrm{PGL}(2,p^2)) = \pi(G) = \{2,3,5,p\}$, using Lemma 2.16 implies that $p = 7$ and the proof is completed. □

Theorem 3.2. *Let G be a finite group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,7^2))$. Then one of the following holds:*

(1) *G is isomorphic to a Frobenius group $K:H$, where K is a 7-group and H contains a normal subgroup H_0 such that $|H:H_0| \leq 2$ and $H_0 \cong \mathrm{SL}(2,5)$;*

(2) *$G \cong \mathrm{PGL}(2,7^2)$, $U_3(5)$, $U_3(5).2$, $U_4(3).2_2$ or $U_4(3).2_3$;*

(3) *$G/O_2(G)$ is isomorphic to A_7, S_7, $L_3(4)$, $L_3(4).2_1$ or $L_3(4).2_3$ for $O_2(G) \neq 1$ and G is isomorphic to S_7 or $L_3(4).2_3$ for $O_2(G) = 1$.*

Moreover $\mathrm{PGL}(2,7^2)$ is unrecognizable by prime graph.

Proof: Applying Lemma 3.1, we obtain either G is a nonsolvable Frobenius group or there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G.

If $G = K:H$ is a Frobenius group, then by Lemma 3.1, $\pi(K) = \{p\} = \{7\}$; and there is a normal subgroup H_0 of H such that $|H:H_0| \leq 2$ and $H_0 \cong \mathrm{SL}(2,5)$. In what follows we show that there are infinitely many Frobenius groups with the above properties and then we obtain $\mathrm{PGL}(2,7^2)$ is unrecognizable by prime graph. Let F be a finite field with $\mathrm{char}(F) = 7$. Since F has prime subfield isomorphic to $\mathbb{Z}_7$, there are elements α and β in F such that $\alpha^2 = -1$ and $\beta^2 = 5$. So $\sqrt{5}, \sqrt{-1} \in F$. Therefore, if V is a vector space of dimension two over F (note that V is an elementary Abelian 7-group), by Proposition 6.1.2 of [16], the group $\mathrm{VSL}(2,5)$ is Frobenius with kernel V and complement $\mathrm{SL}(2,5)$. Since $\Gamma(\mathrm{SL}(2,5)) = \{2 \sim 3, 2 \sim 5\}$ and V is a 7-group, then $\Gamma(\mathrm{VSL}(2,5)) = \Gamma(\mathrm{PGL}(2,7^2)) = \{2 \sim 3, 2 \sim 5, 7\}$. We know that there are infinitely many field F with the above properties and therefore we can construct infinitely many Frobenius group $\mathrm{VSL}(2,5)$. This implies that $\mathrm{PGL}(2,7^2)$ is unrecognizable by prime graph.

If G is not a Frobenius group, there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G. We have $\Gamma(G) = \{2 \sim 3, 2 \sim 5, 7\}$ and $S(G) = \{\pi_1 = \{2,3,5\}, \pi_2 = \{7\}\}$. Then K is a $\{2,3,5\}$-subgroup of G and 7 is an isolated vertex in $\Gamma(G)$.

All simple groups S with $\pi(S) \subseteq \{2,3,5,7\}$ are listed in Table 1, taken from [25]. It is noteworthy that, given the order of the groups S and the order of their outer automorphisms, $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ implies that 5 belongs to $\pi(K)$ in some cases, which we will mention.

Now we study each of the items in Table 1.

1. Let $S \cong A_5, A_6$ or $S_4(3)$. We have $7 \in \pi(G)$ and $|G|/|K|$ divides $|S| \times |\mathrm{Out}(S)|$. Therefore, since K is a $\{2,3,5\}$-group, $7 \in \pi(|S| \cdot |\mathrm{Out}(S)|)$, which is a contradiction.

2. Let $S \cong L_2(7)$. By Lemma 2.14, $L_2(7)$ has a Frobenius subgroup $7:3$. Also $5 \in \pi(K)$, suppose that $K_5 \in \mathrm{Syl}_5(K)$. Since K is nilpotent, $K = O_{5'}(K) \times K_5$ and $O_{5'}(K) \times \Phi(K_5) \trianglelefteq G$. Therefore $G/K \cong [G/O_{5'}(K) \times \Phi(K_5)]/[K/O_{5'}(K) \times \Phi(K_5)]$, and $K/O_{5'}(K) \times \Phi(K_5) \cong K_5/\Phi(K_5)$ is an elementary Abelian 5-group.

So without loss of generality, we may assume that K is an elementary Abelian 5-group. Therefore by Lemma 2.6, $5 \sim 3$ in $\Gamma(G)$, which is a contradiction.

