On sets of discontinuities of functions continuous on all lines

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Abstract. Answering a question asked by K. C. Ciesielski and T. Glatzer in 2013, we construct a C^1-smooth function f on $[0,1]$ and a closed set $M \subset \operatorname{graph} f$ nowhere dense in $\operatorname{graph} f$ such that there does not exist any linearly continuous function on $\mathbb{R}^2$ (i.e., function continuous on all lines) which is discontinuous at each point of M. We substantially use a recent full characterization of sets of discontinuity points of linearly continuous functions on $\mathbb{R}^n$ proved by T. Banakh and O. Maslyuchenko in 2020. As an easy consequence of our result, we prove that the necessary condition for such sets of discontinuities proved by S. G. Slobodnik in 1976 is not sufficient. We also prove an analogue of this Slobodnik's result in separable Banach spaces.

Keywords: linear continuity; discontinuity sets; Banach space

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1. Introduction

Separately continuous functions on $\mathbb{R}^n$ (i.e., functions continuous on all lines parallel to a coordinate axis) and also linearly continuous functions (i.e., functions continuous on all lines) were investigated in a number of articles, see the survey [5]. Note that linearly continuous functions are well-defined in any linear space and recent articles [13] and [1] investigate them also in Banach (and even more general) spaces.

It appears that linearly continuous functions are much "more close to continuous functions" than separately continuous functions. First note that, by Lebesgue's result of [9],

(1.1) each separately continuous function on $\mathbb{R}^n$ belongs to the $(n-1)$th Baire class

(and that the number $n-1$ is optimal, cf. [5]). On the other hand, it was proved independently (answering a question posed in [5]) in [13] and [1] that each linearly continuous function on $\mathbb{R}^n$ belongs to the first Baire class.

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A natural question how small must be members of the families (where $D(f)$ denotes the set of discontinuity points of f)

$$\mathcal{D}_s^n := \{D(f)\colon f \text{ is a separately continuous function on } \mathbb{R}^n\}$$

and

$$\mathcal{D}_l^n := \{D(f)\colon f \text{ is a linearly continuous function on } \mathbb{R}^n\}$$

was considered in several works, see [5]. Clearly, $D_l^n \subset \mathcal{D}_s^n$ and each set from $\mathcal{D}_s^n$ is an F_σ set. It appears that the sets from D_l^n "must be essentially smaller" than those from $\mathcal{D}_s^n$.

The following complete characterization of sets from $\mathcal{D}_s^n$ was given in [7] (and was proved independently by another method in [11]).

Theorem 1.1 (R. Kershner, 1943)**.** *A set $M \subset \mathbb{R}^n$ belongs to $\mathcal{D}_s^n$ if and only if M is an F_σ set and the orthogonal projection of M onto each $(n-1)$-dimensional coordinate hyperplane is a first category (= meager) set.*

This characterization shows that each member of $\mathcal{D}_s^n$ is a first category set, but it can have positive Lebesgue measure (even its complement can be Lebesgue null, cf. [5]). On the other hand (see Remark 1.3 (a) below) all members of D_l^n are Lebesgue null.

Probably the first result concerning the system D_l^n was published in 1910 by W. H. Young and G. C. Young in [12]; they constructed a linearly continuous function on $[0,1]^2$ for which $D(f)$ is uncountable in every nonempty open set.

A. S. Kronrod in 1945, see [10, page 268] and [5, page 28], considered the natural problem to find a complete characterization of sets from the system D_l^2.

As a partial solution of (n-dimensional) Kronrod's problem, S. G. Slobodnik proved Theorem 6 in [11] whose obvious reformulation reads as follows.

Theorem 1.2 (S. G. Slobodnik, 1976)**.** *Let $M \in D_l^n$. Then we can write $M = \bigcup_{k=1}^{\infty} B_k$, where each B_k has the following properties:*

(i) *B_k is a compact subset of a Lipschitz hypersurface L_k.*

(ii) *The orthogonal projection of B_k onto each $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ is nowhere dense in H.*

(iii) *For each $c \in \mathbb{R}^n \setminus B_k$, the set $\{(x-c)/\|x-c\|\colon x \in B_k\}$ is nowhere dense in the unit sphere $S_{\mathbb{R}^n}$.*

Remark 1.3. (a) For the definition of a Lipschitz hypersurface see Definition 2.1. Property (i) clearly implies that $M \subset \mathbb{R}^n$ is Lebesgue null.

(b) The article [10] (written independently of [11]) contains results which are very close to Theorem 1.2 with $n = 2$.

(c) Conditions (i) and (ii) clearly imply that in (i) we can write that B_k is nowhere dense in L_k.

(d) An equivalent reformulation of (iii) (used in [11]) is the following:

(iii)* For each $c \in \mathbb{R}^n \setminus B_k$ and each hyperplane $H \subset \mathbb{R}^n \setminus \{c\}$, the central projection from c of B_k onto H is nowhere dense in H.

Further interesting contributions to Kronrod's problem were proved in [3] and [4]. Main results of [3] read as follows.

Theorem 1.4 (K. Ch. Ciesielski and T. Glatzer, 2012).

(i) *If $n \geq 2$ and $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is convex and $M \subset \operatorname{graph} f$ is nowhere dense in $\operatorname{graph} f$, then there exists a linearly continuous function g on $\mathbb{R}^n$ such that $M \subset D(g)$.*

(ii) *If $f\colon \mathbb{R} \to \mathbb{R}$ is C^2 smooth and $M \subset \operatorname{graph} f$ is nowhere dense in $\operatorname{graph} f$, then there exists a linearly continuous function g on $\mathbb{R}^2$ such that $M \subset D(g)$.*

(iii) *There exists $f\colon \mathbb{R} \to \mathbb{R}$ having bounded derivative and $M \subset \operatorname{graph} f$ which is nowhere dense in $\operatorname{graph} f$ such that there does not exist any linearly continuous function g on $\mathbb{R}^2$ such that $M \subset D(g)$.*

The article [4] contains a full characterization of sets from D_l^2. However, this solution of Kronrod's problem (in $\mathbb{R}^2$) is not quite satisfactory, cf. [5, page 29], since it uses the topology on the set of all lines in $\mathbb{R}^2$ (and its applicability is unclear).

A nice applicable solution of Kronrod's problem in $\mathbb{R}^n$ was proved by T. Banakh and O. Maslyuchenko in [1]. It asserts that a subset of $\mathbb{R}^n$ belongs to D_l^n if and only if it is "$\overline{\sigma}$-l-miserable", see Subsection 2.3 for details. In [1], several applications of this characterization are shown and other two applications are contained in the present article.

In Section 3 we use the Banakh–Maslyuchenko characterization as the main ingredient in the proof of the main result of the present article, Theorem 3.4, which shows, answering the first part of Problem 5.3 from [3], that the function f from Theorem 1.4 (iii) can be even C^1-smooth. More precisely, we use this characterization in the proof of the basic Lemma 3.1. It seems that any proof of Theorem 3.4 based on Lemma 3.1 needs a nontrivial inductive construction. The idea of our construction based on Lemma 2.3 and Lemma 3.2 is not difficult, but the detailed proof is unfortunately rather long and slightly technical.

As an easy but interesting consequence of our Theorem 3.4, we obtain in Section 4 that Slobodnik's necessary condition for sets from D_l^n is not sufficient (which supports the opinion that there exists no characterization of sets from $\mathcal{D}_l^n$ similar to Kershner's characterization of sets from $\mathcal{D}_s^n$).

