Sigma order continuity and best approximation in L_ϱ-spaces

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Abstract. In this paper we give a characterization of σ-order continuity of modular function spaces L_ϱ in terms of the existence of best approximants by elements of order closed sublattices of L_ϱ. We consider separately the case of Musielak–Orlicz spaces generated by non-σ-finite measures. Such spaces are not modular function spaces and the proofs require somewhat different methods.

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Introduction.

The notion of the σ-order continuity plays a central role in the theory of spaces of measurable functions. Many of the properties of these spaces depend on the size of the subspace consisting of those functions having σ-order continuous norms or F-norms. These properties include the existence and the form of linear functionals, reflexivity, uniform convexity, the relationships among different types of convergence, the density of the simple functions, separability and so on. An enormous amount of literature relating to these topics is available; see e.g. [4]–[16].

In modular function spaces, the σ-order continuity of $\|\cdot\|_\varrho$ can be characterized by the Δ_2-condition. See [4]. In special cases of Orlicz spaces and their generalizations, various formulations of the Δ_2-condition have been studied since the 1930's. See e.g. [2,], [7], [12], [13].

Many of the properties of L^p-spaces derive from the fact that L^p-spaces have a σ-order continuous norm. In general, σ-order continuous spaces of measurable functions have a structure similar to that of L^p-spaces and enjoy many of these properties, for example, analogues of Lebesgue's and Vitali's convergence theorems hold. See e.g. [4]. In [1], [14], several results showing the existence of best approximants by elements of closed sublattices of L^p-spaces were presented. In standard Orlicz spaces L^φ, with L^φ convex, [8] showed that the existence of best approximants is closely related to the Δ_2-condition. It is therefore natural to expect that in more general situations, σ-order continuity should be characterized by the existence of best approximants by elements of closed sublattices.

In modular function spaces, [3] showed the existence of best aproximants in order closed sublattices of L_ϱ-spaces and most of our present results will be proved in the same context. However, the interesting special case of Orlicz spaces, the so called Musielak–Orlicz spaces, is covered by general results only in the σ-finite case. Therefore in Section 2, we sketch some of the results necessary to adapt our general methods to the non-σ-finite case.

Preliminaries.

We start with some definitions and basic facts. For proofs and details see [3]–[6].

Let X be a nonempty set, Let Σ be a σ-algebra of subsets of X and let $\mathcal{P} \subset \Sigma$ be a δ-ring such that $E \cap A \in \mathcal{P}$ whenever $E \in \mathcal{P}$ and $A \in \Sigma$, and such that there exists a nondecreasing sequence of sets $\{X_k\}_1^\infty \subset \mathcal{P}$ with $X = \bigcup_{k=1}^\infty X_k$. By $\mathcal{E}$ we mean the set of all simple functions of the form $s = \Sigma_{k=1}^n \alpha_k 1_{A_k}$, where each $A_k \in \mathcal{P}$ and each $\alpha_k \in \mathbb{R}$. A mapping $\varrho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a *function semimodular*, if it satisfies the following properties:

(1) $\varrho(0, A) = 0$ for each $A \in \Sigma$.
(2) $\varrho(f, A) \le \varrho(g, A)$, if $|f| \le |g|$ on $A \in \Sigma$.
(3) $A \mapsto \varrho(f, A) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for each $f \in \mathcal{E}$.
(4) $\varrho(\alpha, A) \to 0$ whenever $\alpha \to 0$ for every $A \in \mathcal{P}$. (Here α denotes the constant function with value α.)
(5) $\varrho(\alpha, A_n) \to 0$ for every $\alpha \in \mathbb{R}$ whenever $A_n \downarrow \Phi$ and $\{A_n\}_1^\infty \subset \mathcal{P}$.
(6) There exists $\alpha_0 \ge 0$ such that $\varrho(\beta, A) = 0$ for every $\beta \in \mathbb{R}$ whenever $A \in \mathcal{P}$ and $\varrho(\alpha, A) = 0$ for some $\alpha > \alpha_0$.