S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6

TABLE 1. Nonabelian simple group S with $\pi(S) \subseteq \{2, 3, 5, 7\}$.

3. Let $S \cong L_2(8)$. By Lemma 2.14, $L_2(8)$ has a Frobenius subgroup $8 : 7$. Also $5 \in \pi(K)$, so we may assume that K is an elementary Abelian 5-group. Therefore by Lemma 2.6, $5 \sim 7$ in $\Gamma(G)$, which is a contradiction.

4. Let $S \cong U_3(3)$. By [5], $L_2(7) \leqslant U_3(3)$; also $5 \in \pi(K)$ in this case. Therefore by 2 we get a contradiction.

5. Let $S \cong A_8, A_9, J_2, A_{10}, S_6(2)$ or $O_8^+(2)$. By [5], $15 \in \omega(S)$, then $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

6. Let $S \cong S_4(7)$. By Lemma 2.2, $\mu(S_4(7)) = \{25, 24, 56, 42\}$. Then 7 is not an isolated vertex in $\Gamma(G)$, which is a contradiction.

Then $S \cong A_7$, $L_2(49)$, $U_3(5)$, $L_3(4)$ or $U_4(3)$.

Case 1. Let $S \cong A_7$. Suppose that $\pi(K)$ contains a prime $r \in \{3, 5\}$, we may assume that K is an elementary Abelian r-group.

By [5], $L_2(7) \leqslant A_7$, and by Lemma 2.14, $L_2(7)$ has a Frobenius subgroup $7 : 3$. So if $r = 5$, by Lemma 2.6, $5 \sim 3$ in $\Gamma(G)$, which is a contradiction. Let $r = 3$. Suppose that x is an element of order 5 in A_7 and let $X = \langle x \rangle$. By the table of 3-modular characters of A_7, see [28], we get $\langle \varphi\mid_{\langle x\rangle}, 1\mid_{\langle x\rangle}\rangle > 0$ for every irreducible character φ of A_7 (mod 3) as follows:

$$\langle 1_S \mid_X, 1\mid_X\rangle = 1$$

$$\langle 6 \mid_X, 1\mid_X\rangle = \frac{1}{5}(6+1+1+1+1) = 2$$

$$\langle 10_1 \mid_X, 1\mid_X\rangle = \langle 10_2 \mid_X, 1\mid_X\rangle = \frac{1}{5}(10+0+0+0+0) = 2$$

$$\langle 13 \mid_X, 1 \mid_X \rangle = \frac{1}{5}(13 - 2 - 2 - 2 - 2) = 1$$
$$\langle 15 \mid_X, 1 \mid_X \rangle = \frac{1}{5}(15 + 0 + 0 + 0 + 0) = 3.$$

Now Lemma 2.12 implies that $3 \cdot 5 \in \omega(G)$, which is a contradiction. Then K is a 2-group; therefore by Table 2, $G/O_2(G) \cong A_7$ or $A_7.2 \cong S_7$. Assume that $O_2(G) = 1$, since $\Gamma(S_7) = \Gamma(\mathrm{PGL}(2, 7^2))$, then $G \cong S_7$.

Case 2. Let $S \cong L_2(49) =: L$. We know $\mu(\mathrm{PSL}(2, p^2)) = \{(p^2-1)/2, p, (p^2+1)/2\}$, where p is an odd prime. Then $\mu(L) = \{24, 7, 25\}$. Let $K \neq 1$, then $\pi(K)$ contains a prime $r \in \pi_1 = \{2, 3, 5\}$. Suppose that $P \in \mathrm{Syl}_r(K)$ and $B \in \mathrm{Syl}_7(L)$. Since K is nilpotent, P is a characteristic subgroup of K; therefore $PB \leqslant G$. But $7 \nsim r$ in $\Gamma(G)$, hence B acts fixed point freely on P. So $P : B$ is a Frobenius subgroup of G with the kernel P and complement B. By Lemma 2.5 (d), B is cyclic, because $|B|$ is odd, which is a contradiction. So we obtain $K = 1$ and $L \leqslant G \leqslant \mathrm{Aut}(L)$. Then, G is isomorphic to L, $L : 2_1$, $L : 2_2$, $L \cdot 2_3$ or $L \cdot 2^2$. But $2 \nsim 5$ in $\Gamma(L)$; also $\Gamma(L \cdot 2_3) = \Gamma(L)$ by [10]. Then $G \ncong L$ and $L \cdot 2_3$. By [3], $C_L(2_2) = \mathrm{PSL}(2, p)$. Therefore $2 \sim p$ in $\Gamma(P\Sigma L(2, p^2))$, which implies that $2 \sim 7$ in $\Gamma(P\Sigma L(2, 7^2))$. Hence $G \ncong L : 2_2$; as a result $G \ncong L \cdot 2^2$. So $G \cong L : 2_1 \cong \mathrm{PGL}(2, 7^2)$.