In Section 5, we prove an analogue of Slobodnik's result in separable Banach spaces for functions having the Baire property (which improves [13, Corollary 4.2]). This result (Proposition 5.1) which easily implies Slobodnik's theorem in $\mathbb{R}^n$ is an easy consequence of [13, Corollary 4.2] and the Banakh–Maslyuchenko characterization.

2. Preliminaries

2.1 Basic notation. In the following, by a Banach space we mean a real Banach space with a norm $\|\cdot\|$. If X is a Banach space, we set $S_X := \{x \in X \colon \|x\| = 1\}$. By $C[0,1]$ and $C^1[0,1]$ we denote the set of all continuous and C^1-smooth functions on $[0,1]$, respectively. If $f \in C[0,1]$, then $\|f\|$ always denotes the supremum norm of f.

The symbol $B(x,r)$ will denote the open ball with center x and radius r. By $\overline{M}$ and $\operatorname{int} M$ we denote the closure and the interior of a set M, respectively. We say that a set Q is an ε-net of a subset A of a metric space, if $Q \subset A \subset \bigcup_{x\in Q} B(x,\varepsilon)$.

The oscillation of a function f on a set M is $\operatorname{osc}(f,M) := \sup\{|f(x) - f(y)|\colon x,y \in M\}$. By $\operatorname{graph} f$, $\operatorname{supp} f$ and $D(f)$, we denote the graph, the closed support $\overline{\{x\colon f(x) \neq 0\}}$ and the set of discontinuity points of f, respectively. We will write $f_n \rightrightarrows f$ if the sequence (f_n) uniformly converges to f. The Lebesgue measure on $\mathbb{R}$ is denoted by λ.

In a metric space X, the system of all sets with the Baire property is the smallest σ-algebra containing all open sets and all first category sets. We say that a function f on X has the Baire property if $f^{-1}(B)$ has the Baire property for all Borel sets $B \subset Y$, see [8, § 32]. We will use the following definition.

Definition 2.1. Let X be a Banach space. We say that $A \subset X$ is a Lipschitz hypersurface if there exists a 1-dimensional linear space $V \subset X$, its topological complement Y and a Lipschitz mapping $\varphi\colon Y \to V$ such that $A = \{y + \varphi(y)\colon y \in Y\}$.

It is easy to see that each Lipschitz hypersurface is a closed set and that, if $X = \mathbb{R}^n$, then we can demand that Y is an orthogonal complement of V.

2.2 Notation and two lemmas concerning tangent lines. If $f \in C^1[0,1]$, $z \in [0,1]$ and $Z \subset [0,1]$, then we use the following notation:

(i) By $A_{f,z}$ we denote the affine function

$$A_{f,z}(x) = f(z) + f'(z)(x - z), \qquad x \in \mathbb{R}.$$

(ii) By $T_{f,z}$ we denote the tangent line to $\operatorname{graph} f$ at the point $(z, f(z))$, i.e.,

$$T_{f,z} := \operatorname{graph}(A_{f,z}).$$

(iii) We set

$$T_{f,Z} := \bigcup_{z\in Z} T_{f,z}.$$

We will need the following easy lemma.

Lemma 2.2. *Let functions f and $f_1, f_2, \dots$ belong to $C^1[0,1]$. Suppose that z and $z_1, z_2, \dots$ belong to $[0,1]$, $(x,y) \in \mathbb{R}^2$ and*

$$f_n \rightrightarrows f, \qquad f_n' \rightrightarrows f', \qquad z_n \to z.$$

Then the conditions $(x,y) \in T_{f_n,z_n}$, $n = 1, 2, \dots$, imply $(x,y) \in T_{f,z}$.

Proof: By the assumptions, we have $f_n(z_n) + f_n'(z_n)(x - z_n) = y$, $n \in \mathbb{N}$. Since $f_n(z_n) \to f(z)$ and $f_n'(z_n) \to f'(z)$, see, e.g., [6, Theorem 7.5, page 268], we obtain $f(z) + f'(z)(x - z) = y$. □

The following lemma is also rather easy but is an important ingredient in our proof of Theorem 3.4.

Lemma 2.3. *Suppose that $f \in C^1[0,1]$ and f' has infinite variation on an interval $[\alpha, \beta] \subset [0,1]$. Then there exist numbers e, w such that $\alpha < e < w < \beta$ and $(w, f(w)) \in T_{f,e}$.*

Proof: First observe that there exist numbers e_0, w_0, e_1, w_1 such that $\alpha < e_0 < w_0 < \beta$, $\alpha < e_1 < w_1 < \beta$ and

$$f(w_0) < f(e_0) + f'(e_0)(w_0 - e_0), \qquad f(w_1) > f(e_1) + f'(e_1)(w_1 - e_1). \tag{2.1}$$

To construct e_0 and w_0, note that f' is not nondecreasing on (α, β) and thus we can choose numbers $\alpha < e^* < w_0 < \beta$ with $f'(e^*) > f'(w_0)$. Set $e_0 := \max\{x \in [e^*, w_0] \colon f'(x) = f'(e^*)\}$. Then we have

$$f(w_0) = f(e_0) + \int_{e_0}^{w_0} f' < f(e_0) + f'(e_0)(w_0 - e_0)$$

and so e_0 and w_0 satisfy (2.1). The existence of e_1 and w_1 follows quite analogously.

Now set

$$e(t) := te_1 + (1-t)e_0, \quad w(t) := tw_1 + (1-t)w_0, \qquad t \in [0,1].$$

Then clearly $e(0) = e_0$, $w(0) = w_0$, $e(1) = e_1$, $w(1) = w_1$ and $e(t) < w(t)$, $t \in [0,1]$. The function $g(t) := f(w(t)) - f(e(t)) - f'(e(t))(w(t) - e(t))$, $t \in [0,1]$, is clearly continuous, $g(0) < 0$ and $g(1) > 0$. Consequently there exists $t^* \in (0,1)$ such that $g(t^*) = 0$ and so $e := e(t^*)$ and $w := w(t^*)$ have the required property. □

2.3 Banakh–Maslyuchenko characterization. The authors of [1] work in "Baire cosmic vector spaces" but we work in the present article in the more special context of separable Banach spaces; so we present basic definitions from [1] in Banach spaces only.

Definition 2.4. Let X be a Banach space and $A \subset X$.

(i) A set $V \subset X$ is called an l-neighborhood of A if for any $a \in A$ and $v \in X$ there exists $\varepsilon > 0$ such that $a + [0, \varepsilon) \cdot v \subset V$.

(ii) The set A is called l-miserable if $A \subset \overline{X \setminus L}$ for some closed l-neighborhood L of A.

(iii) The set A is called $\overline{\sigma}$-l-miserable if A is a countable union of closed l-miserable sets.

An immediate consequence of [1, Theorem 1.5.] is the following result.

Theorem 2.5 (T. Banakh and O. Maslyuchenko)**.** *Let X be a separable Banach space and $M \subset X$. Then the following conditions are equivalent.*

(i) *$M = D(f)$ for some linearly continuous function f on X which has the Baire property.*

(ii) *M is $\overline{\sigma}$-l-miserable.*

Note that if $X = \mathbb{R}^n$, then each linearly continuous function on X has the Baire property by (1.1) and so Theorem 2.5 gives a full characterization of sets of discontinuities of linearly continuous functions on $\mathbb{R}^n$.

