Properties of forcing preserved by finite support iterations

Miroslav Repický

Abstract. We shall investigate some properties of forcing which are preserved by finite support iterations and which ensure that unbounded families in given partially ordered sets remain unbounded.

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0. Introduction.

J.I. Ihoda and S. Shelah [6] proved that the meager forcing keeps unbounded families of functions. Moreover, this property of forcing is preserved by finite support iterations. Instead of the family of functions ${}^{\omega}\omega$ with the eventual ordering we can study arbitrary definable partially ordered set and ask which notions of forcing keep unbounded families in this ordering.

We show that all Boolean algebras with finitely additive measure keep unbounded families in a partial order closely connected with Lebesgue measure (Theorem 2). This class of Boolean algebras is closed under finite support iterations [8].

Let $\mathcal{F}$ be a definable partially ordered set. We say that $\mathcal{F}$ is absolute if for every transitive model M of ZFC, $\mathcal{F}^M = \mathcal{F} \cap M$. Now we introduce the central notion of the paper.

Definition. Let $\mathcal{F}$ be an absolutely definable partially ordered set. Let P be a notion of forcing and let $\mathcal{B} = r.o.P$. We say that P is $\mathcal{F}$-good if for every name $\boldsymbol{f} \in V^{\mathcal{B}}$ such that $\Vdash_P \boldsymbol{f} \in \mathcal{F}$ there exists $g \in \mathcal{F}$ such that $h \le g$ whenever $h \in \mathcal{F}$ and $p \Vdash \check{h} \le \boldsymbol{f}$ for some $p \in P$.

In the forcing language the letter $\mathcal{F}$ stands for the set defined by a corresponding formula and it has a different meaning from $\check{\mathcal{F}}$. The definition does not make sense when $\mathcal{F}$ is not directed. For $\mathcal{F}$ directed let us denote:

$$\mathbf{b}(\mathcal{F}) = \min\{|\mathcal{F}_0| : \mathcal{F}_0 \subseteq \mathcal{F} \ \ \& \ \ (\forall f \in \mathcal{F})(\exists g \in \mathcal{F}_0)\, g \not\le f\},$$
$$\mathbf{d}(\mathcal{F}) = \min\{|\mathcal{F}_0| : \mathcal{F}_0 \subseteq \mathcal{F} \ \ \& \ \ (\forall f \in \mathcal{F})(\exists g \in \mathcal{F}_0)\, f \le g\}.$$

The next simple lemma justifies the definition:

Lemma. *Let $\kappa = \mathbf{b}(\mathcal{F})$ and $\lambda = \mathbf{d}(\mathcal{F})$. If P is a c.c.c. $\mathcal{F}$-good notion of forcing then $\Vdash_P$ "$\mathbf{b}(\mathcal{F}) \le \check{\kappa}$ & $\check{\lambda} \le \mathbf{d}(\mathcal{F})$".*

We shall concentrate our attention to these partially ordered sets:

$${}^{\omega}\omega \text{ ordered by } f \leq^* g \text{ iff } (\exists m)(\forall n \geq m)\, f(n) \leq g(n);$$

$$\mathcal{K} = \{f \in {}^{\omega}\omega : \sum_{n\in\omega} \frac{1}{f(n)} \leq 1\},\ f \leq^* g \text{ iff } (\exists m)(\forall n \geq m)\, f(n) \leq g(n);$$

$$\mathcal{S} = \{f \in {}^{\omega}\mathcal{P}(\omega) : \sum_{n\in\omega} |f(n)| 2^{-n} \leq 1\} \text{ and}$$

$$\mathcal{L} = \{f \in {}^{\omega}\mathcal{P}(\omega) : \lim_{n\to\infty} \frac{f(n)}{n+1} = 0\}$$
$$\text{ordered by } f \leq^* g \text{ iff } (\exists m)(\forall n \geq m)\, f(n) \subseteq g(n);$$

$$\mathcal{M} = \{f \in {}^{\omega}\mathcal{P}({}^{<\omega}2) : (\forall n)\ f(n) \text{ is open dense in } {}^{<\omega}2\},$$
$$\text{ordered by } f \leq^* g \text{ iff } (\exists m)(\forall n \geq m)\, g(n) \subseteq f(n).$$