L	G	elements of $\mu(G)$	L	G	elements of $\mu(G)$
$L_3(4)$	L	$7, 5, 4, 3$	$U_4(3)$	L	$12, 9, 8, 7, 5$
	$L.2_1$	$8, 7, 6, 5$		$L.2_1$	$14, 12, 10, 9, 8$
	$L.3$	$21, 15, 6, 4$		$L.4$	$28, 24, 20, 9$
	$L.6$	$21, 15, 12, 8$		$L.2_2$	$18, 12, 10, 8, 7$
	$L.2_2$	$14, 8, 6, 5$		$L.(2^2)_{122}$	$18, 14, 12, 10, 8$
	$L.3.2_2$	$21, 15, 14, 12, 8$		$L.2_3$	$24, 10, 9, 7$
	$L.2_3$	$10, 8, 7, 6$		$L.(2^2)_{133}$	$24, 14, 10, 9$
	$L.3.2_3$	$21, 15, 10, 8, 6$		$L.D_8$	$28, 24, 20, 18$
	$L.2^2$	$14, 10, 8, 6$	A_7	L	$7, 6, 5, 4$
	$L.D_{12}$	$21, 15, 14, 12, 10, 8$		$L.2$	$12, 10, 7$
$U_3(5)$	L	$10, 8, 7, 6$			
	$L.2$	$20, 12, 8, 7$			
	$L.3$	$30, 24, 21$			
	$L.S_3$	$30, 24, 21, 20$			

TABLE 2. Almost simple groups $L \leqslant G \leqslant \mathrm{Aut}(L)$ for some simple groups L.

Case 3. Let $S \cong U_3(5)$ or $U_4(3)$. Suppose that $3 \in \pi(K)$, we may assume that K is an elementary Abelian 3-group. Let $S \cong U_3(5)$. Assume that $P \in \mathrm{Syl}_3(K)$ and $B \in \mathrm{Syl}_5(S)$. Since $3 \nsim 5$ in $\Gamma(G)$, by the similar argument as in Case 2, we get a Frobenius subgroup $P : B$ of G. By Lemma 2.5 (d), B is cyclic. Then S must have an element of order 5^3, which is a contradiction, because $\mu(U_3(5)) = \{10, 8, 7, 6\}$, see Table 2. Let $S \cong U_4(3)$, by Lemma 2.13 (1), we get $15 \in \omega(K : U_4(3))$. By [27, Lemma 10], we have $\omega(K : U_4(3)) \subseteq \omega(G)$. Therefore $15 \in \omega(G)$, which is a contradiction, so $3 \nmid |K|$.

Now suppose that $K \neq 1$, then $\pi(K)$ contains a prime $r \in \{2, 5\}$. We may assume that K is an elementary Abelian r-group. Let x be an element of order 7 in S and $X = \langle x \rangle$. By the table of r-modular characters of S, see [28], similarly to Case 1, we get $\langle \varphi|_{\langle x \rangle}, 1|_{\langle x \rangle} \rangle > 0$ for every irreducible character φ of S (mod r). Now Lemma 2.12 implies that $7r \in \omega(G)$, which is a contradiction. Then $K = 1$. Now we conclude:

If $S \cong U_3(5) := L$, by Table 2, $G \cong L$, $L.2$, $L.3$ or $L.S_3$. By the structure of $\Gamma(G)$, $G \cong L$ or $L.2$.

If $S \cong U_4(3) := L$, by Table 2, $G \cong L$, $L.2_1$, $L.4$, $L.2_2$, $L.(2^2)_{122}$, $L.2_3$, $L.(2^2)_{133}$ or $L.D_8$. Again by $\Gamma(G)$, $G \cong L.2_2$ or $L.2_3$.

Case 4. Let $S \cong L_3(4) =: L$. Suppose that $\pi(K)$ contains a prime $r \in \{3, 5\}$. We may assume that K is an elementary Abelian r-group.