3. Main result

In this section we prove our main Theorem 3.4 using the following basic lemma whose rather easy proof is based on the Banakh–Maslyuchenko characterization, Theorem 2.5.

Lemma 3.1. *Let $f \in C^1[0,1]$, let $\emptyset \neq P \subset (0,1)$ be a perfect nowhere dense set and let $D \subset P$ be a countable dense subset of P. Let, for each $d \in D$, two points $u_d, v_d \in (0,1)$ be given such that $u_d < v_d < d$,*

$$(d, f(d)) \in \operatorname{int} T_{f,[u_d,v_d]\cap P} \tag{3.1}$$

and

$$\text{the set } D_\varepsilon := \{d \in D\colon\ d - u_d > \varepsilon\} \text{ is finite for each } \varepsilon > 0. \tag{3.2}$$

Then there does not exist any linearly continuous function g on $\mathbb{R}^2$ which is discontinuous at each point of the set $\operatorname{graph}(f|_P)$.

Proof: Suppose to the contrary that such a function g exists. By Theorem 2.5 there exists a $\overline{\sigma}$-l-miserable set $A \subset \mathbb{R}^2$ such that $\operatorname{graph}(f|_P) \subset A$. Let $A_1, A_2, \dots$ be closed l-miserable subsets of $\mathbb{R}^2$ such that $A = \bigcup_{n=1}^{\infty} A_n$. Since the set $\operatorname{graph}(f|_P)$ is closed in $\mathbb{R}^2$, by the Baire theorem there exists $k \in \mathbb{N}$ such that the closed set $A_k \cap \operatorname{graph}(f|_P)$ is not nowhere dense in $\operatorname{graph}(f|_P)$ and so there exists an interval $(a,b) \subset [0,1]$ such that $P \cap (a,b) \neq \emptyset$ and $\operatorname{graph}(f|_{P\cap(a,b)}) \subset A_k$. Since A_k is l-miserable, we can choose a closed l-neighbourhood L of A_k such that $A_k \subset \overline{H}$, where $H := \mathbb{R}^2 \setminus L$. Further choose $K > 0$ such that $\|f'\| \leq K$.

Now we will construct inductively a sequence of intervals $[a_n, b_n]$, $n = 0, 1, \dots$, such that for each $n \geq 0$ the following two conditions hold:

(C1) $[a_n, b_n] \subset (a,b)$, $(a_n, b_n) \cap P \neq \emptyset$ and $n(b_n - a_n) < 1$.

(C2) If $n \geq 1$, then $[a_n, b_n] \subset (a_{n-1}, b_{n-1})$ and for each $x \in [a_n, b_n]$ there exists a point $z_x^n \in T_{f,x} \cap H$ such that $\|z_x^n - (x, f(x))\| < 3(K+1)/n$.

We can clearly choose $[a_0, b_0]$ such that condition (C1) holds for $n = 0$.

Further suppose that $n \geq 1$ and we have defined $[a_{n-1}, b_{n-1}]$ such that condition (C1) holds for $n-1$ instead of n. Choose $p_n \in P \cap (a_{n-1}, b_{n-1})$ and $\delta_n > 0$ such that $[p_n - \delta_n, p_n + \delta_n] \subset (a_{n-1}, b_{n-1})$. Since P is perfect, the set $D \cap (p_n - \delta_n, p_n + \delta_n)$ is infinite and so by (3.2) we can choose $d_n \in D \cap (p_n - \delta_n, p_n + \delta_n)$ such that $[u_{d_n}, v_{d_n}] \subset (a_{n-1}, b_{n-1})$ and $d_n - u_{d_n} < 1/n$.

We know that $(d_n, f(d_n)) \in \operatorname{graph} f|_{P\cap(a,b)} \subset A_k \subset \overline{H}$. Consequently, since $(d_n, f(d_n)) \in \operatorname{int} T_{f,[u_{d_n}, v_{d_n}]\cap P}$ by (3.1) and H is open, we can choose an open set $W \neq \emptyset$ such that

$$W \subset B((d_n, f(d_n)), 1/n) \cap H \cap T_{f,[u_{d_n}, v_{d_n}]\cap P}. \tag{3.3}$$

Consequently we can choose $x_n \in [u_{d_n}, v_{d_n}] \cap P$ for which $T_{f,x_n} \cap W \neq \emptyset$. Since $f \in C^1[0,1]$ and W is open, it is easy to see that there exists an open neighbourhood (a_n, b_n) of x_n such that $[a_n, b_n] \subset (a_{n-1}, b_{n-1})$, $b_n - a_n < 1/n$ and $T_{f,x} \cap W \neq \emptyset$ for each $x \in [a_n, b_n]$. So we can choose for each $x \in [a_n, b_n]$ a point $z_x^n = (\alpha_x^n, \beta_x^n) \in T_{f,x_n} \cap W$. Obviously (C1) holds and $[a_n, b_n] \subset (a_{n-1}, b_{n-1})$ and thus it is sufficient to check that $\|z_x^n - (x, f(x))\| < 3(K+1)/n$. To this end first observe that $|x - \alpha_x^n| < 3/n$ since it is easy to see that $|d_n - x_n| < 1/n$, $|x_n - x| < 1/n$, and (3.3) with $z_x^n \in W$ imply $|d_n - \alpha_x^n| < 1/n$. Since the absolute value of the slope of the tangent $T_{f,x}$ is at most K, we obtain that

$$\|z_x^n - (x, f(x))\| \leq \sqrt{|x - \alpha_x^n|^2 + (K|x - \alpha_x^n|)^2} \leq (1+K)|x - \alpha_x^n| < \frac{3(K+1)}{n}.$$

So we have finished our inductive costruction.

Now observe that the closedness of P and condition (C1) imply that $\{p\} = \bigcap_{n=0}^{\infty} [a_n, b_n]$ for some $p \in P \cap (a,b)$. Applying (C2) to p for each $n \geq 1$ we obtain points $z_p^n \in T_{f,p} \cap H$, $n \geq 1$, such that $z_p^n \to (p, f(p)) \in A_k$. Consequently $L = \mathbb{R}^2 \setminus H$ is not an l-neigbourhood of A_k, which is a contradiction. □

We will need also the following technical lemma.

Lemma 3.2. *Let G and $\widetilde{G}$ be functions from $C^1[0,1]$ and $g := G'$, $\tilde{g} := (\widetilde{G})'$. Let numbers $0 < u < z < v < x < 1$, $y \in \mathbb{R}$ and $\varepsilon > 0$, $\delta > 0$, $\eta > 0$ have the following properties:*

$$v + \delta + \eta < x, \quad v - u < \eta \quad \text{and} \quad 6\eta < \varepsilon\,\delta, \tag{3.4}$$

$$(x, y) \in T_{G,z}, \tag{3.5}$$

$$\|G - \widetilde{G}\| \leq \eta, \tag{3.6}$$

$$\|g\| \leq 1 \quad \textit{and} \quad \operatorname{osc}(g, [u, v]) \leq \eta, \tag{3.7}$$

$$\begin{gathered} \textit{there exist } u < s_1 < s_2 < v \textit{ such that } \tilde{g}(s_1) = g(s_1) - \varepsilon \\ \textit{and } \tilde{g}(s_2) = g(s_2) + \varepsilon. \end{gathered} \tag{3.8}$$

Then we have

$$B((x, y), \eta) \subset T_{\widetilde{G},(u,v)}. \tag{3.9}$$

Proof: Consider an arbitrary point $(\overline{x}, \overline{y}) \in B((x, y), \eta)$. Then $x - \eta < \overline{x} < x + \eta$ and $y - \eta < \overline{y} < y + \eta$. Let s_1, s_2 be as in (3.8). Using (3.4), we obtain

$$\overline{x} - s_1 > x - \eta - v > \delta. \tag{3.10}$$

Now set

$$h(s) := A_{\widetilde{G},s}(\overline{x}) = \widetilde{G}(s) + \tilde{g}(s) \cdot (\overline{x} - s), \qquad s \in [s_1, s_2].$$

It is sufficient to prove that

$$h(s_1) \leq y - \eta \qquad \text{and} \qquad h(s_2) \geq y + \eta. \tag{3.11}$$

Indeed, (3.11) and the continuity of the function h imply that there exists $\overline{s} \in (s_1, s_2)$ such that $h(\overline{s}) = \overline{y}$; consequently $(\overline{x}, \overline{y}) \in T_{\widetilde{G},(u,v)}$ and (3.9) follows.

