Sets invariant under projections onto two dimensional subspaces

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Abstract. The Blaschke–Kakutani result characterizes inner product spaces E, among normed spaces of dimension at least 3, by the property that for every 2 dimensional subspace F there is a norm 1 linear projection onto F. In this paper, we determine which closed neighborhoods B of zero in a real locally convex space E of dimension at least 3 have the property that for every 2 dimensional subspace F there is a continuous linear projection P onto F with $P(B) \subseteq B$.

Keywords: inner product space, two dimensional subspace, projection

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1. Introduction.

As mentioned in the summary, if B is the closed unit ball in a normed space E and for every 2 dimensional subspace F there is a linear projection P of E onto F with $P(B) \subseteq B$, then the norm is given by inner product, as explained in Chapter 12 of Amir's book [1]. A natural question is to see, if there are other sets B such that for every 2 dimensional F there is a linear projection onto F under which B is invariant, or whether we characterize the ball in an inner product space by this property, among a wider class of sets B.

Restricting ourselves to closed neighborhoods of zero, we find B is the inverse image under a continuous linear map of: a closed neighborhood of 0 in $\mathbb{R}$, a unit ball in $\mathbb{R}^2$, or a unit ball in an inner product space.

The reader will note that a similar problem motivates the paper [3].

2. Two dimensional results.

The following result appears as Theorem 8 of [3].

Lemma 2.1. *Let B be a closed nonempty subset of $\mathbb{R}^2$ and suppose there is $w \in \mathbb{R}^2, w \neq 0$ and $\lambda_n \to \infty$, such that $\lambda_n^{-1} w \in B$ or $\lambda_n w \notin B$. For every one dimensional subspace m, there exists a linear projection $P : \mathbb{R}^2 \to m$ with $P(B) \subseteq B$ iff B is one of:*

(a) *a subset, containing* 0, *of a line through* 0,
(b) *a union of parallel lines, containing* 0,
(c) *a bounded convex symmetric neighborhood of* 0.

Lemma 2.2. *Let B be a closed subset of $\mathbb{R}^2$ such that for any vertical line $x = c$ there is a $v \in \mathbb{R}^2$ such that projecting affinitely onto $x = c$ along $\mathbb{R}v$ takes B to B. Then B is either*

(a) *a union of lines, all parallel, or*
(b) *the epigraph of a convex function $h : \mathbb{R} \to \mathbb{R}$, or the negative of such a set.*

Proof: One possibility is that B is empty. Otherwise, we consider two cases, depending on whether $\mathrm{cocl}(B)$ is equal to $\mathbb{R}^2$ or not.

(a) <u>$K = \mathrm{cocl}(B) \neq \mathbb{R}^2$</u>. Suppose u is an extreme point of K. We claim $u \in B$. For if not, take $B(u,r) \subseteq B'$, $r > 0$, with $\partial B(u,r)$ intersecting ∂K in two points u and w, noting $K \neq \{u\}$ since B intersects every vertical line. Now $u \notin \mathrm{aff}\, v, w$, since it is extreme, so u is in the open half space given by $\mathrm{aff}\{v, w\}$which does not intersects B. This contradicts $u \in \mathrm{cocl}(B)$.

Suppose $(a, b) \in \mathbb{R}^2$ is a point in ∂K. To fix ideas, suppose $c < b$ implies $(a, c) \notin K$, by relabelling the y axis. Suppose there is a nonempty open interval $(e, f) \subseteq (b, \infty)$ with $(a, g) \notin B$, if $g \in (e, f)$. Then projecting onto $\{(x, y) : x = a\}$ along a line of slope $\alpha(a)$ gives the open strip $\{(x, y) \in \mathbb{R}^2 : y \in (e, f) + \alpha(a)(\alpha - a)\} \subseteq B'$.

Suppose for the purpose of obtaining a contradiction that this intersects ∂K. Points in the intersection must be nonextreme points, giving a nonempty open line interval in $\partial K \cap B'$, having slope β say. Taking $(p, q) \in \mathbb{R}^2$ in this interval, a projection onto $x = p$ taking B to B must be along the line with slope β. But there is an end of the closed line segment in ∂K with slope β which must be an extreme point, hence in B, and which projects onto (p, q), a contradiction.