Evidently, these partially ordered sets are absolute. There are important connections of these partial orders with Lebesgue measure and category. Let us recall that $\operatorname{add} \mathbf{L}$ is the least cardinal κ for which the ideal $\mathbf{L}$ of Lebesgue measure zero sets is not κ-additive and $\operatorname{cof} \mathbf{L}$ is minimal cardinality of a base of $\mathbf{L}$. The cardinals $\operatorname{add} \mathbf{K}$ and $\operatorname{cof} \mathbf{K}$ are defined similarly for the ideal $\mathbf{K}$ of sets of first category. Particularly we have:

$$\operatorname{add} \mathbf{L} = \mathbf{b}(\mathcal{K}) = \mathbf{b}(\mathcal{S}) = \mathbf{b}(\mathcal{L}),$$
$$\operatorname{cof} \mathbf{L} = \mathbf{d}(\mathcal{K}) = \mathbf{d}(\mathcal{S}) = \mathbf{d}(\mathcal{L}).$$

The characterization using convergent series is due to T. Bartoszyński [1]. The second characterization is due to T. Bartoszyński [1] and J. Stern, J. Raisonier [11]. Their characterization is in terms of localization of functions from ${}^{\omega}\omega$ by members of $\mathcal{S}$ and their proof works also in this case. The last characterization is a consequence of previous (see e.g. [12]). Similar characterization holds true for category:

$$\operatorname{add} \mathbf{K} = \mathbf{b}(\mathcal{M}), \quad \operatorname{cof} \mathbf{K} = \mathbf{d}(\mathcal{M}).$$

This can be easily proved using some ideas of [13].

1. ${}^{\omega}\omega$-goodness.

Measure algebra is weakly (ω,ω)-distributive and so it is ${}^{\omega}\omega$-good. Several other examples of ${}^{\omega}\omega$-good notions of forcing detects the following theorem. Meager forcing [6], forcing for eventually different reals [10], Dirichlet–Minkowski forcing [3] and Cohen forcing fulfill the assumptions of this theorem.

Theorem 1 (J.I. Ihoda, S. Shelah)**.** *Let* $P = \bigcup_{n,k\in\omega} P_{n,k}$ *be a notion of forcing such that*

(i) *for every* n, $P_n = \bigcup_{k\in\omega} P_{n,k}$ *is centered; and*

(ii) *for every* $n, k \in \omega$ *and for every open dense set* $D \subseteq P$ *there exists an integer* $m = m(n,k,D)$ *such that* $(\forall p \in A_{n,k})(\exists i < m)(\exists q \in P_i \cap D)\, q \leq p$.

Then P is ${}^\omega\omega$-good.

Proof: Let $\Vdash_P \boldsymbol{f} \in {}^\omega\omega$. For every $j \in \omega$, the set

$$D_j = \{q \in P : (\exists l \in \omega)\, q \Vdash \boldsymbol{f}(\check{j}) = \check{l}\}$$

is open dense in P. By (ii), for every $n, k \in \omega$ the set

$$a(n,k,j) = \{l \in \omega : (\exists i < m(n,k,D_j))(\exists q \in P_i)\, q \Vdash \boldsymbol{f}(\check{j}) = \check{l}\}$$

is nonempty and by (i) finite. Set

$$g(j) = \max \bigcup_{n,k \le j} a(n,k,j), \quad j \in \omega.$$

Now, assume that $h \in {}^\omega\omega$ and $p \Vdash (\forall j \ge \check{j}_0)\, \check{h}(j) \le \boldsymbol{f}(j)$ for some $p \in P$ and $j_0 \in \omega$. There are $n, k \in \omega$ such that $p \in P_{n,k}$. Therefore $h(j) \le \max a(n,k,j)$ for $j \ge j_0$ and so $h \le^* g$. □

2. $\mathcal{S}$-goodness, $\mathcal{K}$-goodness and $\mathcal{L}$-goodness.