Recall that $h(s_1) = \widetilde{G}(s_1) + \tilde{g}(s_1) \cdot (\overline{x} - s_1)$, $h(s_2) = \widetilde{G}(s_2) + \tilde{g}(s_2) \cdot (\overline{x} - s_2)$. By (3.5),

$$\begin{aligned} y &= G(z) + g(z)(x - z) = G(z) + g(s_1)(x - z) + (g(z) - g(s_1))(x - z) \\ &= G(z) + g(s_1)((\overline{x} - s_1) + (x - \overline{x}) + (s_1 - z)) + (g(z) - g(s_1))(x - z). \end{aligned}$$

Using these equalities, (3.4), (3.6), (3.7), (3.10), the choice of s_1 and the inequality $|G(z) - G(s_1)| < \eta$ which follows from (3.7) and $|z - s_1| < \eta$ by the mean value theorem, we obtain

$$\begin{aligned} y - h(s_1) = &\ (G(z) - G(s_1)) + (G(s_1) - \widetilde{G}(s_1)) + (g(s_1) - \tilde{g}(s_1))(\overline{x} - s_1) \\ &+ g(s_1)(x - \overline{x}) + g(s_1)(s_1 - z) + (g(z) - g(s_1))(x - z) \\ \geq &\ -\eta - \eta + \varepsilon\delta - \eta - \eta - \eta \geq \eta. \end{aligned}$$

Quite analogously we obtain $y - h(s_2) \leq -\eta$ and (3.11) follows. □

The proof of our main Theorem 3.4 is based on the construction of $f \in C^1[0,1]$, $P \subset [0,1]$ (and also D and u_d, v_d) which satisfy assumptions of Lemma 3.1. We will set $f := \lim_{n\to\infty} f_n$ and $P := \bigcap_{n=1}^{\infty} P_n$, where (f_n) and (P_n) are defined by a nontrivial inductive construction, in which each P_n is a finite union of compact intervals. For sets P_n we will use the following notation.

Definition 3.3. If $\emptyset \neq P \subset \mathbb{R}$ is a finite union of nondegenerate compact intervals,

(i) we denote by $\mathcal{C}(P)$ the set of all components of P, and set
(ii) $R(P) := \{d \in \mathbb{R} \colon d$ is a right endpoint of some $I \in \mathcal{C}(P)\}$,
(iii) $\nu(P) := \max\{\lambda(I) \colon I \in \mathcal{C}(P)\}$.

Theorem 3.4. *There exist $f \in C^1[0,1]$ and a closed set $M \subset \operatorname{graph} f$ which is nowhere dense in $\operatorname{graph} f$ such that there does not exist any linearly continuous g on $\mathbb{R}^2$ which is discontinuous at each point of M.*

Proof: We will define sequences $(\eta_k)_{k=1}^{\infty}$, $(P_k)_{k=1}^{\infty}$, $(f_k)_{k=1}^{\infty}$ such that the following seven conditions hold for each $k \in \mathbb{N}$:

(3.12) $$\eta_1 = 1 \quad \text{and} \quad 0 < \eta_k < \frac{\eta_{k-1}}{2}, \qquad k \geq 2.$$

(3.13) $\emptyset \neq P_k \subset (0,1)$ is a finite union of nondegenerate compact intervals.

(3.14) $$\nu(P_k) \leq \frac{1}{k}, \quad P_k \subset P_{k-1} \quad \text{and} \quad R(P_{k-1}) \subset R(P_k) \qquad \text{if } k \geq 2.$$

(3.15) Each $I \in \mathcal{C}(P_{k-1})$ contains at least two elements of $\mathcal{C}(P_k)$ if $k \geq 2$.

(3.16) $f_k \in C^1[0,1]$ and f_k' has infinite variation on each interval $[\alpha, \beta] \subset [0,1]$.

(3.17) $$f_k(0) = 0 \quad \text{and} \quad \|f_1'\| \leq \frac{1}{2}.$$

(3.18) $$\|f_k - f_{k-1}\| < \frac{\eta_k}{2} \quad \text{and} \quad \|f_k' - f_{k-1}'\| = 2^{-k} \qquad \text{if } k \geq 2.$$

Moreover, for each $k \in \mathbb{N}$ and each point d from the set R_k^*, where

(3.19) $$R_1^* := R(P_1) \quad \text{and} \quad R_k^* := R(P_k) \setminus R(P_{k-1}) \qquad \text{for } k \geq 2,$$

we will define an interval $[u_d, v_d]$ such that the following two conditions hold:

$$0 < u_d < v_d < d, \quad [u_d, d] \subset P_k \quad \text{and} \quad 3\eta_{k+1} < d - v_d. \tag{3.20}$$

$$\text{If } k < l, \ l \in \mathbb{N}, \text{ then } T_{f_l,(u_d,v_d)\cap \operatorname{int} P_l} \supset B((d, f_k(d)), \eta_{k+1}). \tag{3.21}$$

In the formulation of (3.20) and (3.21) we have used that $R_1^*, R_2^*, \dots$ are pairwise disjoint by (3.14) and so k is uniquely determined by d.

In our inductive construction we will have defined, after the nth step, $n \in \mathbb{N}$, of the construction, the numbers $\eta_1, \dots, \eta_n$, the sets $P_1, \dots, P_n$, the functions $f_1, \dots, f_n$ and, if $n \geq 2$ for each $1 \leq k \leq n-1$ and $d \in R_k^*$, see (3.19), we will have defined an interval $[u_d, v_d]$, such that

$$\text{seven conditions (3.12)–(3.18) hold whenever } 1 \leq k \leq n, \tag{3.22}$$

$$\begin{gathered}\text{conditions (3.20) and (3.21) hold} \\ \text{whenever } 1 \leq k \leq n-1, \ d \in R_k^* \text{ and } l \leq n\end{gathered} \tag{3.23}$$

and

$$\begin{gathered}\text{for any point } d \in R_n^* \text{ there exists } 0 < e < d \\ \text{such that } [e, d] \subset P_n, \ e \in \operatorname{int} P_n \text{ and } (d, f_n(d)) \in T_{f_n,e}.\end{gathered} \tag{3.24}$$

The first step. We set $\eta_1 := 1$. Choose (using e.g. [2, Corollary 2.2, page 143]) a nowhere differentiable function $g \in C[0,1]$ with $\|g\| \leq 1/2$ and set $f_1(x) := \int_0^x g, \ x \in [0,1]$. Then $f_1' = g$ has infinite variation on each interval $[\alpha, \beta] \subset [0,1]$. Using Lemma 2.3 with $f := f_1$, $\alpha := 0$ and $\beta := 1$, we can choose $0 < e < w < 1$ such that $(w, f_1(w)) \in T_{f_1,e}$ and set $P_1 := [e/2, w]$. It is easy to check that conditions (3.22), (3.23) and (3.24) hold for $n = 1$.