Hence either ∂K has slope $\alpha(a)$, or $(a, c) \in B$ for all $c > b$. In the first case, projecting onto any line $x = c$, taking B to B, must take ∂K to ∂K and be along the line slope $\alpha(a)$, giving B as the union of lines with slope $\alpha(a)$. In the second case, B being closed is equal to K, which is the epigraph of a convex function from $\mathbb{R}$ to $\mathbb{R}$. Without our assumption that the lower half of $x = a$ was in B' we could reverse the direction of the y axis to give B as the negative of such an epigraph.

(b) <u>$\mathrm{cocl}(B) = \mathbb{R}^2$</u>. If a whole vertical line is in B, then $B = \mathbb{R}^2$. Suppose now that for all $c \in \mathbb{R}$, if $S(c) = \{y : (c, y) \in B\}$ then $S(c) \neq \mathbb{R}$. Note for all c, $S(c)$ is not bounded above or below. We have for all $c, \alpha(c)$ such that for all d,

$$S(d) + \alpha(c)(c - d) \subseteq S(c). \tag{1}$$

We take two cases, depending on whether α is either nondecreasing or nonincreasing, or not. If α is nonincreasing, by renaming we may assume it is nondecreasing.

(b1) <u>α is nondecreasing</u>. We define $p(x) = \int_0^x \alpha(x)\, dx$, which gives the epigraph H of p of a closed convex set such that for all c and d, $S(d)+\alpha(c)(c-d) \subseteq S(c)$.

Since $S(c) \neq \mathbb{R}$ and $S(c)$ is not bounded above or below for all c, $S(c)$ has more than one component, so that there is a bounded open interval (d, e) in $S(c)'$, with the points (c, d) and (c, e) in B. Let H_b be a vertical translate of H with $(c, d) \in H_b$. Now $H_b \cap B$ is invariant under projections onto lines $x = c$ along lines with slope $\alpha(c)$, and by (a), since $(c, (d + e)/2) \notin B, H_b \cap B$ is a union of lines, with slope α

say. Thus the line through (c,d) with slope α is in ∂K, and so $\alpha(d)=\alpha$ for all d. Hence, by (1), since $S(d)+\alpha(c-d)\subseteq S(c)$ and $S(c)+\alpha(d-c)\subseteq S(d)$, we have $S(d)+\alpha(c-d)=S(c)$ and B is a union of lines with slope α.

(b2) There are $z,y,w\in\mathbb{R}$, $z<y<w$, such that $\alpha(z)>\alpha(y)<\alpha(w)$. (If we had $\alpha(z)<\alpha(y)>\alpha(w)$, we could relabel the y axis to obtain this assumption.) By (1), $S(w)+\alpha(y)(y-w)\subseteq S(y)$, and $S(y)+\alpha(w)(w-y)\subseteq S(w)$, so $S(y)+(\alpha(w)-\alpha(y))(w-y)\subseteq S(y)$. Let $x_1=(\alpha(w)-\alpha(y))(w-y)>0$. Let $x_2=(\alpha(z)-\alpha(y))(z-y)>0$. We have two cases; x_1/x_2 is rational or irrational.

(b2a) $x_1/x_2\in\mathbb{Q}$. Let $x_1=kd, x_2=hd$, $k,h\in\mathbb{N}$, $d>0$. Then $s(y)-khd\subseteq S(y)$ and $S(y)+khd\subseteq S(y)$. Hence the map $x\to x+khd$ is onto $S(y)$, since $x\in S(y)$ gives $x=(x-khd)+(khd)$. Now let $g:S(y)\to S(w)$ be given by $z=g(z)+\alpha(y)(w-y)$, and let $f:S(w)\to S(y)$ be given by $x=f(x)+\alpha(w)(y-w)$. The map $x\to x+khd$ is the composite $(f\circ g)^k$, so g and f are bijections,