Theorem 2. *Let $\mathcal{B}$ be a complete Boolean algebra with strictly positive finitely additive measure. Then $\mathcal{B}^+ = \mathcal{B} - \{\mathbf{0}\}$ is $\mathcal{S}$-good.*

In the proof of the theorem we will need this result:

Lemma 1 (A. Krawczyk)**.** *Let $\mathcal{B}$ be a Boolean algebra with finitely additive measure μ. Let $n, m \in \omega$ and let $\{b_i : i < n\} \subseteq \mathcal{B}$ be such that $\bigwedge_{i \in a} b_i = \mathbf{0}$ for every $a \in [n]^m$. Then $\sum_{i<n} \mu(b_i) \le (m-1)\mu(\mathbf{1})$.*

Proof: By induction on $n \in \omega$ simultaneously for all Boolean algebras with finitely additive measure:

$$\sum_{i<n} \mu(b_i) = \mu(b_{n-1}) + \sum_{i<n-1} \mu(b_i \wedge b_{n-1}) + \sum_{i<n-1} \mu(b_i - b_{n-1}) \le$$
$$\le \mu(b_{n-1}) + (m-2)\mu(b_{n-1}) + (m-1)\mu(-b_{n-1}) = (m-1)\mu(\mathbf{1}).$$

□

Proof of Theorem 2: Let μ be a strictly positive finitely additive measure on $\mathcal{B}$ and let $\mu(\mathbf{1}) = 1$. Let $\boldsymbol{f} \in V^{\mathrm{r.o.}\mathcal{B}}$ and $\|\boldsymbol{f} \in \mathcal{S}\| = \mathbf{1}$, i.e.

$$\left\| \sum_{n \in \omega} |\boldsymbol{f}(n)| 2^{-n} \le 1 \right\| = \mathbf{1}.$$

Denote $b_{n,i} = \|\check{i} \in \boldsymbol{f}(\check{n})\|$ and let $g_m(n) = \{i \in \omega : \mu(b_{n,i}) > \frac{1}{m+1}\}$. We shall show that

$$\sum_{n \in \omega} |g_m(n)| 2^{-n} < \infty, \tag{1}$$

for all $m \in \omega$.

Let us assume that (1) does not hold for some m. Then there is an $N \in \omega$ such that

$$\sum_{n<N} |g_m(n)| 2^{-n} > m+1. \tag{2}$$

Let $A = \{(n,i,j) : n < N \;\;\&\;\; i \in g_m(n) \;\;\&\;\; j < 2^{N-n}\}$ and let us denote $b_{n,i,j} = b_{n,i}$ for $(n,i,j) \in A$. Then by (2),

$$\sum_{(n,i,j)\in A} \mu(b_{n,i,j}) > \frac{1}{m+1} \sum_{n<N} |g_m(n)| 2^{N-n} > 2^N.$$

According to Lemma 1 there exists $B \subseteq A$, $|B| > 2^N$ such that

$$b = \bigwedge_{(n,i,j)\in B} b_{n,i,j} \neq \mathbf{0}.$$

Then

$$b \Vdash \text{“} \sum_{n<N} |\boldsymbol{f}(n)| 2^{-n} \geq \sum_{n<N} |\{i : (\exists j)\ (n,i,j) \in B\}| 2^{-n} \geq$$
$$\geq \sum_{n<N} |\{(i,j) : (n,i,j) \in B\} 2^{n-N} 2^{-n} = |B| 2^{-n} > 1\text{”}.$$

This is a contradiction and therefore (1) holds true. Choose $g \in \mathcal{S}$ such that $g_m \leq^* g$ for every $m \in \omega$.