The inductive step. We suppose that $m \geq 2$ and the $(m-1)$th step of the construction was accomplished. In particular, we know that conditions (3.22), (3.23) and (3.24) hold for $n = m - 1$.

Our aim is to construct η_m, P_m, f_m and an interval $[u_d, v_d]$ for each $d \in R_{m-1}^*$ such that (3.22), (3.23) and (3.24) hold for $n = m$.

First we choose, by the validity of (3.24) for $n = m-1$, for each $d \in R_{m-1}^*$ a point $e =: z_d$ such that

$$0 < z_d < d, \ \ [z_d, d] \subset P_{m-1}, \ \ z_d \in \operatorname{int} P_{m-1} \ \text{ and } \ (d, f_{m-1}(d)) \in T_{f_{m-1},z_d} \tag{3.25}$$

and set

$$Z_1^m := \{z_d \colon d \in R_{m-1}^*\}.$$

Further choose $\delta_m > 0$ so small, that

$$\delta_m < \frac{d - z_d}{3} \text{ for each } d \in R^*_{m-1} \text{ and } \delta_m < \min\left(\frac{1}{2m}, \eta_{m-1}\right). \tag{3.26}$$

Now we set $Z_2^m := \emptyset$ if $m = 2$ and, if $m \geq 3$, we define Z_2^m as follows. In this case $R^*_k \neq \emptyset$ for each $1 \leq k \leq m-2$, see (3.22) and (3.15), and for such k and $d \in R^*_k$ we have defined an interval (u_d, v_d) such that, by the validity of (3.23) for $n = m-1$, (3.20) holds and

$$T_{f_{m-1},(u_d,v_d)\cap \operatorname{int} P_{m-1}} \supset B((d, f_k(d)), \eta_{k+1}). \tag{3.27}$$

Choose $\eta_m > 0$ so small that

$$6\eta_m < 2^{-m}\delta_m, \quad \eta_m < \frac{\eta_{m-1}}{2} \quad \text{and} \tag{3.28}$$

$$6\eta_m < 2^{-m}\frac{d - v_d}{3}, \text{ whenever } 1 \leq k \leq m-2 \text{ and } d \in R^*_k. \tag{3.29}$$

Further, for every fixed $1 \leq k \leq m-2$ and $d \in R^*_k$, we choose a finite η_m-net Q_d of the ball $B((d, f_k(d)), \eta_{k+1})$ and for each $q \in Q_d$ choose by (3.27) a point $z_{q,d} \in (u_d, v_d) \cap \operatorname{int} P_{m-1}$ such that $q \in T_{f_{m-1}, z_{q,d}}$. Now we define $Z_2^m := \{z_{q,d}\colon 1 \leq k \leq m-2,\ d \in R^*_k,\ q \in Q_d\}$.

(Note that, as above, k is uniquely determined by d; however our construction allows cases when $(q_1, d_1) \neq (q_2, d_2)$ and $z_{q_1,d_1} = z_{q_2,d_2}$.)

Further choose Z_3^m as an arbitrary finite set $Z_3^m \subset \operatorname{int} P_{m-1}$ such that

$$Z_3^m \cap \operatorname{int} I \neq \emptyset \qquad \text{for each } I \in \mathcal{C}(P_{m-1}) \tag{3.30}$$

and set $Z^m := Z_1^m \cup Z_2^m \cup Z_3^m$.

Choose $0 < \delta^*_m < \delta_m$ so small that

$$[z_{q,d} - \delta^*_m, z_{q,d} + \delta^*_m] \subset (u_d, v_d) \quad \text{whenever } 1 \leq k \leq m-2,\ d \in R^*_k \text{ and } q \in Q_d, \tag{3.31}$$

$$\text{the intervals } \{[z - \delta^*_m, z + \delta^*_m]\colon z \in Z^m\} \text{ are pairwise disjoint subsets of } \operatorname{int} P_{m-1}, \tag{3.32}$$

$$\lambda\Bigg(\bigcup_{z \in Z^m} [z - \delta^*_m, z + \delta^*_m]\Bigg) < \frac{\eta_m}{2}, \quad \text{and} \tag{3.33}$$

$$\operatorname{osc}(f'_{m-1}, [z - \delta^*_m, z + \delta^*_m]) \leq \eta_m \qquad \text{for each } z \in Z^m. \tag{3.34}$$

Further choose a piecewise linear function $h_m \in C[0,1]$ such that

(3.35) $$\|h_m\| = 2^{-m}, \quad \operatorname{supp} h_m \subset \bigcup_{z \in Z^m} [z - \delta_m^*, z + \delta_m^*] \quad \text{and}$$

(3.36) $$\text{for every } z \in Z^m \text{ there exist } z - \delta_m^* < s_1^z < s_2^z < z \text{ with } h_m(s_1^z) = -2^{-m},\ h_m(s_2^z) = 2^{-m}.$$

Now we define f_m by

(3.37) $$f_m(x) = f_{m-1}(x) + \int_0^x h_m, \qquad x \in [0,1].$$

Then clearly $f_m \in C^1[0,1]$ and, using (3.16) for $k = m-1$, it is easy to see that $f_m' = f_{m-1}' + h_m$ has infinite variation on each interval $[\alpha, \beta] \subset [0,1]$.

So by Lemma 2.3 we can find for each $z \in Z^m$ points $z < e_z < w_z < z + \delta_m^*$ such that

(3.38) $$(w_z, f_m(w_z)) \in T_{f_m, e_z}.$$

For each $d \in R_{m-1}^*$, set

(3.39) $$u_d := z_d - \delta_m^*, \qquad v_d := w_{z_d}.$$

To define P_m, assign to each $d \in R(P_{m-1})$ a point $c_d < d$ such that $c_d \in \operatorname{int} P_{m-1}$, $[c_d, d] \subset P_{m-1}$, $d - c_d < 1/m$ and $[c_d, d] \cap \bigcup_{z \in Z^m} [z - \delta_m^*, z + \delta_m^*] = \emptyset$, and define

(3.40) $$P_m := \bigcup_{z \in Z^m} [z - \delta_m^*, w_z] \cup \bigcup_{d \in R(P_{m-1})} [c_d, d].$$

Thus we have constructed η_m, f_m, P_m, and an interval $[u_d, v_d]$ for each $d \in R_{m-1}^*$.

Our aim is now to prove that properties (3.22), (3.23) and (3.24) hold for $n = m$.

First note that, by the above construction,

(3.41) (3.40) gives the decomposition of P_m into its components.

So, using (3.26), $\delta_m^* < \delta_m$ and $d - c_d < 1/m$ ($d \in R(P_{m-1})$), we obtain

(3.42) $$\nu(P_m) \le 1/m.$$

To prove that (3.22) holds for $n = m$, it is sufficient (since we know that (3.22) holds for $n = m-1$) to verify that conditions (3.12)–(3.18) hold for $k = m$. These facts easily follow from the construction:

Conditions (3.12) and (3.13) follow from (3.28) and (3.40), respectively. Condition (3.14) follows from (3.42), (3.41) and (3.32). Condition (3.15) follows from (3.41) and (3.30). Condition (3.16) is stated just after (3.37). Condition (3.17) follows from (3.37) and the validity of (3.17) for $k = m - 1$.

