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More on the product of pseudo radial spaces

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Abstract. It is proved that the product of two pseudo radial compact spaces is pseudo radial provided that one of them is monolithic.

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It is well known that the product of two pseudo radial, or even radial, spaces can easily fail to be pseudo radial. For instance, in [5] there is described an example of a compact radial space and a Lindelöf radial space whose product is not pseudo radial. The situation changes assuming that the spaces under consideration are both compact. It was observed by J. Gerlits, in a talk presented at the 1987 Baku Topological Conference, that the product of two compact radial spaces is always pseudo radial, whereas it is not clear if the same holds for two compact pseudo radial spaces. A partial answer to this question was given by Z. Frolík and G. Tironi in[5], where they showed that the product of a compact radial space and a compact pseudo radial space is pseudo radial.

In the present note, we give another partial answer to the above question proving that the product of two pseudo radial compact spaces is pseudo radial provided that one of them is monolithic.

In what follows, λ and κ are cardinal numbers and α, β, μ and ν ordinal numbers. κ^+ is the cardinal successor of κ. Compact means compact Hausdorff and all spaces considered here are assumed at least T_1. π_X $(\pi_Y) : X \times Y \to X$ (Y) denote the projections.

A sequence $\{x_\alpha : \alpha \in \lambda\}$ of points in the space X is said to be strictly convergent to x provided that λ is regular, $\{x_\alpha : \alpha \in \lambda\}$ converges to x and $x \notin \overline{\{x_\beta : \beta \in \alpha\}}$ for any $\alpha \in \lambda$.

A space X is said to be pseudo (almost) radial provided that for any non closed subset A of X there exists a sequence $\{x_\alpha : \alpha \in \lambda\}$ in A which converges (strictly converges) to a point $x \in \overline{A} \setminus A$. X is radial provided the requirement of the previous definition is satisfied for any $x \in \overline{A} \setminus A$. Pseudo radial spaces are also called chain–net spaces.

The class of almost radial spaces (see [2]) properly lies between the classes of radial and pseudo radial spaces. Moreover, each sequential space is almost radial.

The chain character $\sigma_c(X)$ of the pseudo radial space X is the smallest cardinal κ such that for any non closed subset A of X there exists a sequence of length not exceeding κ in A which converges to some point outside A. Notice that if X is almost radial, then in the definition of the chain character we can replace convergent sequences by strictly convergent sequences.

Given a subset A of the topological space X we denote by $[A]_{ch}$ the set of all points which are limits of convergent sequences in A. Inductively we define $[A]_{ch}^{\alpha+1} = [[A]_{ch}^{\alpha}]_{ch}$ and $[A]_{ch}^{\alpha} = \cup_{\beta\in\alpha}[A]_{ch}^{\beta}$ whenever α is limit. X is pseudo radial, if and only if $\overline{A} = [A]_{ch}^{\alpha(A)}$ for some $\alpha(A)$ and any A subset of X. In this case it is easily seen that we can take $\sigma_c(X)^+$ as a common $\alpha(A)$.

$nw(X)$ and $\psi(X)$ denote respectively the netweight and the pseudo character of the topological space X (for more details see [7]).

We begin with a lemma which gives a useful sufficient condition for a pseudo radial space to be almost radial.

Lemma 1. *If X is a pseudo radial space such that $\sigma_c(\overline{A}) \le |A|$ for any subset A of X, then X is almost radial.*

Proof: Let A be a non closed subset of X and let λ be the smallest length of a sequence in A converging to a point in $\overline{A}\setminus A$. Fix a point $x \in \overline{A}\setminus A$ and a sequence $\{x_\alpha : \alpha\in\lambda\}$ in A converging to x. We claim that this sequence strictly converges to x. Assume the contrary and let $\alpha\in\lambda$ be such that $x\in\overline{\{x_\beta : \beta\in\alpha\}}$. Put $|\alpha| = \kappa$ and $B = \{x_\beta : \beta\in\alpha\}$. We have $\sigma_c(\overline{B}) \le \kappa$ and $\overline{B} = \cup_{\nu\in\kappa^+}[B]_{ch}^{\nu}$. Let ν_0 be the least $\nu\in\kappa^+$ such that $[B]_{ch}^{\nu}\setminus A \ne \emptyset$. Clearly ν_0 is a successor ordinal and we write it as $\mu+1$. Since $[B]_{ch}^{\mu}\subset A$, there exists a sequence of length not exceeding κ in A converging to a point in $\overline{A}\setminus A$. As $\kappa<\lambda$, we are in contradiction with the supposed minimality of λ and the proof is complete. □