$$S(w)=S(y)+\alpha(y)(w-y). \tag{2}$$

(b2b) $x_1/x_2=\alpha\notin\mathbb{Q}$. There are sequences n_i,m_i in $\mathbb{N}$ with $|n_i\alpha-m_i|\le\frac{1}{n_i}$. So $y\to y+\alpha x_2$ and $y\to y-x_2$ take $S(y)$ to $S(y)$. Hence for $y\in S(y), y_i=y-m_ix_2+(n_i-1)x_1\in S(y)$ and $y_i\to y-x_1$, giving $y-x_1\in S(y)$ since $S(y)$ is closed. Hence, as in (b2a), the map $g:S(y)\to S(w)$ is a bijection, or $S(w)=S(y)+\alpha(y)(w-y)$, so (2) holds for all x_1 and x_2. We either have: (c) for all $z<y,\alpha(z)>\alpha(y)$, or (d) there is $z_0<y,\alpha(z_0)<\alpha(y)$. In case (d) we have for all $w>y$, (2) holds, by using z above, if $\alpha(w)>\alpha(y)$ and z_0, if $\alpha(w)<\alpha(y)$, and noting (2) holds, if $\alpha(w)=\alpha(y)$. And in case (c), we replace (2) by $S(z)=S(y)+\alpha(y)(w-y)$ for all $z<y$. In case (d), we have $\alpha(w)=\alpha(y)$ for $w>y$ and in case (c) we have $\alpha(z)=\alpha(y)$ for all $z<y$, a contradiction to (b2). □

3. Three dimensional results.

Lemma 3.1. *Let B be a closed subset of $\mathbb{R}^3$, N a two dimensional subspace, $d\in N, d\neq 0$. Suppose any plane M containing 0 but not $\mathbb{R}d$ is the range of a projection P with $P(B)\subseteq B$ and $P(N)\subseteq N$. Then B is a union of translates of $\mathbb{R}d$, or $B\subseteq N$.*

Proof: Let $b\in B\setminus N$. Any line m in $b+N$ not parallel to $\mathbb{R}d$ is the range of an affine projection in $b+N$. By Lemma 1.2, $B\cap(b+N)$ is a union of parallel lines or a convex set $K_b\neq b+B$ intersecting every translate of m in $b+N$. Supposing the latter and not the former, we have a contradiction by taking m to be a supporting line to K_b not parallel to $\mathbb{R}d$. Hence $B\cap(b+N)$ is a union of translates of a line k in $b+N$. If k is not parallel to $\mathbb{R}d$ and $B\cap(b+N)\neq b+N$, we may take a translate of k contained in the complement of B in $b+N$, to obtain a contradiction. □

The following result of Blaschke is proved simply in [2, Lemma 1] except that p is assumed to be a norm.

Lemma 3.2. *Let X be a real three dimensional normed space with the basis $\{e_1, e_2, e_3\}$, where e_i is a unit vector. Suppose every two dimensional subspace which contains e_1 is the range of a nonexpansive projection along a vector in span $\{e_2, e_3\}$. Then there is a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that for all $x_i \in \mathbb{R}$, $\|x_1e_1+x_2e_2+x_3e_3\| = F(x_1, \|x_2e_2 + x_3e_3\|)$.*

Theorem 3.3. *Let B be a closed neighborhood of 0 in $\mathbb{R}^3$. For all planes M through 0, there exists a linear projection P of $\mathbb{R}^3$ onto M with $P(B) \subseteq B$ iff B is one of:*

(a) *the closed unit ball given by an inner product,*
(b) *a union of parallel planes,*
(c) *$K + \mathbb{R}v$, where K is a bounded convex symmetric neighborhood of 0 in a plane M through 0 and $\mathbb{R}v$ is a line not in M.*

Proof: We let $C = \mathrm{cocl}(B)$ and consider four distinct cases:

(i) C contains no lines,
(ii) C contains a line but no planes,
(iii) C contains a plane by not $\mathbb{R}^3$,
(iv) $C = \mathbb{R}^3$.

(i) Let $D = C \cap -C$. Then D is a closed convex bounded symmetric neighborhood of 0, invariant under projections onto all 2 dimensional subspaces, and hence the unit ball given by an inner product, by the Blaschke–Kakutani theorem.

Take any 2 dimensional subspace M, and consider $\partial D \cap M$ and $\partial C \cap M$. Let $\mathbb{R}e$ be perpendicular to M under the inner product. Any plane through $\mathbb{R}e$ is the range of a projection taking C to C, hence D to D, hence is along a direction in M. We can parametrize $\partial D \cap M$ and $\partial C \cap M$ to give radius $d(\theta)$ and $c(\theta)$ say as functions of angle θ; these functions are absolutely continuous and their derivative is equal for angles, where $d(\theta)$ and $c(\theta)$ have a unique tangent, i.e. almost everywhere. Hence, if $d(\theta)$ and $c(\theta)$ are equal to θ_0, they are equal near θ_0, and $M \cap \partial C \cap \partial D$ is open in $M \cap \partial D$. Since $M \cap \partial C \cap \partial D$ is also closed in $M \cap \partial D$, and nonempty, and $M \cap \partial D$ is connected, $M \cap \partial C = M \cap \partial D$. Hence $C = D$.