To prove (3.18), observe that by (3.37) and (3.35) we have $\|f'_m - f'_{m-1}\| = \|h_m\| = 2^{-m}$ and, using also (3.35) and (3.33), we obtain

$$|(f_m - f_{m-1})(x)| = \left| \int_0^x h_m \right| \le \left| \int_{\operatorname{supp} h_m} |h_m| \right| < \frac{\eta_m}{2}, \qquad x \in [0,1],$$

and so $\|f_m - f_{m-1}\| < \eta_m/2$.

Now we will show that (3.23) holds for $n = m$. Since we know that (3.23) holds for $n = m - 1$, it is sufficient to verify that

(3.43) $$\text{(3.20) holds if } k = m-1 \text{ and } d \in R^*_k$$

and

(3.44) $$\text{(3.21) holds if } 1 \le k \le m-1,\ l = m \text{ and } d \in R^*_k.$$

To prove (3.43), consider an arbitrary $d \in R^*_{m-1}$ and recall that $z_d \in Z^m_1 \subset Z^m$ and $u_d = z_d - \delta^*_m$, $v_d = w_{z_d}$, see (3.39). By (3.25) we have $[z_d, d] \subset P_{m-1}$ and so (3.32) and $z_d < w_{z_d} < z_d + \delta^*_m$ imply $0 < u_d < v_d < d$ and $[u_d, d] \subset P_{m-1}$. To prove $3\eta_m < d - v_d$, observe that (3.26) gives $d - z_d > 3\delta_m$ and so, using also $\delta^*_m < \delta_m$ and (3.28), we obtain

$$d - v_d > d - z_d - \delta^*_m > 3\delta_m - \delta_m > 3\eta_m.$$

So (3.43) is proved.

To prove (3.44), we will distinguish cases **a)** $k = m - 1$ and **b)** $1 \le k < m - 1$.

a) Consider an arbitrary $d \in R^*_{m-1}$. Then $z_d \in Z^m_1$ and $u_d = z_d - \delta^*_m$, $v_d = w_{z_d}$. By (3.25) we have

(3.45) $$(d, f_{m-1}(d)) \in T_{f_{m-1}, z_d}.$$

Now we will show that the assumptions of Lemma 3.2 are satisfied for

$$G = f_{m-1}, \qquad \widetilde{G} = f_m, \qquad u = u_d, \qquad z = z_d, \qquad v = v_d,$$
$$x = d, \qquad y = f_{m-1}(d), \qquad \varepsilon = 2^{-m}, \qquad \delta = \delta_m, \qquad \eta = \eta_m.$$

First we show that inequalities (3.4) hold:

Using (3.28) and (3.26) we obtain

$$v_d + \delta_m + \eta_m < (z_d + \delta^*_m) + \delta_m + \frac{\delta_m}{6} < z_d + 3\delta_m < d.$$

Further we obtain $v_d - u_d < 2\delta^*_m < \eta_m$ by (3.33) and $6\eta_m < 2^{-m}\delta_m$ by (3.28).

Condition (3.5) coincides with (3.45) and (3.6) holds since we know that (3.18) is valid for $k = m$. Condition (3.7) follows from (3.34) and the validity of (3.17) and (3.18) for each $k \le m-1$. Finally, condition (3.8) follows from (3.36) since $\tilde{g} - g = h_m$.

Consequently conclusion (3.9) of Lemma 3.2 holds, i.e., $B((d, f_{m-1}(d)), \eta_m) \subset T_{f_m,(u_d,v_d)}$. Since $(u_d, v_d) \subset \operatorname{int} P_m$ by (3.39) and (3.40), we obtain that (3.21) holds for $k = m-1$, $l = m$ and our d.

b) Consider arbitrary $1 \le k < m-1$ and $d \in R_k^*$. Then we have defined a finite η_m-net Q_d of the ball $B((d, f_k(d)), \eta_{k+1})$ and for each $q \in Q_d$ we have defined a point $z_{q,d} \in (u_d, v_d) \cap \operatorname{int} P_{m-1}$ such that $q \in T_{f_{m-1}, z_{q,d}}$.

Now we will show that, for an arbitrary $q =: (x_q, y_q) \in Q_d$, the assumptions of Lemma 3.2 are satisfied for

$$G = f_{m-1}, \qquad \widetilde{G} = f_m, \qquad u = z_{q,d} - \delta_m^*, \qquad z = z_{q,d}, \qquad v = w_{z_{q,d}},$$
$$x = x_q, \qquad y = y_q, \qquad \varepsilon = 2^{-m}, \qquad \delta = \frac{d - v_d}{3}, \qquad \eta = \eta_m.$$

First we show that inequalities (3.4) hold:

Using (3.31), (3.29) and (3.20) we obtain

$$\begin{aligned} v + \delta + \eta = w_{z_{q,d}} + \frac{d - v_d}{3} + \eta_m < v_d + \frac{d - v_d}{3} + \frac{d - v_d}{3} \\ = d - \frac{d - v_d}{3} < d - \eta_{k+1} < x_q = x. \end{aligned}$$

Further we obtain the inequalities $v - u = w_{z_{q,d}} - (z_{q,d} - \delta_m^*) < 2\delta_m^* < \eta_m = \eta$ by (3.33) and $6\eta = 6\eta_m < 2^{-m}(d - v_d)/3 = \varepsilon\delta$ by (3.29).

Condition (3.5) holds since $q \in T_{f_{m-1}, z_{q,d}}$ and (3.6) holds since we know that (3.18) is valid for $k = m$. Condition (3.7) follows from (3.34) and the validity of (3.17) and (3.18) for each $k \le m-1$. Finally, condition (3.8) follows from (3.36) since $\tilde{g} - g = h_m$.

Thus assertion (3.9) of Lemma 3.2 holds, i.e., $B(q, \eta_m) \subset T_{f_m,(z_{q,d} - \delta_m^*, w_{z_{q,d}})}$. Note that $z_{q,d} \in Z_2^m \subset Z^m$ and so $(z_{q,d} - \delta_m^*, w_{z_{q,d}}) \subset (u_d, v_d) \cap \operatorname{int} P_m$ by (3.31) and (3.40).

Since Q_d is a η_m-net of $B((d, f_k(d)), \eta_{k+1})$ we obtain that (3.21) holds for $l = m$ and our k and d.

So we have proved (3.44). It remains to prove that (3.24) holds for $n = m$. So consider an arbitrary $d \in R_m^*$. By (3.41) we obtain that there exists $\tilde{z} \in Z^m$ such that $d = w_{\tilde{z}}$ and (3.38) shows that (3.24) holds for $n = m$ (since the choice $e := e_{\tilde{z}}$ works for our d).

So we have finished our inductive construction. It is easy to see that we have defined the sequences $(\eta_k)_{k=1}^\infty$, $(P_k)_{k=1}^\infty$, $(f_k)_{k=1}^\infty$ and intervals $[u_d, v_d]$, whenever

$d \in R_k^*$, $k = 1, 2, \dots$, such that all nine properties (3.12)–(3.18), (3.20) and (3.21) hold for each $k \in \mathbb{N}$.