A function semimodular ϱ satisfying (6) above with $\alpha_0 = 0$ is called a *function modular*. The definition of ϱ is then extended to M, the set of all real-valued measurable functions f, and to all $E \in \Sigma$ by defining that

$$\varrho(f, E) = \sup\{\varrho(g, E) : g \in \mathcal{E} \text{ and } |g| \le |f| \text{ on } E\}.$$

For the sake of simplicity, $\varrho(f)$ is written in place of $\varrho(f, X)$.

Let ϱ be a function semimodular on $(X, \Sigma, \mathcal{P})$. We define $m_\varrho = \sup\{\varrho(g) : g \in M\} \in [0, \infty]$, and for each $f \in M$, $\beta_f = \sup\{\beta \ge 0 : \varrho(\beta f) < m_\varrho\} \in [0, \infty]$. The function $r_f : [0, \beta f] \to [0, \infty]$ is defined by $r_f(t) = \varrho(tf)$ and $\mathcal{R}_s$, respectively $\mathcal{R}_m$, is the set of all nonzero function semimodulars, respectively function modulars, ϱ such that for every $f \in M$, r_f is continuous. If we assume that $\varrho \in \mathcal{R}_s$ or $\mathcal{R}_m$, it follows immediately that ϱ is a left continuous semimodular and therefore has the Fatou property; that is when each $f_n \ge 0$, $\varrho(\liminf_n f_n) \le \liminf_n \varrho(f_n)$.

The set of functions,

$$L_\varrho = \{f \in M : \varrho(\lambda f) \to 0 \text{ as } \lambda \to 0\},$$

forms a vector subspace of M and is denoted as a *modular function space*.

A set $E \in \Sigma$ is called *ϱ-null*, if $\varrho(\alpha, E) = 0$ for each $\alpha > 0$. We say that ϱ has *property* (K), if ϱ satisfies

$$\sup_{f \in L_\varrho} \varrho(f, X) = \sup_{f \in L_\varrho} \varrho(f, E),$$

whenever E is not ϱ-null. See [3, Definition 1.19].

Let ϱ be given by

$$\varrho(f, A) = \int_A \varphi(x, f(x))\, d\mu(x),$$

where μ, a measure on X, and $\varphi : X \times \mathbb{R} \to [0, \infty)$ satisfy the following:

(1) $u \mapsto \varphi(x, u)$ is a nondecreasing continuous even function such that $\varphi(x, 0) = 0, \varphi(x, u) > 0$ for $u \neq 0$, and $\varphi(x, u) \to \infty$, as $u \to \infty$.

(2) $x \mapsto \varphi(x, u)$ is a locally integrable function for each $u \in \mathbb{R}$; that is a measurable function such that $\int_A \varphi(x, u)\, d\mu(x) < \infty$ when $u \in \mathbb{R}$ and $\mu(A) < \infty$.

In this case ϱ is called a *Musielak–Orlicz modular*. If we define $\mathcal{P}$ to be σ-ring of all sets of finite measure then the corresponding Musielak–Orlicz modular is a function modular if and only if μ is σ-finite. If μ is not σ-finite, then $L^\varphi(X, \Sigma, \mathcal{P}, \mu)$ is not a modular function space, since X cannot be represented as a countable union of sets from $\mathcal{P}$.

If ϱ is a function semimodular or a Musielak–Orlicz modular, then the formula

$$\|f\|_\varrho = \inf\{\alpha > 0 : \varrho(f/\alpha) \le \alpha\}$$

defines an F-norm under which the metric space L_ϱ, with $d(f, g) = \|f - g\|_\varrho$ is complete. Moreover, if ϱ is a semimodular, $\|\cdot\|_\varrho$ is a function modular; that is ξ defined by

$$\xi(f, A) = \|f 1_A\|_\varrho$$

is a function modular, and $L_\xi = L_\varrho$.