We claim $B = D$. If $x \in \partial D$, but $x \notin B$, then $x \notin \mathrm{cocl}(B)$, a contradiction, giving $\partial D \subseteq B$. If $x \in \mathrm{int}(D)$, take P a projection onto M, a 2 dimensional subspace containing x, with $P(B) \subseteq B$. Then $x \in P(\partial D) \subseteq B$. Hence $D \subseteq B$, giving $B = D$.

(ii) C may be represented as $K + \mathbb{R}v$, where K is a closed convex set, not containing a line, in a plane M, and $v \notin M$. All projections onto planes not containing $\mathbb{R}v$ are along $\mathbb{R}v$, so $B \setminus \mathbb{R}v$ is a union of lines parallel to $\mathbb{R}v$. Let $B_1 = B \cup \mathbb{R}v$. Now in $\mathbb{R}^3/\mathbb{R}v$, we have all lines through 0 being the range of a projection taking the quotient $B_1/\mathbb{R}v$ to itself.

By Lemma 1.1 and our hypotheses, it must be a closed bounded convex symmetric neighborhood of 0. Hence $B_1 = K + \mathbb{R}v$, with K a closed bounded symmetric convex neighborhood of 0 in M, $v \notin M$. Hence, $\mathbb{R}v \subseteq B$, and $B = K + \mathbb{R}v$.

(iii) Let N be a plane through O with a translate of N contained in c. Now any plane M through O, $M \neq N$, is the range of projection along a direction in N.

Hence for $b \in B \setminus N$, any line $b + N$ is the range of an affine projection in $b + N$ taking B to B.

By Lemma 1.2, $B \cap (b + N)$ is a convex set not equal to $b + N$ but meeting all lines, which is impossible, or is a union of parallel lines. Hence $b + N \subseteq B$.

(iv) We assume B is not a union of parallel lines.

(a) We claim that for any line $\mathbb{R}w, w \neq 0$, and any $M \in \mathbb{R}, B$ intersects $[M, \infty)w$. For, take a plane $\mathbb{R}w + \mathbb{R}v$, and project onto it along u. Suppose we project onto $\mathbb{R}w + \mathbb{R}u$ along y. B intersects $[M, \infty)w + \mathbb{R}y + \mathbb{R}u$. Projecting onto $\mathbb{R}w + \mathbb{R}u$ gives $[M, \infty)w$ intersecting B.

(b) Since $B \neq \mathbb{R}^3$, take $a \in B', a \neq 0$. Take a plane N through $\mathbb{R}a$, and project along b, so $B(a, \delta) + \mathbb{R}b \subseteq B'$. Take the plane $\mathbb{R}a + brb$ and project along c onto it. For $\delta > 0$ small, $B(a, \delta) + \mathbb{R}b + \mathbb{R}c \subseteq B'$. Let us call the set between two parallel planes a "slice".

(c) We claim there is a basis (f_1, f_2, f_3) and a nonempty open ball $B(c, \delta)$ with the three slices $B(c, \delta) + \mathbb{R}f_1 + \mathbb{R}f_2, B(c, \delta) + \mathbb{R}f_2 + \mathbb{R}f_3, B(c, \delta) + \mathbb{R}f_1 + \mathbb{R}f_3$ all contained in B'. Since we are assuming B not a union of parallel lines, take the slice $B(a, \delta) + M \subseteq B', \delta > 0, M$ on a plane through 0 and by Lemma 3.1 take $N \neq M$ a plane through 0 with projection along $r \notin M$. By Lemma 3.1, take Q another plane through 0, not containing $N \cap M$, with projection along $s \notin M$. Let c be the point of intersection of $a + M, N$ and Q. We take the three planes through c: $c + M, c + \mathbb{R}r + (M \cap N), c + \mathbb{R}s + (M \cap Q)$. These are all contained in B', together with slices containing them, and the intersection is $\{c\}$. Together they give f_i as required.

(d) We claim there is a sequence of projections P_n onto planes through 0 with $\|P_n\| \to \infty$. Assume by renaming that c is the positive octant. For $\delta > 0$, let $f_\delta = f_3^* - \delta(f_1^* + f_2^*)$, where (f_1^*, f_2^*, f_3^*) is the dual basis to (f_1, f_2, f_3).