Theorem. *If X is a compact pseudo radial space such that $\sigma_c(\overline{A}) \le |A|$ for any $A\subset X$ and Y is a compact pseudo radial space, then $X\times Y$ is pseudo radial.*

Proof: Let us suppose by contradiction that $X\times Y$ is not pseudo radial. Then the family $\mathcal{C}$ of all non closed sequentially closed subsets of $X\times Y$ is not empty. If $C\in\mathcal{C}$, then there exists a point $(x,y)\in\overline{C}\setminus C$ and we can select a closed neigbourhood $U\times V$ of (x,y) in such a way that $x\notin\pi_X(C\cap U\times V)$. This means that the family $\mathcal{C}'$ of all $C\in\mathcal{C}$, for which $\pi_X(C)$ is not closed, is not empty. By virtue of Lemma 1, for any $C\in\mathcal{C}'$ let $\lambda(C)$ be the smallest length of a sequence in $\pi_X(C)$ which strictly converges to a point outside $\pi_X(C)$. Let λ be the minimum of all such $\lambda(C)$ and fix a set $C\in\mathcal{C}'$ such that $\lambda=\lambda(C)$. Denote with $\{x_\alpha : \alpha\in\lambda\}$ a sequence in $\pi_X(C)$ which strictly converges to a point $x\in\overline{\pi_X(C)}\setminus\pi_X(C)$. Put $Z=\overline{\{x_\alpha : \alpha\in\lambda\}}$. We want to construct now a sequence $\{U_\alpha : \alpha\in\lambda\}$ of open subsets of Z in such a way that $\cap_{\alpha\in\lambda}U_\alpha = \{x\}$ and $\overline{U_\beta}\subset U_\alpha\setminus\{x_\alpha\}$ for any $\alpha\in\beta$. Assume that U_β has been defined for any $\beta\in\alpha$. If $\alpha=\beta+1$, then by the regularity of Z we can immediately define U_α. Therefore let α be a limit ordinal and for any $\beta\in\alpha$ pick $\alpha_\beta\in\lambda$ in such a way that $x_\nu\in U_\beta$ for any $\nu\in\lambda\setminus\alpha_\beta$. Since λ is regular, there exists $\alpha^*\in\lambda$ such that $\alpha_\beta\in\alpha^*$ for any $\beta\in\alpha$. We have $\overline{Z\setminus\overline{U_\beta}}\subset\overline{\{x_\nu : \nu\in\alpha^*\}}$ and consequently $Z\setminus\cap_{\beta\in\alpha}U_\beta = Z\setminus\cap_{\beta\in\alpha}\overline{U_\beta}\subset\overline{\{x_\nu : \nu\in\alpha^*\}}$. Letting $\alpha^{**}=\max\{\alpha^*,\alpha+1\}$, because of the strict convergence of the sequence $\{x_\alpha : \alpha\in\lambda\}$, we have $x\notin\overline{\{x_\nu : \nu\in\alpha^{**}\}}$ and again for the regularity of Z we can select an open subset U_α of Z so that