Now, if $h \in \mathcal{S}$ is such that $b \Vdash (\forall n > \check{n}_0)\, \check{h}(n) \subseteq \boldsymbol{f}(n)$ for some $b \in \mathcal{B}^+$ and $n_0 \in \omega$, then $h(n) \subseteq g_m(n)$ whenever $n > n_0$ and $\mu(b) > \frac{1}{m+1}$. Therefore $h \leq^* g$. □

One can easily prove:

Lemma 2. *Let $\mathcal{F}$, $\mathcal{H}$ be absolute partially ordered sets. Let mappings $\alpha : \mathcal{F} \to \mathcal{H}$, $\beta : \mathcal{H} \to \mathcal{F}$ be such that*

(i) *α is absolute and monotone,*
(ii) *$\alpha(f) \leq h$ implies $f \leq \beta(h)$.*

Then $\mathcal{H}$-goodness implies $\mathcal{F}$-goodness.

Almost immediate consequence of this lemma is that $\mathcal{L}$-goodness implies ${}^{\omega}\omega$-goodness (see e.g. [12, Lemma 2.1] and the proof) and $\mathcal{S}$-goodness implies $\mathcal{K}$-goodness. Thus every complete Boolean algebra with strictly positive finitely additive measure is $\mathcal{K}$-good. Dominating algebra [13] is not ${}^{\omega}\omega$-good and since it is σ-centered, it carries a finite additive measure. Therefore $\mathcal{B}^+$ need not be $\mathcal{L}$-good. This also shows that the condition (i) in Lemma 2 cannot be dropped since one can easily construct absolute mappings $\alpha : {}^{\omega}\omega \to \mathcal{K}$, $\beta : \mathcal{K} \to {}^{\omega}\omega$ satisfying condition (ii).

Theorem 3. *Let $\mathcal{B}$ be a measure algebra. Then $\mathcal{B}^+$ is an $\mathcal{L}$-good notion of forcing.*

PROOF: Let μ be a σ-additive measure on $\mathcal{B}$ and let $\mu(\mathbf{1}) = 1$. Let $\boldsymbol{f} \in V^{\mathrm{r.o.}\,\mathcal{B}}$ and $\|\boldsymbol{f} \in \mathcal{L}\| = \mathbf{1}$. Since $\mathcal{B}$ is weakly (ω,ω)-distributive and satisfies c.c.c., there exists $\varphi \in {}^\omega\omega$ such that $\lim\limits_{n\to\infty} \frac{\varphi(n)}{n+1} = 0$ and $\|(\exists m)(\forall n > m)\,|\boldsymbol{f}(n)| \leq \varphi(n)\| = \mathbf{1}$. Let $\{a_m : m \in \omega\}$ be a partition of $\mathbf{1}$ in $\mathcal{B}$ such that $a_m \Vdash (\forall n > \check{m})\,|\boldsymbol{f}(n)| \leq \check{\varphi}(n)$. Find a monotone unbounded function $\psi \in \omega$ such that $\lim\limits_{n\to\infty} \frac{\varphi(n)\psi(n)}{n+1} = \mathbf{0}$. Let us denote $b_{n,i} = \|\check{i} \in \boldsymbol{f}(\check{n})\|$ and set $g_m(n) = \{i : \mu(b_{n,i} \wedge a_m) > \frac{1}{\psi(n)}\}$. According to Lemma 1, $|g_m(n)| \leq \varphi(n)\psi(n)$ for $n > m$ and so $g_m \in \mathcal{L}$. Choose $g \in \mathcal{L}$ such that $g_m \leq^* g$ for every $m \in \omega$.

If $h \in \mathcal{L}$ and if $b \Vdash (\forall n > \check{n}_0)\,\check{h}(n) \subseteq \boldsymbol{f}(n)$ for some $m, n_0 \in \omega$ and $b \leq a_m$ then $h(n) \subseteq g_m(n)$ whenever $n > \max\{n_0, m\}$ and $\mu(b) > \frac{1}{\psi(n)}$. Hence $h \leq^* g$. □

3. $\mathcal{M}$-goodness.