Suppose there is $\delta > 0$ with $\{x = (f_\delta, x) \geq 0\} \cap B \cap \{x : x_1 \geq c_1, x_2 \geq c_2, x_3 \leq c_3\}$ nonempty. Then by compactness there is a maximal such $\delta, d(\max)$, and an $e \in B$ with $(f_{\delta(\max)}, e) = 0, e_1 \geq c_1, e_2 \geq c_2, e_3 \leq c_3$. For $\delta > \delta(\max)$ there is no such e. If there is no $\delta > 0$, take $\delta(\max) = 0$ and in this case by (a) there is $e \in B$ with $e_1 \geq c_1, e_2 \geq c_2$ and $e_3 = 0$.

Let $\delta(n) \to \delta(\max)^+$ and let P_n be a projection on $N(f_{\delta(n)})$. If $P_{n(m)}$ is a bounded subsequence, then $P_{n(m)}e \to e$, giving $P_{n(m)}e$ in B, with $(P_{n(m)}e)_1 \geq c_1, (P_{n(m)}e)_2 \geq c_2, (P_{n(m)}e)_3 \leq c_3$, contradicting the maximality of $\delta(\max)$. Hence $\|P_n\| \to \infty$.

(e) We derive a contradiction, showing B is a union of parallel lines. Since $\|P_n\| \to \infty$, and $P_n(B)$ contains the symmetric convex set $P_n B(0, \varepsilon)$ for some $\varepsilon > 0$, we have $P_n(B)$ intersecting $c + \mathbb{R}f_i + \mathbb{R}f_j$ for n large, for some i and j.

(f) We claim B is a union of parallel planes. Since B is a union of parallel lines, there is $q \neq 0$, so B is a union of translates of $\mathbb{R}q$. By 2.1 applied to $\mathbb{R}^3/\mathbb{R}q$, we have $B/\mathbb{R}q$ a union of parallel lines, since its convex closure is $\mathbb{R}^3/\mathbb{R}q$, and it is a neighborhood of 0. This gives B a union of parallel planes. $\square$

4. Higher dimensions.

Theorem 4.1. *Suppose B is a closed neighborhood of 0 in a real locally convex topological vector space X of dimension ≥ 3. For all two dimensional subspaces M there is a continuous linear projection P of X onto M with $P(B) \subseteq B$, iff B is the inverse image under a continuous linear map T of:*

(a) *the closed unit ball in an inner product H,*
(b) *the closed unit ball given by a norm on $\mathbb{R}^2$, or*
(c) *a closed neighborhood of 0 in $\mathbb{R}$.*

Proof: $\Longrightarrow$ (1) We suppose that for al 3 dimensional subspaces F of X $F \cap B$ is a union of parallel planes. We claim B is a union of parallel closed hyperplanes, so (c) holds.

For H a closed subspace of codimension ≥ 2 with $H \subseteq B$, we claim there is a closed subspace H_{+1} with $H_{+1} \subseteq B$ and H of codimension 1 in H_{+1}. Let H_{-1} be a closed subspace of H of codimension 1 and let E be a three dimensional subspace of X with $E \cap H_{-1} = \{0\}$. Let M be a two dimensional subspace of E contained in B. Given $h \in H_{-1}, h \neq 0, (\mathbb{R}h + M) \cap B$ is a union of translates of M, so $h + M \subseteq B$, giving $H_{-1} + M \subseteq B$. Take $H_{+1} = H_{-1} + M$. By Zorn's lemma, a closed subspace H of codimension ≤ 1 in X is contained in B. If $x \in B \setminus H$, and $h \in H$, let E be a three dimensional space containing x, h and a two dimensional subspace M of H. Then $x + M \subseteq B$, giving $x + h \in B$. Thus for $x \in B, x + H \subseteq B$, and the claim is proved, $B = \cup\{x + H : x \in B\}$.

(2) We now suppose there exists a 3 dimensional subspace F_0 such that $F_0 \cap B$ contains no plane, and we suppose that for all three dimensional subspaces $F, F \cap B$ contains a line. We claim B is convex, contains a 2 codimensional closed subspace E, and with E_2 a complementary subspace, $B \cap E_2$ is a bounded symmetric neighborhood of 0 in E_2. We take $E_2 \subseteq F_0$ with $E_2 \cap B$ a bounded symmetric neighborhood K of 0. Let $e \in E_2, e \neq 0, B \cap \mathbb{R}e = \{\lambda e : |\lambda| \leq 1\}$.