$x \in U_\alpha$ and $\overline{U_\alpha} \cap \overline{\{x_\nu : \nu \in \alpha^{**}\}} = \emptyset$. This completes the inductive construction. We need only to check that $\cap_{\alpha\in\lambda} U_\alpha = \{x\}$. To this end let $z \in Z \setminus \{x\}$ and choose a closed neighbourhood V of x such that $z \notin V$. For some α we have $\overline{\{x_\nu : \nu \in \lambda \setminus \alpha\}} \subset V$ and therefore $z \in \overline{\{x_\nu : \nu \in \alpha+1\}} \subset Z \setminus U_\alpha$. Now we want to prove that $C \cap (Z \setminus U_\alpha) \times Y$ is closed for any $\alpha \in \lambda$. Clearly, this set is sequentially closed and if it is not closed then, as shown at the beginning of the present proof, we can find a set $C' \subset C \cap (Z \setminus U_\alpha) \times Y$ which is a member of $\mathcal{C}'$. $\pi_X(C')$ is a subset of $Z \setminus U_\alpha$ which is a subspace of density less than λ and thus we have $\sigma_c(Z \setminus U_\alpha) < \lambda$. Because $\pi_X(C')$ is not closed, this would imply the existence in it of a sequence having length shorter than λ which strictly converges outside $\pi_X(C')$, in contrast with the minimality of λ.

To finish, let $F_\alpha = \pi_Y(C \cap \overline{U_\alpha} \times Y)$ and observe that there must be $\cap_{\alpha\in\lambda} F_\alpha = \emptyset$. Indeed if $y \in \cap_{\alpha\in\lambda} F_\alpha$, then for any α there exists $z_\alpha \in \overline{U_\alpha}$ such that $(z_\alpha, y) \in C$. But taking a generic open neighbourhood $U \times V$ of (x,y) in $X \times Y$, the compactness of $Z \setminus U$ implies the existence of some α such that $\overline{U_\alpha} \subset U$ and consequently the sequence $\{(z_\alpha, y) : \alpha \in \lambda\}$ converges to (x,y), in contrast with the fact that $(x,y) \notin C$. Since $\cap_{\alpha\in\lambda} F_\alpha = \emptyset$ and $\{F_\alpha : \alpha \in \lambda\}$ is decreasing, it follows that for some $\alpha^* \in \lambda$ the set F_{α^*} is not closed. Select a sequence $\{y_\nu : \nu \in \kappa\}$ in F_{α^*} which converges to a point $y \notin F_{\alpha^*}$ and assume κ be a regular cardinal. For any $\alpha \geq \alpha^*$, the set $C \cap (\overline{U_{\alpha^*}} \setminus U_\alpha) \times Y = C \cap (Z \setminus U_\alpha) \times Y \cap \overline{U_{\alpha^*}} \times Y$ is closed and therefore also the projection $\pi_Y(C \cap (\overline{U_{\alpha^*}} \setminus U_\alpha) \times Y) = F'_\alpha$ is closed. Hence the set $\Gamma_\alpha = \{\nu : y_\nu \in F'_\alpha\}$ has cardinality less than κ. Since λ and κ are regular and $\cup_{\alpha\geq\alpha^*} \Gamma_\alpha = \kappa$, it follows that $\lambda = \kappa$. Owing to the previous observation for any $\alpha \in \lambda$ we can pick a point $z_\alpha \in U_\alpha$ and an ordinal $\nu_\alpha \in \lambda = \kappa$ in such a way that $(z_\alpha, y_{\nu_\alpha}) \in C$ and $\{\nu_\alpha : \alpha \in \lambda\}$ is increasing. It is clear that the sequence $\{(z_\alpha, y_{\nu_\alpha}) : \alpha \in \lambda\}$ converges to $(x,y) \notin C$ and this contradicts the fact that C is sequentially closed. □

Recall (see [1]) that a space X is said to be monolithic provided that $nw(\overline{A}) \leq |A|$ for any subset A of X.

Lemma 2. *If X is a pseudo radial Hausdorff monolithic space, then $\sigma_c(\overline{A}) \leq |A|$ for any subset A of X.*

Proof: The result follows from the inequality $\psi(X) \leq nw(X)$, true in any Hausdorff space, and the inequality (see [4, Theorem 1]) $\sigma_c(X) \leq \psi(X)$, true in any pseudo radial space. □

Incidentally, notice that Lemmas 1 and 2 imply that any pseudo radial Hausdorff monolithic space is almost radial.

Combining Lemmas and the Theorem we have:

Corollary. *The product of a pseudo radial compact monolithic space and a pseudo radial compact space is pseudo radial.*

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