We understand the *ϱ-distance*, respectively the *$\|\cdot\|_\varrho$-distance*, from an $f \in L_\varrho$ to a set $D \subset L_\varrho$ to be

$$\operatorname{dist}_\varrho(f, D) = \inf\{\varrho(f - h) : h \in D\},$$

respectively

$$\operatorname{dist}_{\|\cdot\|_\varrho}(f, D) = \inf\{\|f - g\|_\varrho : g \in D\}.$$

The set of all *best ϱ-approximants*, respectively *best $\|\cdot\|_\varrho$-approximants* of f, with respect to D will be denoted by

$$P_\varrho(f, D) = \{g \in D : \varrho(f - g) = \operatorname{dist}_\varrho(f, D)\},$$

respectively

$$P_{\|\cdot\|_\varrho}(f, D) = \{g \in D : \|f - g\|_\varrho = \operatorname{dist}_{\|\cdot\|_\varrho}(f, D)\}.$$

If D satisfies $P_\varrho(f, D) \neq 0$, respectively $P_{\|\cdot\|_\varrho}(f, D) \neq \Phi$, for every $f \in L_\varrho$, we say D is *ϱ-proximinal*, respectively *$\|\cdot\|_\varrho$-proximinal*.

Definition 0.1.

(a) A set function $\eta : \Sigma \to [0, \infty]$ is called *order continuous* if whenever $A_k \downarrow \Phi, \eta(A_k) \to 0$.

(b) $E_\varrho = \{f \in M : E \mapsto \varrho(\alpha f, E)$ is order continuous on $\Sigma \,\forall\, \alpha > 0\} =$
$= \{f \in M : \|f 1_{A_k}\|_\varrho \to 0$ as $A_k \downarrow \Phi\}$.

(c) $\|\cdot\|_\varrho$ it is σ-order continuous if $\|f_n\|_\varrho \to 0$ whenever $f_n \downarrow 0$.

E_ϱ is a closed subspace of L_ϱ having some properties similar to those of L^p. For instance, analogues of Lebesgue's dominated convergence theorem and Vitali's theorem hold. See [4]. The following simple result is well known for function semimodulars. We will include a proof that applies Musielak–Orlicz modulars as well.

Theorem 0.2. *If $\varrho \in \mathcal{R}_s$ or if ϱ is a Musielak–Orlicz modular, then $E_\varrho = L_\varrho$, if and only if $\|\cdot\|_\varrho$ is σ-order continuous.*

PROOF: $\Rightarrow$ Suppose that $L_\varrho = E_\varrho$. Let $\{f_n\}_1^\infty \subset L_\varrho$be such that $f_n \downarrow 0$-a.e. By the Lebesgue dominated convergence theorem, using f_1 as the dominating function from E_ϱ, we obtain that $\|f_n\|_\varrho \to 0$.

$\Leftarrow$ Let $f \in L_\varrho$ and let $A_k \downarrow \Phi$. Then $|f1_{A_k}| \downarrow 0\varrho$-a.e. and by (b), $\|f1_{A_k}\|_\varrho \to 0$, showing $f \in E_\varrho$. $\square$

The proof of the following theorem is identical to the proof of Theorem 4.6 in [3]. This proof makes no use of σ-finiteness and hence applies to Musielak–Orlicz modulars as well as to $\varrho \in \mathcal{R}_s$. Recall that $C \subset L_\varrho$ is *order closed* in L_ϱ, if from $f_n \in C$ and $f_n \uparrow f \in L_\varrho$, (or $f_n \downarrow f \in f \in L_\varrho$), it follows that $f \in C$.

Theorem 0.3. *Let $\varrho \in \mathcal{R}_s$ be orthogonally additive and have the property (K) or let ϱ be a Musielak–Orlicz modular. If $C \subset L_\varrho$ is a nonempty order closed lattice, then C is $\|\cdot\|_\varrho$-proximinal.*

Theorem 0.4. *Let $\varrho \in \mathcal{R}_s$ or let ϱ be a Musielak–Orlicz modular. If $C \subset L_\varrho$ is $\|\cdot\|_\varrho$-proximinal, then C is $\|\cdot\|_\varrho$–closed.*

PROOF: Let $\{g_n\}_1^\infty \subset C$, and let $g \in L_\varrho$ be such that $\|g_n - g\|_\varrho \to 0$. Then $\mathrm{dist}_{\|\cdot\|_\varrho}$ (g,C)=0. Since C is $\|\cdot\|_\varrho$-proximinal, there exists $h \in P_{\|\cdot\|_\varrho}(g, C)$, which implies that $\|g - h\|_\varrho = 0$. Therefore $g = h \in C$, completing the proof. $\square$

Section 1.