Let M denote the set of all Borel codes [7] of Borel meager sets. For $a \in M$, B_a is a Borel meager set with code a. M is ordered by $a \leq b$ iff $B_a \subseteq B_b$. Obviously, this ordering is absolute and since the ideal $\mathbf{K}$ has the base consisting of meager F_σ-sets we can "identify" it with $\mathcal{M}$ ordered by $f \leq g$ iff $A_f \subseteq A_g$ (where $A_f = {}^\omega 2 - \bigcap\limits_{n\in\omega} \bigcup\limits_{s\in f(n)} [s]$). So we have two orderings on $\mathcal{M}$ and we will see that they do not differ very much.

Lemma 3. *M-goodness implies ${}^\omega\omega$-goodness.*

PROOF: We show that assumptions of Lemma 2 hold true. For $f \in {}^\omega\omega$ let $\alpha(f)$ be a Borel code (chosen in a standard way) of the set

$$\{x \in {}^\omega 2 : (\exists m)(\forall n > m)(\exists i \leq f'(n))\, x(n+i) = 1\},$$

where $f'(n) = \max\limits_{i\leq n} f(i)$. Evidently, $\alpha : {}^\omega\omega \to M$ is absolute and monotone. We define $\beta : M \to {}^\omega\omega$. Let $a \in M$. There are nowhere dense sets $A_n \subseteq {}^\omega 2$ such that $A_n \subseteq A_{n+1}$ and $B_a \subseteq \bigcup\limits_{n\in\omega} A_n$. Set

$$\beta(a)(n) = \min\{m \in \omega : (\forall s \in {}^n 2)(\exists t \in {}^{n+m}2)\ s \subseteq t\ \ \&\ \ [t] \cap A_n = \emptyset\}.$$

We shall verify the condition (ii) of Lemma 2. Assume that $f(n) > \beta(a)(n)$ for infinitely many $n \in \omega$. Then there exists a sequence $n_k \in \omega$ such that $n_{k+1} > n_k + f'(n_k)$ and $f'(n_k) > \beta(a)(n_k)$. By induction on k define $s_k \in {}^{n_k}2$ such that

$s_k \subseteq s_{k+1}$,
$[s_{k+1}\restriction(n_k + f'(n_k))] \cap A_{n_k} = \emptyset$, and
if $n_k + f'(n_k) \leq i < n_{k+1}$ then $s_{k+1}(i) = 1$.

Set $x = \bigcup\limits_{k\in\omega} s_k$. Then $x \in B_{\alpha(f)} - B_a$ and so $\alpha(f) \not\leq a$. □

Theorem 4. *Let P be a notion of forcing. The following are equivalent:*

(i) *P is $\mathcal{M}$-good,*

(ii) *P is M-good,*

(iii) *for every P-name $\boldsymbol{A}$ such that $\Vdash_P \boldsymbol{A} \in \mathbf{K}$, the set $\{x \in {}^\omega 2 : (\exists p \in P)\, p \Vdash \check{x} \in \boldsymbol{A}\}$ is meager.*

Proof: (i)→(iii). Let $\Vdash_P \boldsymbol{A} \in \mathbf{K}$. Then there exists a P-name $\boldsymbol{f}$ such that

$$\Vdash_P \boldsymbol{f} \in \mathcal{M} \ \ \& \ \ \boldsymbol{A} \subseteq A_{\boldsymbol{f}} \ \ \& \ \ (\forall n \in \omega)\, \boldsymbol{f}(n+1) \subseteq \boldsymbol{f}(n).$$

For every $x \in {}^\omega 2$ denote $f_x(n) = \{s \in {}^{<\omega}2 : x \notin [s]\}$. Then $f_x \in \mathcal{M}$ and if $p \Vdash \check{x} \in \boldsymbol{A}$ then by monotonicity of $\boldsymbol{f}$, $p \Vdash (\exists m)(\forall n > m)\, x \notin \bigcup_{s \in \boldsymbol{f}(n)} [s]$, and so $p \Vdash \check{f}_x \leq^* \boldsymbol{f}$. Hence by $\mathcal{M}$-goodness of P there is $g \in \mathcal{M}$ such that $f_x \leq^* g$ whenever $p \Vdash \check{x} \in \boldsymbol{A}$ for some $p \in P$. Therefore

$$\{x \in {}^\omega 2 : (\exists p \in P)\, p \Vdash \check{x} \in \boldsymbol{A}\} \subseteq A_g \in \mathbf{K}.$$