B is convex since if $a, b \in B$, we take a 3 dimensional space G, containing a, b and e, and note that if $B \cap G$ is a union of planes, it is of the form $M + \lambda e, |\lambda| \leq 1$, and hence $B \cap G$ is convex.

Let H be a closed subspace of X, of codimension > 2, with $H \subseteq B$. Take $f \notin E_2 + H$. Now $(E_2 + \mathbb{R}f) \cap B$ contains a line $\mathbb{R}e$ say, giving $B \supseteq H + \mathbb{R}e$ since it is closed and convex. Hence, by Zorn's lemma there is a closed subspace E of codimension 2 with $E \subseteq B$.

Since B is closed and convex, $K + E \subseteq B$. Let $b \in B$, with $b = b_2 + b_e, b_2 \in E_2, b_e \in E$. We claim $b_2 \in K$. If $b_E \neq 0, B \cap (E_2 + \mathbb{R}b_E)$ is projected onto E_2 taking B to B, hence along b_E, and $b_2 \in K = B \cap E_2$. Thus $B = K + E$, giving (b).

(3) We suppose there exists a three dimensional subspace E_0 such that $E_0 \cap B$ contains no line. Now as in (2) we find B is convex, and the same idea gives B symmetric. By Zorn's lemma, there is a maximal closed subspace $E \subseteq B$. Let $Q : X \to X/E$ be the projection. We see $Q(B)$ is convex, symmetric, and radial. If p is its Minkowski functional, by maximality of E, if $p(Qx) = 0$, then $\mathbb{R}x \in B$ and $x \in E$, so p is a norm.

We claim p is given by an inner product, by the Blaschke–Kakutani theorem. Let M be a 2 dimensional subspace of x/E and take N a two dimensional subspace of X with $QN = M$. Let R be a continuous projection of X onto N with $R(B) \subseteq B$. We define $P : X/E \to M$ by $P(Qx) = QR(x)$; this is well defined for if $Qx = 0$, then $x \in E$ giving $Rx \in E$ and $QRx = 0$. We see P maps $X/E \to M$ and is the identity on M and maps $Q(B)$ to $Q(B)$. Hence $Q(B)$ is the closed unit ball in an inner product space, $Q : X \to X/E$ is continuous and linear, and $B = Q^{-1}(Q(B))$ giving (a).

$\Longleftarrow$ (a) Suppose (a) holds. Let M be a 2 dimensional subspace of X.

(i) Let TM be a 2 dimensional subspace of H. Let R be the projection on TM under which the unit ball $B[0, 1]$ in H is invariant. Let $T\,|_M$ be the restriction, and define $P = (T\,|_M)^{-1}RT$. One checks P takes X to M, is the identity on M, is a continuous linear map and maps $B = T^{-1}(B[0, 1])$ to itself.

(ii) Let TM be a 1 dimensional subspace of H. Take (e_1, e_2) a basis of $M, Te_1 = 0$. Let $S : X \to M$ be a continuous projection, $Sx = x_1(x)e_1 + x_2(x)e_2$. Define $Px = (T\,|_{\mathbb{R}e_2})^{-1}RTx + x_1(x)e_1$, where R is the projection on TM leaving $B[0, 1]$ invariant.

(iii) Let TM be 0 dimensional. Let $S : X \to M$ be as in (ii) and take $P = S$.

(b) Suppose (b) holds. Let M be a 2 dimensional subspace of X. Let $T : X \to \mathbb{R}^2$ be given, $B[0, 1]$ the unit ball in $\mathbb{R}^2$, and $B = T^{-1}B[0, 1]$.

(i) Let $TM = \mathbb{R}^2$. Define $P = (T\,|_M)^{-1}T$.

(ii) Let TM be 1 dimensional. Let R be the projection on $\mathbb{R}^2$ of TM leaving $B[0, 1]$ invariant, and define P as in (a)(ii).

(iii) Let TM be 0 dimensional. Define P as in (a)(iii).

(c) Suppose (c) holds. Let M be a two dimensional subspace of !X. Let $T : X \to \mathbb{R}^2$ be given, A a closed neighborhood of 0 in $\mathbb{R}$ and $B = T^{-1}(A)$.

(i) Let $T(\mathbb{R}m) = \mathbb{R}, m \in M$. Define $P = (T\,|_{\mathbb{R}m})^{-1}T$.

(ii) Let $T(M) = 0$. Define P as in (a)(ii).

□

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