In this section, we give several characterizations of those modular function spaces L_ϱ in which E_ϱ is all of L_ϱ; that is, in which $\|\cdot\|_\varrho$ is σ-order continuous. These characterizations include properties of the sublattices of L_ϱ and the existence of best approximants by elements of those sublattices. In order to keep our notation to a minimum, we make the following definitions:

Definition 1.1. (a) $\mathfrak{L}$ denotes the family of all sublattices of L_ϱ.
(b) $\mathfrak{L}_0$ denotes the family of all order closed lattices in $\mathfrak{L}$.
(c) $\mathfrak{L}_n$ denotes the family of all F-norm closed lattices in $\mathfrak{L}$.
(d) $\mathfrak{L}_s$ denotes $\{C \in \mathfrak{L} : \bigvee_{k=1}^\infty g_k \in C$, whenever $\{g_k\}_1^\infty \subset C\}$.
(e) $\mathfrak{L}_i$ denotes $\{C \in \mathfrak{L} : \bigwedge_{k=1}^\infty g_k \in C$, whenever $\{g_k\}_1^\infty \subset C\}$.
(f) $\mathfrak{L}_\uparrow$ denotes $\{\{g_k\}_1^\infty \subset L_\varrho : g_1 \le g_2 \le \cdots \le g_k \le \dots\}$.
(g) $\mathfrak{L}_\downarrow$ denotes $\{\{g_k\}_1^\infty \subset L_\varrho : g_1 \ge g_2 \ge \cdots \ge g_k \ge \dots\}$.
Observe that $\mathfrak{L}_\uparrow \subset \mathfrak{L}_i$ and $\mathfrak{L}_\downarrow \subset \mathfrak{L}_s$.

Definition 1.2. Let $\mathcal{C} \subset \mathfrak{L}$. We say that $\mathcal{C}$ has the *existence property*, if every order closed $C \in \mathcal{C}$ is $\|\cdot\|_\varrho$-proximinal.

Remark 1.3. If ϱ is orthogonally additive and has the property (K), then $\mathfrak{L}$ has the existence property. See [3, Theorem 4.6].

Remark 1.4. If X is countable and ϱ has the property (K), then $\mathfrak{L}_s \cup \mathfrak{L}_i$ has the existence property. See [3, Theorem 3.6].

We now present the main theorem of this section.

Theorem 1.5. *Let $\varrho \in \mathcal{R}_s$, let $\mathcal{C} \subset \mathfrak{L}$ have the existence property and suppose that either $\mathfrak{L}_\uparrow \subset \mathcal{C}$ or $\mathfrak{L}_\downarrow \subset \mathcal{C}$. Then the following statements are equivalent:*

(a) $L_\varrho = E_\varrho$.
(b) $\|\cdot\|_\varrho$ *is σ-order continuous.*
(c) $\mathcal{C} \cap \mathfrak{L}_0 = \mathcal{C} \cap \mathfrak{L}_n$.
(d) C *is $\|\cdot\|_\varrho$-proximinal for every* $C \in \mathcal{C} \cap \mathfrak{L}_n$.
(e) $P_{\|\cdot\|_\varrho}(f, C) \neq \Phi$ *for every* $f \in E_\varrho$ *and for every* $C \in \mathcal{C} \cap \mathfrak{L}_n$.