(iii)→(ii). Let $\Vdash_P \boldsymbol{a} \in M$. There is $b \in M$ such that

$$\{x \in {}^\omega 2 : (\exists p \in P)\, p \Vdash \check{x} \in B_{\boldsymbol{a}}\} \subseteq B_b.$$

By absoluteness $\Vdash_P \check{B}_c \subseteq B_{\check{c}}$ for $c \in M$. Therefore, if $c \in M$ and $p \Vdash \check{c} \leq \boldsymbol{a}$ then $B_c \subseteq \{x : p \Vdash \check{x} \in B_{\boldsymbol{a}}\} \subseteq B_b$.

(ii)→(i). Let $\Vdash_P \boldsymbol{f} \in \mathcal{M}$. Without loss of generality we can assume that

$$\Vdash_P (\forall n \in \omega)\, \boldsymbol{f}(n+1) \subseteq \boldsymbol{f}(n).$$

Otherwise take $\boldsymbol{f}'(n) = \bigcap_{i \leq n} \boldsymbol{f}(i)$. The forcing P is M-good. Therefore there is such a code $b \in M$ that for every code $a \in M$ such that $p \Vdash B_{\check{a}} \subseteq A_{\boldsymbol{f}}$, for some condition p, we have $B_a \subseteq B_b$. Let $Q = \{r_n : n \in \omega\}$ be a countable dense subset of ${}^\omega 2$ in V disjoint with the meager set B_b. Obviously $\Vdash_P \check{Q} \cap A_{\boldsymbol{f}} = \emptyset$. In $V^{\mathrm{r.o.}\,P}$ (and in V respectively) let us define:

$$\varphi_h(n) = \min\{k \in \omega : r_n{\restriction}k \in h(n)\}, \quad h \in \mathcal{M}.$$

Then $\Vdash_P (\forall n \in \omega) \varphi_{\boldsymbol{f}}(n) < \infty$. By Lemma 3, there is $\psi \in {}^\omega\omega$ such that $\varphi \leq^* \psi$ whenever $\varphi \in {}^\omega\omega$ and $p \Vdash \check{\varphi} \leq^* \varphi_{\boldsymbol{f}}$ for some $p \in P$. Set

$$g(n) = \{s \in {}^{<\omega}\omega : (\exists k \geq n)\, r_k{\restriction}\psi(k) \subseteq s\}.$$

Then $g \in \mathcal{M}$.

Now let $h \in \mathcal{M}$ be arbitrary such that $p \Vdash \check{h} \leq^* \boldsymbol{f}$ for some $p \in P$. There exist $h_0 \in \mathcal{M}$, $m \in \omega$ and $q \leq p$ such that $q \Vdash (\forall n \in \omega)\, \boldsymbol{f}(n) \subseteq \check{h}_0(n)$ and $h_0(n) = h(n)$ for $n > m$. Let us denote $h_1(n) = \bigcap_{i \leq n} h_0(i)$. Then $q \Vdash (\forall n \in \omega)\, \boldsymbol{f}(n) \subseteq \check{h}_1(n)$ and hence $q \Vdash (\forall n > m)\, \varphi_{\check{h}}(n) \leq \varphi_{\check{h}_1}(n) \leq \varphi_{\boldsymbol{f}}(n)$. Therefore $\varphi_h \leq^* \psi$ and so $h \leq^* g$. □

From condition (iii) of Theorem 4 it follows that if P is $\mathcal{M}$-good then for every set $A \subseteq {}^\omega 2$ of second category, $\Vdash_P$ "$\check{A}$ is of second category". Hence, measure algebras are not $\mathcal{M}$-good (see [2]).