By [5], $L_2(7) \leqslant L_3(4)$; therefore if $r = 5$, similarly to Case 1, we get a contradiction. By Lemma 2.14, $L_3(4)$ has a Frobenius subgroup $4^2 : 5$. Then if $r = 3$, by Lemma 2.6, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. So K is a 2-group; therefore according to Table 2, $G/O_2(G) \cong L$, $L.2_1$, $L.3$, $L.6$, $L.2_2$, $L.3.2_2$, $L.2_3$, $L.3.2_3$, $L.2^2$ or $L.D_{12}$. But 7 is not an isolated vertex in the prime graph of the groups $L.3$, $L.6$, $L.2_2$, $L.3.2_2$, $L.3.2_3$, $L.2^2$ and $L.D_{12}$. Therefore $G/O_2(G) \cong L$, $L.2_1$ or $L.2_3$. Assume that $O_2(G) = 1$, since $\Gamma(L.2_3) = \Gamma(\mathrm{PGL}(2, 7^2))$, then $G \cong L.2_3$. □

Theorem 3.3. *Let G be a finite group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 11^2))$. Then $G \cong \mathrm{PGL}(2, 11^2)$ or the factor group $G/O_3(G)$ is isomorphic to $P\Sigma L(2, 3^5)$ for $O_3(G) \neq 1$.*

Proof: By Lemma 3.1, there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G. Also $\mu(\mathrm{PGL}(2, 11^2)) = \{11^2 - 1, 11, 11^2 + 1\}$. Therefore $\Gamma(G) = \{2 \sim 3, 2 \sim 5, 3 \sim 5, 2 \sim 61, 11\}$ and $S(G) = \{\pi_1 = \{2, 3, 5, 61\}, \pi_2 = \{11\}\}$. Then K is a $\{2, 3, 5, 61\}$-subgroup of G and 11 is an isolated vertex in $\Gamma(G)$.

Because $\pi(\mathrm{Out}(S)) \subseteq \{2, 3, 5, 7\}$ by Lemma 2.11, 11 must belong to $\pi(S)$. Here we have listed all possibilities for S in Table 3, taken from [25].

S	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|S\|$	$\|\mathrm{Out}(S)\|$
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(3^5)$	$2^2 \cdot 3^5 \cdot 11^2 \cdot 61$	10
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(11^2)$	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$	4
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_4(11)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 11^4 \cdot 61$	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2			

TABLE 3. Nonabelian simple group S with $11 \in \pi(S) \subseteq \{2, 3, 5, 11, 61\}$.

Now we study each of the items in Table 3:

1. Let $S \cong L_2(11)$. By Lemma 2.14, $L_2(11)$ has a Frobenius subgroup $11 : 5$. But $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ implies that 61 divides $|K|$, so we may assume that K is an elementary Abelian 61-group. Therefore by Lemma 2.6, $61 \sim 5$ in $\Gamma(G)$, which is a contradiction.

2. Let $S \cong M_{11}, M_{12}$ or $U_5(2)$. By [5], $L_2(11) \leqslant S$, and in this case $61 \in \pi(K)$ too. So by 1, we get a contradiction.

3. Let $S \cong S_4(11)$. By Lemma 2.2, we have $\mu(S_4(11)) = \{61, 60, 11 \cdot 12, 11 \cdot 10\}$. Then 11 is not an isolated vertex in $\Gamma(G)$, which is a contradiction.

Then $S \cong L_2(3^5)$ or $L_2(11^2)$.

Case 1. Let $S \cong L_2(3^5) =: L$. By Lemma 2.14, L has a Frobenius subgroup $3^5 : 121$. Let $\pi(K)$ contain a prime $r \in \{2, 5, 61\}$, we may assume that K is an elementary Abelian r-group. Therefore by Lemma 2.6, 11 is not an isolated vertex in $\Gamma(G)$, which is a contradiction. So we obtain K is a 3-group. By Lemma 2.9, $\mathrm{Out}(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$. Then $G/K \cong L$, $L.2$, $L.5$ or $L.10$. If $G/K \cong L$ $(2^2 \cdot 3^5 \cdot 11^2 \cdot 61)$ or $L.2$ $(2^3 \cdot 3^5 \cdot 11^2 \cdot 61)$, we get a contradiction, because $5 \,|\, |G|$ and K is a 3-group. If $G/K \cong L.10 \cong \mathrm{Aut}(L)$, then by Lemma 2.17, $2 \sim 11$ in $\Gamma(G/K)$. Therefore $2 \sim 11$ in $\Gamma(G)$, which is a contradiction. Hence $G/O_3(G) \cong P\Sigma L(2, 3^5)$ (note that $\Gamma(P\Sigma L(2, 3^5)) \subseteq \Gamma(G)$, see Lemma 2.17).