Proof: (a) $\Leftrightarrow$ (b). This is Theorem 0.2.
(b) $\Rightarrow$ (c). Let $C \in \mathcal{C} \cap \mathfrak{L}_n$, and suppose that $\{f_n\}_1^\infty \subset C$ with $f_n \uparrow f \in L_\varrho$, (respectively $f_n \downarrow f \in L_\varrho$). Then $(f - f_n) \downarrow 0$, (respectively $(f_n - f) \downarrow 0$). Hence by σ-order continuity, $\|f - f_n\|_\varrho \to 0$. Since C is F-norm closed, $f \in C$. This shows that $C \in \mathfrak{L}_0$ and hence that $\mathcal{C} \cap \mathfrak{L}_n \subset \mathcal{C} \cap \mathfrak{L}_0$.

On the other hand, since $\mathcal{C}$ has the existence property, we have by Theorem 0.4 that $\mathcal{C} \cap \mathfrak{L}_0 \subset \mathcal{C} \cap \mathfrak{L}_n$. Thus (c) holds.
(c) $\Rightarrow$ (d). Every C in $\mathcal{C} \cap \mathfrak{L}_n$ is order closed by (c), hence by the existence property of $\mathcal{C}$, C is $\|\cdot\|_\varrho$-proximinal.
(d) $\Rightarrow$ (e). This is obvious, since $E_\varrho \subset L_\varrho$.
(e) $\Rightarrow$ (a). We consider only the case $\mathfrak{L}_\uparrow \subset \mathcal{C}$, since the other case is similar. We assume for contradiction that $E_\varrho \not\subseteq L_\varrho$.

Step 1: Suppose that X is a ϱ-atom; that is, if $A \not\subseteq X$, then A is ϱ-null. Since elements of $\mathcal{P}$ must cover X, we see that $X \in \mathcal{P}$. Let $f \in M$. Then f is finite ϱ-a.e., for otherwise we could decompose X by inverse images. In particular, f is bounded, which implies that $f \in E_\varrho$. Therefore $L_\varrho \subset M \subset E_\varrho \subset L_\varrho$, showing that $L_\varrho = E_\varrho$. We can therefore assume that X can be decomposed into two measurable sets A and B, neither of which are ϱ-null.

Step 2: There exists a nonnegative function $w \in L_\varrho \setminus E_\varrho$. Since at least one of the functions $w1_A$ and $w1_B$ is not in E_ϱ, we may assume that $h = w1_A \in L_\varrho \setminus E_\varrho$.

Choose a sequence of nonnegative simple functions $\{h_k\}_1^\infty \subset \mathcal{E}$, with supports in A, such that $h_k \uparrow h$ ϱ-a.e. Let $u \geq 0$ be any simple function with support in B such that $\varrho(u) > 0$. Choose $c > 0$ so that $3c < \sup_{t \geq 0} \|tu\|_\varrho$. Since $0 \neq h \in L_\varrho$, there exists $\lambda > 0$ such that $0 < \|\lambda h\|_\varrho < c$. For each $k \in \mathbb{N}$ define

$$\varphi_k(t) = \|tu + \lambda h_k\|_\varrho\,.$$

Note that

$$\sup_{t \geq 0} \varphi_k(t) \geq \sup_{t \geq 0} \|tu\|_\varrho - \|\lambda h_k\|_\varrho \geq 3c - c = 2c$$

and that $\varphi_k(0) = \|\lambda h_k\|_\varrho < c$. Let $\{c_k\}_1^\infty \subset (c, 2c]$ satisfy $c_k \downarrow c$. Since for each k, φ_k is continuous, we can choose t_k so that $\varphi_k(t_k) = c_k$.

Define $w_k = -t_k u + \lambda h_k$ for each k and let $C = \{w_k\}_1^\infty$. We claim that $C \in \mathfrak{L}_\uparrow$. For all k,

$$\|t_{k+1}u + \lambda h_{k+1}\|_\varrho = c_{k+1} \leq c_k = \|t_k u + \lambda h_k\|_\varrho \leq \|t_k u + \lambda h_{k+1}\|_\varrho\,,$$

by the monotonicity of the F-norm. Since each term is nonnegative, again by the monotonicity of the F-norm, $t_{k+1} \le t_k$ for every k. From this and the fact that the h_k's are increasing, we infer that $w_k \le w_{k+1}$ for each k, proving our claim.