Corollary. *Let P be a notion of forcing of cardinality less than* $\operatorname{add}\mathbf{K}$. *Then P is $\mathcal{M}$-good.*

Proof: If $\Vdash_P$ "$\boldsymbol{A}_n \subseteq {}^\omega 2$ is nowhere dense" then the set $\{x \in {}^\omega 2 : p \Vdash \check{x} \in \boldsymbol{A}_n\}$ is nowhere dense and

$$\{x \in {}^\omega 2 : (\exists p \in P)\, p \Vdash \check{x} \in \bigcup_{n\in\omega} \boldsymbol{A}_n\} = \bigcup_{p\in P}\bigcup_{n\in\omega}\{x \in {}^\omega 2 : p \Vdash \check{x} \in \boldsymbol{A}_n\}.$$

□

4. Iterated $\mathcal{F}$-good forcing.

Theorem 5. *Let $\mathcal{F} = {}^\omega\omega, \mathcal{K}, \mathcal{L}, \mathcal{S}, \mathcal{M}$. Let $\{P_\xi : \xi \le \alpha\} \cup \{\boldsymbol{Q}_\xi : \xi < \alpha\}$ be a finite support iterated forcing system and let for every $\xi < \alpha$,*

$$\Vdash_{P_\xi} \text{“}\boldsymbol{Q}_\xi \text{ is a c.c.c. } \mathcal{F}\text{-good notion of forcing”}.$$

Then P_α is $\mathcal{F}$-good.

Proof: By induction on α. The case α nonlimit or $\operatorname{cf}\alpha > \omega$ is trivial. Let $\operatorname{cf}\alpha = \omega$, $\alpha = \lim_{n\in\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$. Let $\boldsymbol{f}$ be a P_α-name such that $\Vdash_{P_\alpha} \boldsymbol{f} \in \mathcal{F}$. We will find $g \in \mathcal{F}$ such that if $h \in \mathcal{F}$ and $p \Vdash \check{h} \le^* \boldsymbol{f}$ then $h \le^* g$.

For every $n \in \omega$ there is a P_{α_n}-name $\boldsymbol{f}_n$ such that

if $\mathcal{F} = {}^\omega\omega, \mathcal{K}, \mathcal{S}$ then

$$\Vdash_{P_{\alpha_n}} \text{“there exists a decreasing sequence } \{r_k : k \in \omega\} \subseteq \boldsymbol{P}_{\alpha_n,\alpha} \text{ such that } (\forall k \in \omega)\, r_k \Vdash \boldsymbol{f}(k) = \boldsymbol{f}_n(k)\text{”};$$

if $\mathcal{F} = \mathcal{L}$ then

$$\Vdash_{P_{\alpha_n}} \text{“there exists a decreasing sequence } \{r_k : k \in \omega\} \subseteq \boldsymbol{P}_{\alpha_n,\alpha} \text{ such that}$$

$$(\forall k \in \omega)(\exists m > k)(\forall i \ge m)\ \frac{|\boldsymbol{f}(i)|}{i+1} < \frac{1}{k}\ \ \&\ \ r_k \Vdash \boldsymbol{f}{\restriction}m = \boldsymbol{f}_n{\restriction}m\text{”};$$

if $\mathcal{F} = \mathcal{M}$ then

$$\Vdash_{P_{\alpha_n}} \text{“there exists a sequence } \{r_{k,i} : k, i \in \omega\} \subseteq \boldsymbol{P}_{\alpha_n,\alpha} \text{ such that}$$

$$(\forall k, i \in \omega)\ r_{k,i+1} \le r_{k,i}\ \ \&\ \ (\exists m > i)(\forall s \in {}^i 2)(\exists t \in \boldsymbol{f}(k) \cap {}^m 2)$$

$$s \subseteq t\ \ \&\ \ r_{k,i} \Vdash \boldsymbol{f}_n(k) \cap {}^{\le m}2 = \boldsymbol{f}(k) \cap {}^{\le m}2\text{”}.$$

It is not difficult to verify that $\Vdash_{P_{\alpha_n}} \boldsymbol{f}_n \in \mathcal{F}$. By the induction hypothesis, for every $n \in \omega$ there is a $g_n \in \mathcal{F}$ such that $h \le^* g_n$ whenever $h \in \mathcal{F}$ and $p \Vdash \check{h} \le^* \boldsymbol{f}_n$ for some $p \in P_{\alpha_n}$. Since $\mathbf{b}(\mathcal{F}) > \omega$, there is $g \in \mathcal{F}$ such that $g_n \le^* g$ for every $n \in \omega$. We will show that g has the required properties.