Case 2. Let $S \cong L_2(11^2) =: L$. Let $K \neq 1$, then $\pi(K)$ contains a prime $r \in \pi_1 = \{2, 3, 5, 61\}$. Similar to Theorem 3.2 Case 2, G has a Frobenius subgroup $P : B$, where $P \in \mathrm{Syl}_r(K)$ and $B \in \mathrm{Syl}_{11}(L)$. By Lemma 2.5 (d), B is cyclic, because $|B|$ is odd, which is a contradiction. So we obtain $K = 1$ and $L \leqslant G \leqslant \mathrm{Aut}(L)$. Then, G is isomorphic to L, $L : 2_1$, $L : 2_2$, $L \cdot 2_3$ or $L \cdot 2^2$. But $2 \nsim 61$ in $\Gamma(L)$; also $\Gamma(L \cdot 2_3) = \Gamma(L)$ by [10]. Then $G \ncong L$ and $L \cdot 2_3$. By [3], $C_L(2_2) = \mathrm{PSL}(2, 11)$. Therefore $2 \sim 11$ in $\Gamma(P\Sigma L(2, 11^2))$; hence $G \ncong L : 2_2$; as a result $G \ncong L \cdot 2^2$. So $G \cong L : 2_1 \cong \mathrm{PGL}(2, 11^2)$. □

Remark 3.4. If there are no examples of extensions of $P\Sigma L(2, 3^5)$ by nontrivial 3-groups having the same prime graph as for $\mathrm{PGL}(2, 11^2)$, we can say that $\mathrm{PGL}(2, 11^2)$ is recognizable by prime graph.

Theorem 3.5. *Let G be a finite group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p^2))$ for $p = 13$ or 17. Then $G \cong \mathrm{PGL}(2, p^2)$, in other words $\mathrm{PGL}(2, p^2)$, where $p \in \{13, 17\}$ is recognizable by prime graph.*

Proof: By Lemma 3.1, there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal π_1-subgroup K of G.

Case 1. Let $p = 13$. Then $\mu(\mathrm{PGL}(2, 13^2)) = \{13^2 - 1, 13, 13^2 + 1\}$. Therefore $\Gamma(G) = \{2 \sim 3, 2 \sim 7, 2 \sim 5, 2 \sim 17, 3 \sim 7, 5 \sim 17, 13\}$ and $S(G) = \{\pi_1 = \{2, 3, 5, 7, 17\}, \pi_2 = \{13\}\}$. Then K is a $\{2, 3, 5, 7, 17\}$-subgroup of G and 13 is an isolated vertex in $\Gamma(G)$.

Because $\pi(\mathrm{Out}(S)) \subseteq \{2, 3, 5, 7\}$ by Lemma 2.11, 13 must belong to $\pi(S)$. We have listed all possibilities for S in Table 4, taken from [25].

S	$\lvert S\rvert$	$\lvert \mathrm{Out}(S)\rvert$	S	$\lvert S\rvert$	$\lvert \mathrm{Out}(S)\rvert$
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_z(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4			

Table 4. Nonabelian simple group S with $13 \in \pi(S) \subseteq \{2, 3, 5, 7, 13, 17\}$.

Now we study all of the items in the above table. Note that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$ implies that 1, 2 and 5 to 12, 17 divides $|K|$. Therefore in the mentioned items, we may assume that K is an elementary Abelian 17-group.

1. Let $S \cong L_3(3)$. By [5], $L_3(3)$ has a Frobenius subgroup $13 : 3$. Since $17 \in \pi(K)$, by Lemma 2.6, $17 \sim 3$ in $\Gamma(G)$, which is a contradiction.

2. Let $S \cong L_2(25)$. By Lemma 2.14, $L_2(25)$ has a Frobenius subgroup $25 : 12$. Also $17 \in \pi(K)$, so by Lemma 2.6, $17 \sim 3$ in $\Gamma(G)$, which is a contradiction.

3. Let $S \cong U_3(4)$, $S_6(3)$, $O_7(3)$, $G_2(4)$ or $F_4(2)$. By [5], $15 \in \omega(S)$. Then $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

4. Let $S \cong S_4(5)$ or $U_4(4)$. By Lemma 2.2, $30 \in \mu(S)$, so $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

5. Let $S \cong L_4(3)$. By Lemma 2.14, $L_4(3)$ has a Frobenius subgroup $3^3 : 13$. But $17 \in \pi(K)$, therefore by Lemma 2.6, $17 \sim 13$ in $\Gamma(G)$, which is a contradiction.

6. Let $S \cong {}^2F_4(2)'$. By [5], $L_2(25) \leqslant {}^2F_4(2)'$, and in this case $17 \in \pi(K)$ too. So by 2, we get a contradiction.