Furthermore, since A and B are disjoint, for each k,

$$\|w_k - 0\|_\varrho = \|\,|-t_k u + \lambda h_k|\,\|_\varrho = \|\,|t_k u| + \lambda h_k\|_\varrho = \|t_k u + \lambda h_k\|_\varrho = c_k\,.$$

Step 3: We claim that $C \in \mathfrak{L}_n$. If not, there exists $g \in \mathrm{closure}_{\|\cdot\|_\varrho}(C) \setminus C$. Since $C \subset \mathcal{E} \subset E_\varrho$, which is F-norm closed, $g \in E_\varrho$. There exists a subsequence converging to g in the F-norm and hence a subsequence $\{w_{n_k}\}_1^\infty$ such that $w_{n_k} \to g$ ϱ-a.e. See Proposition 2.3.5 in [4]. In particular, since $w_{n_k} \uparrow \lambda h$ on A, it follows that $g1_A = \lambda h \in L_\varrho \setminus E_\varrho$. This shows that $g \notin E_\varrho$, a contradiction that proves C is F-norm closed.

Furthermore, since for each k, $\|w_k - 0\|_\varrho = c_k$,

$$\mathrm{dist}_{\|\cdot\|_\varrho}(0, C) = c,$$

while for every $w_k \in C$, $\|w_k - 0\|_\varrho = c_k > c$. This shows that $P_{\|\cdot\|_\varrho}(0, C) = \Phi$. Since $\mathfrak{L}_\uparrow \subset \mathcal{C}$, this contradicts (e), proving that $L_\varrho = E_\varrho$ and finishes the proof. □

Considering remarks 1.3 and 1.4, we immediately have the following corollaries:

Theorem 1.6. *If $\varrho \in \mathcal{R}_s$ is orthogonally additive and has the property (K), then the following statements are equivalent:*

(a) $L_\varrho = E_\varrho$.
(b) $\|\cdot\|_\varrho$ *is σ-order continuous.*
(c) $\mathfrak{L}_0 = \mathfrak{L}_n$.
(d) C *is $\|\cdot\|_\varrho$-proximinal for every $C \in \mathfrak{L}_n$.*
(e) $P_{\|\cdot\|_\varrho}(f, C) \ne \Phi$ *for every $f \in E_\varrho$ and for every $C \in \mathfrak{L}_n$.*

Theorem 1.7. *Let X be countable and let $\varrho \in \mathcal{R}_s$ have the property (K). If $\mathcal{C} = \mathfrak{L}_i \cup \mathfrak{L}_s$, then the following statements are equivalent:*

(a) $L_\varrho = E_\varrho$.
(b) $\|\cdot\|_\varrho$ *is σ-order continuous.*
(c) $\mathcal{C} \cap \mathfrak{L}_0 = \mathcal{C} \cap \mathfrak{L}_n$.
(d) C *is $\|\cdot\|_\varrho$-proximinal for every $C \in \mathcal{C} \cap \mathfrak{L}_n$.*
(e) $P_{\|\cdot\|_\varrho}(f, C) \ne \Phi$ *for every $f \in E_\varrho$ and for every $C \in \mathcal{C} \cap \mathfrak{L}_n$.*

These theorems cover many interesting situations. Some examples follow:

Example 1.8. Theorem 1.6 applies to Musielak–Orlicz spaces $L^\varphi(X, \Sigma\mathcal{P}, \mu)$ if μ is σ-finite and $\mathcal{P}$ is a δ-ring of sets of finite measure.

Example 1.9. Theorem 1.7 applies to Lorentz-type L^p-spaces for X countable. Here

$$\varrho(f, A) = \sup_{\mu \in \Gamma} \int_A |f(k)|^p \, d\mu(k),$$

where Γ is a family of positive σ-finite measures on X, such that $\sup_{\mu \in \Gamma} \mu(A) < \infty$ for each finite subset $A \subset X$.