Let $h \in \mathcal{F}$ and let $p \Vdash \check{h} \le^* \boldsymbol{f}$, $p \in P_\alpha$. If $\mathcal{F} = {}^\omega\omega, \mathcal{S}, \mathcal{L}$ then there is $q \le p$ and $k_0 \in \omega$ such that $q \Vdash (\forall k > \check{k}_0)\, \check{h}(k) \subseteq \boldsymbol{f}(k)$. $q \in P_{\alpha_n}$ for some integer n.

Then $q \Vdash_{P_{\alpha_n}} (\forall k > \check{k}_0)\, r_k \Vdash \check{h}(k) \subseteq \boldsymbol{f}(k) = \boldsymbol{f}_n(k)$ and so $q \Vdash \check{h} \leq^* \boldsymbol{f}_n$. Therefore $h \leq^* g_n \leq^* g$. The case $\mathcal{F} = \mathcal{K}$ is similar.

If $\mathcal{F} = \mathcal{M}$ and $q \Vdash_{P_\alpha} (\forall k > \check{k}_0)\, \boldsymbol{f}(k) \subseteq \check{h}(k)$, for some $n \in \omega$ and $q \in P_{\alpha_n}$ then

$$q \Vdash_{P_\alpha} (\forall k > \check{k}_0)(\forall i \in \omega)\ r_{k,i} \Vdash \boldsymbol{f}_n(k) \cap {}^{\leq i}2 = \boldsymbol{f}(k) \cap {}^{\leq i}2 \subseteq \check{h}(k).$$

Hence $q \Vdash (\forall k > \check{k}_0)\, \boldsymbol{f}_n(k) \subseteq \check{h}(k)$ and so $h \leq^* g_n \leq^* g$. □

Corollary. *Cohen forcing is $\mathcal{F}$-good for $\mathcal{F} = {}^\omega\omega, \mathcal{K}, \mathcal{S}, \mathcal{L}, \mathcal{M}$.*

For Cohen forcing we can state another general assertion.

Theorem 6. *Let $\mathcal{F} \subseteq {}^\omega([\omega]^{<\omega})$ be an absolutely definable family ordered by $f \leq^* g$ iff $(\exists m)(\forall n > m)\, f(n) \subseteq g(n)$. Let $\mathcal{F}$ satisfy the following two conditions:*

(i) $\mathbf{b}(\mathcal{F}) > \omega$,

(ii) $(\forall g \in \mathcal{F})(\forall f \in {}^\omega([\omega]^{<\omega})(\forall n \in \omega) f(n) \subseteq g(n) \Rightarrow f \in \mathcal{F}$.

Then Cohen forcing is $\mathcal{F}$-good.

Proof: For $p \in P$ let us denote

$$F_p(n) = \bigcap \{x \subseteq \omega : (\exists q \leq p)\, q \Vdash \boldsymbol{f}(n) = \check{x}\}.$$

Since $p \Vdash (\forall n \in \omega)\, F_p(n) \subseteq \boldsymbol{f}(n)$, by condition (ii) and by absoluteness of $\mathcal{F}$, $F_p \in \mathcal{F}$ for every $p \in P$. Let $g \in \mathcal{F}$ be such that $F_p \leq^* g$ for every $p \in P$.

Now let $h \in \mathcal{F}$ and let $p \Vdash (\forall n > \check{n}_0)\, \check{h}(n) \subseteq \boldsymbol{f}(n)$. Then $h(n) \subseteq F_p(n)$ for every $n > n_0$ and so $h \leq^* g$. □

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Matematický ústav SAV, Jesenná 5, 041 54 Košice, Czechoslovakia

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