Using these properties only, we will show that $f := \lim_{n\to\infty} f_n$, $P := \bigcap_{n=1}^{\infty} P_n$ and $D := \bigcup_{k=1}^{\infty} R_k^* = \bigcup_{k=1}^{\infty} R(P_k)$ satisfy the assumptions of Lemma 3.1.

By (3.18) we obtain that $\|f'_m - f'_l\| \le 2^{-l}$ whenever $1 \le l < m$ and therefore the sequence (f'_k) uniformly converges to a function $\varphi \in C[0,1]$. Since $f_k(x) = \int_0^x f'_k$, $x \in [0,1]$, by (3.17), we obtain $f_k \rightrightarrows f$, where $f(x) := \int_0^x \varphi$, $x \in [0,1]$. Clearly $f \in C^1[0,1]$ and $f'_k \rightrightarrows f' = \varphi$.

By (3.13) and (3.14), $P = \bigcap_{n=1}^{\infty} P_n$ is a nonempty closed set and (3.14) with (3.15) easily imply that P is perfect and nowhere dense.

By (3.14) we easily obtain that the countable set $D \subset P$ is dense in P.

Now consider an arbitrary $d \in D$. Then there exists $k \in \mathbb{N}$ such that $d \in R_k^*$ and so we have defined $u_d, v_d \in (0,1)$ such that $u_d < v_d < d$ and (3.20) and (3.21) hold. Now consider an arbitrary point $(x,y) \in B((d, f_k(d)), \eta_{k+1})$. By (3.21) we can choose, for each $l > k$, a point $p_l \in [u_d, v_d] \cap P_l$ such that $(x,y) \in T_{f_l, p_l}$. Choose a convergent subsequence $(p_{l_i})_{i=1}^{\infty}$ of $(p_l)_{l=k+1}^{\infty}$ with $p_{l_i} \to p$. Then $p \in P \cap [u_d, v_d]$ and Lemma 2.2 gives $(x,y) \in T_{f,p}$. Thus we have proved that

$$B((d, f_k(d)), \eta_{k+1}) \subset T_{f, P\cap[u_d,v_d]}. \tag{3.46}$$

Using (3.18) and (3.12), we obtain

$$\begin{aligned} |f(d) - f_k(d)| &\le \|f_{k+1} - f_k\| + \|f_{k+2} - f_{k+1}\| + \dots \\ &< \frac{\eta_{k+1}}{2} + \frac{\eta_{k+2}}{2} + \frac{\eta_{k+3}}{2} + \dots \\ &\le \frac{1}{2}\Big(\eta_{k+1} + \frac{\eta_{k+1}}{2} + \frac{\eta_{k+1}}{4} + \dots\Big) = \eta_{k+1}. \end{aligned}$$

So (3.46) gives

$$(d, f(d)) \in B((d, f_k(d)), \eta_{k+1}) \subset \operatorname{int} T_{f, P\cap[u_d,v_d]}$$

and (3.1) is proved.

Finally, condition (3.2) holds since each R_k^* is finite and $d - u_d < 1/k$ for $d \in R_k^*$ by (3.14) and (3.20).

So the assertion of our theorem holds for $M := \operatorname{graph} f|_P$ by Lemma 3.1. □

4. Slobodnik's necessary condition is not sufficient

As an easy consequence of Theorem 3.4, we will prove the following result which shows that Slobodnik's necessary condition (for sets from $\mathcal{D}_l^n$) is not suficient.

Proposition 4.1. *There exists a set $M \subset \mathbb{R}^2$ such that $M = \bigcup_{k=1}^{\infty} B_k$, where B_k have properties* (i), (ii) *and* (iii) *from Theorem* 1.2 *for $n = 2$, but $M \notin \mathcal{D}_l^2$.*

Proof: Let M and f be as in Theorem 3.4. So we have $M \notin \mathcal{D}_l^2$ and we have $M = \{(x, f(x)) \colon x \in P\}$ where $P \subset [0,1]$ is nowhere dense. Set $B_k := M,\ k \in \mathbb{N}$; we will show that then properties (i), (ii), (iii) hold.

The property (i) is almost obvious, since we can extend f to a Lipschitz function f^* on $\mathbb{R}$ and thus M is a compact subset of graph f^* which is a Lipschitz hypersurface in $\mathbb{R}^2$.

To prove (ii), consider a hyperplane H in $\mathbb{R}^2$. Then H is a line and we can suppose that it contains the origin. Then the projection onto H is a linear mapping and so it is clearly sufficient to prove that, for each $a, b \in \mathbb{R}$, $Z := \{ax + bf(x) \colon x \in P\}$ is nowhere dense in $\mathbb{R}$. Since $Z = g(P)$, where $g(x) := ax + f(x),\ x \in [0,1]$, is C^1-smooth and P is nowhere dense, we obtain, see [3, Lemma 4.1], that $Z = g(P)$ is nowhere dense.

To prove (iii), consider an arbitrary point $c = (c_1, c_2) \in \mathbb{R}^2 \setminus M$. To prove that the set

$$
\begin{aligned}
E &:= \left\{ \frac{(x,y) - c}{\|(x,y) - c\|} \colon (x,y) \in M \right\} \\
&= \left\{ \left(\frac{x - c_1}{\sqrt{(x-c_1)^2 + (f(x) - c_2)^2}}, \frac{f(x) - c_2}{\sqrt{(x-c_1)^2 + (f(x) - c_2)^2}} \right),\ x \in P \right\}
\end{aligned}
$$

is nowhere dense in $S_{\mathbb{R}^2}$, it is clearly sufficient to show that (its "projection")

$$
F := \left\{ \frac{x - c_1}{\sqrt{(x-c_1)^2 + (f(x) - c_2)^2}},\ x \in P \right\}
$$

is nowhere dense in $\mathbb{R}$. Set

$$
h(x) := \frac{x - c_1}{\sqrt{(x-c_1)^2 + (f(x) - c_2)^2}} \qquad \text{if } (x, f(x)) \neq c.
$$

If $c_1 \in P$ or $c_1 \notin [0,1]$, then h is clearly C^1-smooth on $[0,1]$ and therefore $F = h(P)$ is nowhere dense by [3, Lemma 4.1].

If $c_1 \in [0,1] \setminus P$, then h can be undefined for $x = c_1$. However, then there exist points $0 < t_1 < t_2 < 1$ such that $F = h(P \cap [0, t_1]) \cup h(P \cap [t_2, 1])$ and h is C^1-smooth on $[0, t_1]$ ($[t_2, 1]$, respectively) whenever $P \cap [0, t_1] \neq \emptyset$ ($P \cap [t_2, 1] \neq \emptyset$, respectively). So [3, Lemma 4.1] gives again that F is nowhere dense. □

Remark 4.2. In the above proposition, we can state that sets B_k satisfy the following condition (more general than (iii)):

(iii)* *For each $c \in \mathbb{R}^2$, the set $\{(x - c)/\|x - c\| \colon x \in B_k \setminus \{c\}\}$ is nowhere dense in the unit sphere $S_{\mathbb{R}^2}$.*

The proof is only a slight refinement of the proof of (iii).

5. An analogue of Slobodnik's necessary condition in separable Banach spaces

In this section we will prove the following analogue of Slobodnik's Theorem 1.2.