Example 1.10. Theorem 1.7 applies in the following space: Let $X = \mathbb{N}$, and let Σ be the σ-algebra of all subsets of X. For each n define the probability measure μ_n by

$$\mu_n(\{k\}) = \begin{cases} \frac{1}{n} & \text{for } \ k = 1, 2, \ldots, n \\ 0 & \text{for } \ k = n+1, n+2, \ldots \end{cases}$$

and then ϱ by

$$\begin{aligned} \varrho(h) &= \sup_n \int_X |h| \, d\mu_n \\ &= \sup_n \sum_{k=1}^{\infty} \mu_n(\{k\}) |h(k)| \\ &= \sup_n \frac{1}{n} \sum_{k=1}^{n} |h(k)|. \end{aligned}$$

Section 2.

Theorem 1.6 applies to Musielak–Orlicz spaces $L^\varphi(X, \Sigma, \mathcal{P}, \mu)$ if μ is σ-finite and $\mathcal{P}$ is the Δ-ring of all sets of finite measure. We proceed to show that an analogue to Theorem 1.5 holds for $L^\varphi(X, \Sigma, \mathcal{P}, \mu)$ when μ is not necessarily σ-finite. We first need the following lemma.

Lemma 2.1. *Let (X, μ) be a measure space and let L^φ be a Musielak–Orlicz space. If $\{f_n\}_1^\infty \subset L^\varphi$ and $\|f_n\|_\varrho \to 0$ as $n \to \infty$, then there exists a subsequence $\{f_n\}_1^\infty$ such that $f_{n_k} \to 0$ μ-a.e. as $k \to \infty$.*

Proof: Since $\varrho(f_n) \to 0$ as $n \to \infty$,

$$\int_X \varphi(x, f_n(x)) \, d\mu(x) = \varrho(f_n) < \infty$$

for sufficiently large n. Hence we may assume that $\Phi_n \in \text{Ł}^1(\mu)$ for each n, where $\Phi_n(x) = \varphi(x, f_n(x))$. By hypothesis $\varphi(x, u) > 0$, unless $u = 0$. Thus for each n

$$\operatorname{supp} f_n = \operatorname{supp} \Phi_n$$

which must be σ-finite, since each $\Phi_n \in L^1(\mu)$.

Let $S_n = \operatorname{supp} f_n$ and define $S = \bigcup_{n=1}^\infty S_n$. Then S is σ-finite as well and $\mu_{|S}$ is a σ-finite measure.

Let $D \subset S$ be any measurable set such that $\mu(D) < \infty$. We claim that on D $f_n \to 0$ in measure. To that end let $\varepsilon > 0$. Since $\mu_{|S}$ is absolutely continuous with respect to the measure defined by

$$\nu(A) = \int_{A \cap D} \varphi(x, \varepsilon) \, d\mu(x),$$

there exists $\delta > 0$ such that when $\nu(A) < \delta$, $\mu_{|S}(A) < \varepsilon$. There exists N such that whenever $n \geq N$, $\varrho(f_n) < \delta$. Let

$$A_n = \{t \in D : |f_n(t)| \geq \varepsilon\}.$$

Then

$$\nu(A_n) = \int_{A_n} \varphi(x, \varepsilon)\, d\mu(x) \leq \int_{A_n} \varphi(x, f_n(x))\, d\mu(x) < \delta$$

when $n \geq N$, and consequently $\mu(A_n) < \varepsilon$ for such n, proving our claim.