7. Let $S \cong L_2(13)$. By Lemma 2.14, $L_2(13)$ has a Frobenius subgroup $13 : 6$. Since $17 \in \pi(K)$, therefore $17 \sim 3$ in $\Gamma(G)$ by Lemma 2.6, which is a contradiction.

8. Let $S \cong L_2(27)$. By Lemma 2.14, $L_2(27)$ has a Frobenius subgroup $27 : 13$. Also $17 \in \pi(K)$, so by Lemma 2.6, $17 \sim 13$ in $\Gamma(G)$, which is a contradiction.

9. Let $S \cong G_2(3)$. By [5], $L_2(13) \leqslant G_2(3)$; also $17 \in \pi(K)$. Then by 7, we get a contradiction.

10. Let $S \cong {}^3D_4(2)$. By [5], $L_2(8) \leqslant {}^3D_4(2)$, by Lemma 2.14, $L_2(8)$ has a Frobenius subgroup $8 : 7$; also $17 \in \pi(K)$. So by Lemma 2.6, $17 \sim 7$ in $\Gamma(G)$, which is a contradiction.

11. Let $S \cong Sz(8)$. By [5], $Sz(8)$ has a Frobenius subgroup $2^{3+3} : 7$. Since $17 \in \pi(K)$, therefore $17 \sim 7$ in $\Gamma(G)$ by Lemma 2.6, which is a contradiction.

12. Let $S \cong L_2(64)$. By Lemma 2.14, $L_2(64)$ has a Frobenius subgroup $64 : 63$. Since $17 \in \pi(K)$, then $17 \sim 3$ and $17 \sim 7$ in $\Gamma(G)$ by Lemma 2.6, which is a contradiction.

13. Let $S \cong U_4(5)$. By Lemma 2.17, $15 \in \omega(U_4(5))$; hence $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

14. Let $S \cong L_3(9)$. By Lemma 2.2, $91 \in \mu(L_3(9))$. Then $7 \sim 13$ in $\Gamma(G)$, which is a contradiction.

15. Let $S \cong S_4(8)$. By Lemma 2.17, $65 \in \omega(S_4(8))$; hence $5 \sim 13$ in $\Gamma(G)$, which is a contradiction.

16. Let $S \cong O_8^+(3)$. By [5], $O_7(3) \leqslant O_8^+(3)$; so by 3 we get a contradiction.

17. Let $S \cong U_3(17)$. By Lemma 2.2, $17 \cdot 18 \in \mu(U_3(17))$; therefore $17 \sim 3$ in $\Gamma(G)$, which is a contradiction.

18. Let $S \cong S_4(13)$. By Lemma 2.2, $13 \cdot 14 \in \mu(S_4(13))$; therefore 13 is not an isolated vertex in $\Gamma(G)$, which is a contradiction.

19. Let $S \cong L_3(16), S_6(4)$ or $O_8^+(4)$. By Lemma 2.17, $15 \in \omega(S)$; hence $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

Then $S \cong L_2(13^2) := L$ and $L \leqslant G/K \leqslant \mathrm{Aut}(L)$. Let $K \neq 1$, then $\pi(K)$ contains a prime $r \in \pi_1 = \{2, 3, 5, 7, 17\}$. Similarly to Theorem 3.2 Case 2, G has a Frobenius subgroup $P : B$, where $P \in \mathrm{Syl}_r(K)$ and $B \in \mathrm{Syl}_{13}(L)$. By Lemma 2.5 (d), B is cyclic, because $|B|$ is odd, which is a contradiction. So we obtain $K = 1$ and $L \leqslant G \leqslant \mathrm{Aut}(L)$. Then, G is isomorphic to L, $L : 2_1$, $L : 2_2$, $L \cdot 2_3$ or $L \cdot 2^2$. But $2 \nsim 5$ and $2 \nsim 17$ in $\Gamma(L)$; also $\Gamma(L \cdot 2_3) = \Gamma(L)$

by [10]. Then $G \ncong L$ and $L \cdot 2_3$. By [3], $C_L(2_2) = \mathrm{PSL}(2,13)$. Therefore $2 \sim 13$ in $\Gamma(P\Sigma L(2,13^2))$; hence $G \ncong L : 2_2$; as a result $G \ncong L \cdot 2^2$. So $G \cong L : 2_1 \cong \mathrm{PGL}(2,13^2)$.

Case 2. Let $p = 17$. Then $\mu(\mathrm{PGL}(2,17^2)) = \{17^2-1, 17, 17^2+1\}$. Therefore $\Gamma(G) = \{2 \sim 3, 2 \sim 5, 2 \sim 29, 5 \sim 29, 17\}$ and $S(G) = \{\pi_1 = \{2,3,5,29\}, \pi_2 = \{17\}\}$. Then K is a $\{2,3,5,29\}$-subgroup of G and 17 is an isolated vertex in $\Gamma(G)$.