Proposition 5.1. *Let X be a separable Banach space ($\dim X \geq 2$) and let $f\colon X \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then the set $D(f)$ of all discontinuity points of f can be written as $D(f) = \bigcup_{k=1}^{\infty} B_k$ where each B_k has the following properties:*

- (i) *B_k is a bounded closed subset of a Lipschitz hypersurface $L_k \subset X$ which is nowhere dense in L_k.*
- (ii) *Any linear projection of B_k onto any hyperplane $0 \in H \subset X$ is a first category subset of H.*
- (iii) *For each $c \in X \setminus B_k$, the set $\{(x-c)/\|x-c\|\colon x \in B_k\}$ is a first category set in S_X.*

This result improves [13, Corollary 4.2] which only asserts that $D(f)$ can be covered by countably many Lipschitz hypersurfaces (and implies that $D(f)$ is a null subset of X in any usual sense).

We infer Proposition 5.1 easily from [13, Corollary 4.2], the Banakh–Maslyuchenko characterization (Theorem 2.5) and simple Lemma 5.2 below.

Although our Proposition 5.1 is not a direct generalization of Slobodnik's Theorem 1.2, *it easily implies Slobodnik's result.* Indeed, if $X = \mathbb{R}^n$, then each linearly continuous f has the Baire property by (1.1) and all B_k and their projections (in (ii) and (iii)) are compact. So it is sufficient to use the fact that each closed first category subset of a complete metric space is nowhere dense.

Lemma 5.2. *Let X be a separable Banach space with $\dim X \geq 2$, $v \in S_X$ and let Y be a topological complement of $V := \operatorname{span}\{v\}$. Let $A \subset X$ be an l-miserable set. Then*

- (i) *the projection $\pi_Y(A)$ of A onto Y in the direction of V is of the first category in Y, and*
- (ii) *for each $c \in X \setminus A$, the set $\{(x-c)/\|x-c\|\colon\ x \in A\}$ is of the first category in S_X.*

Proof: Choose a closed l-neighbourhood L of A such that $A \subset \overline{X \setminus L}$. The proofs of (i) and (ii) are quite analogous.

(i) For each $n \in \mathbb{N}$, set $A_n := \{x \in A\colon \{x + tv\colon t \in [-1/n, 1/n]\} \subset L\}$. Since L is an l-neighbourhood of A, we have $A = \bigcup_{n\in\mathbb{N}} A_n$. Now choose, for each $n \in \mathbb{N}$, a covering of V by open sets $V_{n,1}, V_{n,2}, \dots$ such that each $V_{n,k} \subset V$ is an open subset of V with $\operatorname{diam}(V_{n,k}) < 1/n$. Denote by π_V the projection of X onto V in the direction of Y and set $A_{n,k} := A_n \cap (\pi_V)^{-1}(V_{n,k})$, $n, k \in \mathbb{N}$. Using the definitions of sets A_n and $V_{n,k}$, it is easy to see that

$$(5.1)\qquad (\pi_Y)^{-1}(\pi_Y(x)) \cap (\pi_V)^{-1}(V_{n,k}) \subset L \qquad \text{whenever } n,k \in \mathbb{N},\ x \in A_{n,k}.$$

Since $A = \bigcup_{n,k\in\mathbb{N}} A_{n,k}$, it is sufficient to prove that each set $\pi_Y(A_{n,k})$ is nowhere dense in the space Y. So suppose, to the contrary, that there exist n, k and a nonempty set $G \subset Y$ which is open in Y and $\pi_Y(A_{n,k}) \cap G$ is dense in G. Using (5.1) and the closedness of L, we obtain $(\pi_Y)^{-1}(G) \cap (\pi_V)^{-1}(V_{n,k}) \subset L$. Since $(\pi_Y)^{-1}(G) \cap (\pi_V)^{-1}(V_{n,k})$ is open and contains a point $a \in A_{n,k} \subset A$, we obtain $a \notin \overline{X \setminus L}$ which contradicts to $A \subset \overline{X \setminus L}$.

(ii) Fix a point $c \in X \setminus A$ and define the "projection" $\pi_S \colon X \setminus \{c\} \to S_X$ by $\pi_S(x) := (x-c)/\|x-c\|$, $x \in X \setminus \{c\}$. For each $n \in \mathbb{N}$, denote $A_n := \{x \in A\colon \{x + t\pi_S(x)\colon t \in [-1/n, 1/n]\} \subset L\}$. Since L is an l-neighbourhood of A, we have $A = \bigcup_{n\in\mathbb{N}} A_n$. Now choose, for each $n \in \mathbb{N}$, a covering of $(0,\infty)$ by open subsets $H_{n,1}, H_{n,2}, \dots$ with $\operatorname{diam}(H_{n,k}) < 1/n$. Set $A_{n,k} := A_n \cap \{x \in X\colon \|x-c\| \in H_{n,k}\}$, $n, k \in \mathbb{N}$. Using the definitions of sets A_n and $H_{n,k}$, it is easy to see that

$$(5.2)\qquad \begin{aligned}(\pi_S)^{-1}(\pi_S(x)) \cap \{x \in X\colon \|x-c\| \in H_{n,k}\} \subset L \\ \text{whenever } n,k \in \mathbb{N},\ x \in A_{n,k}.\end{aligned}$$

Since $A = \bigcup_{n,k\in\mathbb{N}} A_{n,k}$, it is sufficient to prove that each set $\pi_S(A_{n,k})$ is nowhere dense in the sphere S_X. So suppose, to the contrary, that there exist n, k and a nonempty set $G \subset S_X$ which is open in S_X and $\pi_S(A_{n,k}) \cap G$ is dense in G. Using (5.2) and the closedness of L, we obtain $(\pi_S)^{-1}(G) \cap \{x \in X\colon \|x-c\| \in H_{n,k}\} \subset L$. Since $(\pi_S)^{-1}(G) \cap \{x \in X\colon \|x-c\| \in H_{n,k}\}$ is open and contains a point $a \in A_{n,k} \subset A$, we obtain $a \notin \overline{X \setminus L}$ which contradicts to $A \subset \overline{X \setminus L}$. ☐

Proof of Proposition 5.1: By Theorem 2.5 there exist closed l-miserable sets $F_m \subset X$, $m \in \mathbb{N}$, such that $D(f) = \bigcup_{m\in\mathbb{N}} F_m$. By [13, Corollary 4.2] there exist Lipschitz hypersurfaces M_n, $n \in \mathbb{N}$, such that $D(f) \subset \bigcup_{n\in\mathbb{N}} M_n$. Set

$$Z_{m,n,p} := F_m \cap M_n \cap \{x \in X\colon \|x\| \le p\}, \qquad m,n,p \in \mathbb{N},$$

and let $(B_k)_{k=1}^{\infty}$ be a sequence of all elements of the set $\{Z_{m,n,p}\colon m,n,p \in \mathbb{N}\}$. Now consider an arbitrary $k \in \mathbb{N}$. Obviously, B_k is closed and bounded. It is also l-miserable since it is a subset of an l-miserable set F_m and so we obtain by Lemma 5.2 that conditions (ii) and (iii) hold. Further B_k is contained in some Lipschitz hypersurface M_n. Let V and Y be linear spaces corresponding to M_n as in Definition 2.1 and denote by π_Y the projection onto Y in the direction of V. Since $\pi_Y|_{M_n}\colon M_n \to Y$ is clearly a homeomorphism, we obtain that $\pi_Y(B_k)$ is a closed subset of the complete space Y and so it is nowhere dense in Y, because it is a first category set in Y by condition (ii). Consequently B_k is nowhere dense in M_n and thus condition (i) holds (with $L_k := M_n$). □

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