Because $\pi(\mathrm{Out}(S)) \subseteq \{2,3,5,7\}$ by Lemma 2.11, 17 must belong to $\pi(S)$. We have listed all possibilities for S in Table 5, taken from [25].

S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$L_2(17^2)$	$2^5 \cdot 3^2 \cdot 5 \cdot 17^2 \cdot 29$	4
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	$S_4(17)$	$2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$	2
$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4			

TABLE 5. Nonabelian simple group S with $17 \in \pi(S) \subseteq \{2,3,5,17,29\}$.

Now we study each of the items in the above table separately:

1. Let $S \cong L_2(16)$. By Lemma 2.14, $L_2(16)$ has a Frobenius subgroup with kernel of order 16 and cyclic complement of order 15. Then $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

2. Let $S \cong S_4(4)$. By [5], $15 \in \omega(S_4(4))$. Then $3 \sim 5$ in $\Gamma(G)$, which is a contradiction.

3. Let $S \cong S_4(17)$. By Lemma 2.2, $17 \cdot 18 \in \mu(S_4(17))$; therefore 17 is not an isolated vertex in $\Gamma(G)$, which is a contradiction.

4. Let $S \cong L_2(17)$. By Table 5, $\{5,29\} \subseteq \pi(K)$. Therefore we may assume that K is an elementary Abelian 29-group. Since 29 does not belong to $L_2(17)$, the ordinary character table of $L_2(17)$ implies that either an element of order 3 or an element of order 17 has a fixed point in K, see [24, Lemma 2.17]. Then $3 \sim 29$ or $17 \sim 29$ in $\Gamma(G)$, which is a contradiction.

Then $S \cong L_2(17^2) := L$ and $L \leqslant G/K \leqslant \mathrm{Aut}(L)$. Let $K \neq 1$, then $\pi(K)$ contains a prime $r \in \pi_1 = \{2,3,5,29\}$. Similarly to Theorem 3.2 Case 2, G has a Frobenius subgroup $P : B$, where $P \in \mathrm{Syl}_r(K)$ and $B \in \mathrm{Syl}_{17}(L)$. By Lemma 2.5 (d), B is cyclic, because $\lvert B\rvert$ is odd, which is a contradiction. So we obtain $K = 1$ and $L \leqslant G \leqslant \mathrm{Aut}(L)$. Then, G is isomorphic to L, $L : 2_1$, $L : 2_2$, $L \cdot 2_3$ or $L \cdot 2^2$. But $2 \nsim 5$ and $2 \nsim 29$ in $\Gamma(L)$; also $\Gamma(L \cdot 2_3) = \Gamma(L)$ by [10]. Then $G \ncong L$ and $L \cdot 2_3$. By [3], $C_L(2_2) = \mathrm{PSL}(2,17)$. Therefore $2 \sim 17$ in $\Gamma(P\Sigma L(2,17^2))$; hence $G \ncong L : 2_2$; as a result $G \ncong L \cdot 2^2$. So $G \cong L : 2_1 \cong \mathrm{PGL}(2,17^2)$. □

Acknowledgement. The author expresses her deep gratitude to the referee for valuable, helpful and detailed comments. Also the author would like to thank professor A. Hulpke and professor S. Kohl for their constant support in using GAP.

References

[1] Akhlaghi Z., Khosravi B., Khatami M., *Characterization by prime graph of* $\mathrm{PGL}(2,p^k)$ *where* p *and* $k>1$ *are odd*, Internat. J. Algebra Comput. **20** (2010), no. 7, 847–873.

[2] Aleeva M. R., *On composition factors of finite groups having the same set of element orders as the group* $U_3(q)$, Sibirsk. Mat. Zh. **43** (2002), no. 2, 249–267 (Russian); translation in Siberian Math. J. **43** (2002), no. 2, 195–211.

[3] Aschbacher M., Seitz G. M., *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.

[4] Buturlakin A. A., *Spectra of finite linear and unitary groups*, Algebra Logika **47** (2008), no. 2, 157–173, 264 (Russian); translation in Algebra Logic **47** (2008), no. 2, 91–99.

[5] Conway J. H., Curtis R. T., Norton S. P., Parker R. A., Wilson R. A., *Atlas of Finite Groups*, Clarendon Press (Oxford), London, 1985.

[6] Darafsheh M. R., Farjami Y., Sadrudini A., *A characterization property of the simple group* $\mathrm{PSL}_4(5)$ *by the set of its element orders*, Arch. Math. (Brno) **43** (2007), no. 1, 31–37.

[7] Higman G., *Finite groups in which every element has prime power order*, J. Lond. Math. Soc. **32** (1957), 335–342.

[8] Khatami M., Khosravi B., Akhlaghi Z., *NCF-distinguishability by prime graph of* $PGL(2,p)$*, where* p *is a prime*, Rocky Mountain J. Math. **41** (2011), no. 5, 1523–1545.

[9] Kleidman P. B., Liebeck M. W., *The Subgroup Structure of the Finite Classical Groups*, London Mathematical Society Lecture Note Series, 129, Cambridge University Press, Cambridge, 1990.

[10] Lucido M. S., *Prime graph components of finite almost simple groups*, Rend. Sem. Mat. Univ. Padova **102** (1999), 1–22.

[11] Mahmoudifar A., *On finite groups with the same prime graph as the projective general linear group* $PGL(2,81)$, Transactions on Algebra and Its Applications **2** (2016), 43–49.

[12] Mahmoudifar A., *On the unrecognizability by prime graph for the almost simple group* $\mathrm{PGL}(2,9)$, Discuss. Math. Gen. Algebra Appl. **36** (2016), no. 2, 223–228.

[13] Mahmoudifar A., *Recognition by prime graph of the almost simple group* $\mathrm{PGL}(2,25)$, J. Linear. Topol. Algebra **5** (2016), no. 1, 63–66.

[14] Mazurov V. D., *Characterization of finite groups by sets of orders of their elements*, Algebra i Logika **36** (1997), no. 1, 37–53, 117 (Russian); translation in Algebra and Logic **36** (1997), no. 1, 23–32.

[15] Passman D. S., *Permutation Groups*, W. A. Benjamin, New York, 1968.

[16] Perumal P., *On the Theory of the Frobenius Groups*, Ph.D. Dissertation, University of Kwa-Zulu Natal, Pietermaritzburg, 2012.

[17] Praeger C. E., Shi W. J., *A characterization of some alternating and symmetric groups*, Comm. Algebra **22** (1994), no. 5, 1507–1530.

[18] Sajjadi M., Bibak M., Rezaeezadeh G. R., *Characterization of some projective special linear groups in dimension four by their orders and degree patterns*, Bull. Iranian Math. Soc. **42** (2016), no. 1, 27–36.

[19] Shi W. J., *On simple K4-groups*, Chinese Sci. Bull. **36** (1991), no. 7, 1281–1283 (Chinese).

[20] Simpson W. A., Frame J. S., *The character tables for* $\mathrm{SL}(3,q)$, $\mathrm{SL}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, Canadian. J. Math. **25** (1973), no. 3, 486–494.
[21] Srinivasan B., *The characters of the finite symplectic group* $\mathrm{Sp}(4,q)$, Trans. Am. Math. Soc. **131** (1968), no. 2, 488–525.
[22] Vasil'ev A. V., Grechkoseeva M. A., *On recognition by spectrum of finite simple linear groups over fields of characteristic* 2, Sibirsk. Mat. Zh. **46** (2005), no. 4, 749–758 (Russian); translation in Siberian Math. J. **46** (2005), no. 4, 593–600.
[23] Williams J. S., *Prime graph components of finite groups*, J. Algebra **69** (1981), no. 2, 487–513.
[24] Yang N., Grechkoseeva M. A., Vasil'ev A. V., *On the nilpotency of the solvable radical of a finite group isospectral to a simple group*, J. Group Theory **23** (2020), no. 3, 447–470.
[25] Zavarnitsine A. V., *Finite simple groups with narrow prime spectrum*, Sib. Èlektron. Mat. Izv. **6** (2009), 1–12.
[26] Zavarnitsine A. V., *Fixed points of large prime-order elements in the equicharacteristic action of linear and unitary groups*, Sib. Èlektron. Mat. Izv. **8** (2011), 333–340.
[27] Zavarnitsine A. V., Mazurov V. D., *Element orders in coverings of symmetric and alternating groups*, Algebra Log. **38** (1999), no. 3, 296–315, 378 (Russian); translation in Algebra and Logic **38** (1999), no. 3, 159–170.
[28] *The GAP Group, GAP – Groups, Algorithms, and Programming, Version* 4.11.1, 2021, https://www.gap-system.org.

M. Sajjadi:
Department of Mathematics, Payame Noor University (PNU), Nakhl St, P. O. Box 19395-4697, Tehran, RG23+F4X, Iran

E-mail: m.sajjadi@pnu.ac.ir

(*Received* September 28, 2021, *revised* January 11, 2022)