Since S is σ-finite, there exist mutually disjoint measurable sets B_k with $S = \bigcup_{k=1}^{\infty} B_k$ such that $\mu_{|S}(B_k) < \infty$ for each k. Since $\varrho(f_n 1_{B_1}) \leq \varrho(f_n) \to 0$, by the above claim $f_n 1_{B_1} \to 0$ in measure. By Riesz's theorem, there exists a subsequence converging to zero μ-a.e. on B_1. By continuing inductively, a diagonal argument produces a subsequence converging to zero μ-a.e. on S and hence on X. This completes the proof of the lemma. $\square$

Theorem 2.2. *Let $L^{\varphi}(X, \Sigma, \mathcal{P}, \mu)$ be a Musielak–Orlicz space and let ϱ be the Musielak–Orlicz function modular induced by φ. Then the following statements are equivalent:*

(a) $L_\varrho = E_\varrho$.
(b) $\|\cdot\|_\varrho$ *is σ-order continuous.*
(c) $\mathfrak{L}_0 = \mathfrak{L}_n$.
(d) *Every $C \in \mathfrak{L}_n$ is $\|\cdot\|_\varrho$-proximinal.*
(e) $P_{\|\cdot\|_\varrho}(f, C) \neq \Phi$ *for every $f \in E_\varrho$ and for every $C \in \mathfrak{L}_n$.*

Proof: (a) $\Leftrightarrow$ (b). This is Theorem 0.2.
(b) $\Rightarrow$ (c). Let $C \in \mathfrak{L}_n$, and suppose that $\{f_n\}_1^{\infty} \subset C$ with $f_n \uparrow f \in L_\varrho$, (respectively $f_n \downarrow f \in L_\varrho$). Then $(f - f_n) \downarrow 0$, (respectively $(f_n - f) \downarrow 0$), hence by σ-order continuity, $\|f - f_n\|_\varrho \to 0$. Since C is F-norm closed, $f \in C$. This shows that $C \in \mathfrak{L}_0$ and hence that $\mathfrak{L}_n \subset \mathfrak{L}_0$. On the other hand, Theorem 0.3 implies that each $C \in \mathfrak{L}_0$ is $\|\cdot\|_\varrho$-proximinal and Theorem 0.4 determines that if C is $\|\cdot\|_\varrho$-proximinal, then $C \in \mathfrak{L}_n$, completing the proof.
(c) $\Rightarrow$ (d). This follows immediately from Theorem 0.3.
(d) $\Rightarrow$ (e). This is obvious, since $E_\varrho \subset L_\varrho$.
(e) $\Rightarrow$ (a). Suppose for contradiction that $E_\varrho \neq L_\varrho$.

Step 1: We claim that X can be decomposed into two disjoint non-null subsets. Fix $g \in L_\varrho \setminus E_\varrho$ and let $S = \operatorname{supp} g$. Note that $\mu(S) > 0$. If $\mu(X \setminus S) > 0$, then we have the desired decomposition. Assuming $\mu(X \setminus S) = 0$, it suffices to decompose S.

Since φ is locally integrable and $g \notin E_\varrho$, either g is not bounded on S or S is not of finite measure. If g is not bounded on S, then in particular g is not constant on S and we can decompose S as desired by inverse image.

Let us then assume that $\mu(S) = \infty$. Since $g \neq 0$, $\varphi(x, \lambda g(x)) \neq 0$ for each $\lambda > 0$, and hence there exists $\varepsilon > 0$ and $Z \subset S$ such that Z is of positive measure and

$\varphi(x, g(x)) \geq \varepsilon$ for each $x \in Z$. Suppose that $\mu(Z) = \infty$. Then for each $\lambda > 0$,

$$\varrho(\lambda g) = \int_S \varphi(x, \lambda g(x))\, d\mu(x) \geq \int_Z \varphi(x, \lambda g(x))\, d\mu(x) \geq \varepsilon\mu(Z) = \infty\,.$$

But $g \in L_\varrho$, so $\varrho(\lambda g) \to 0$ as $\lambda \downarrow 0$. This contradiction proves that $\mu(Z) < \infty$. Therefore Z and $S \setminus Z$ decompose S.

Step 2: By Step 1 there exist disjoint sets A and B, each of positive measure, such that $X = A \cup B$. Since $E_\varrho \neq L_\varrho$, There exists a nonnegative function $w \in L_\varrho \setminus E_\varrho$. Since at least on of the functions $w1_A$ and $w1_B$ is not in E_ϱ, we may assume that $h = w1_A \in L_\varrho \setminus E_\varrho$.

We claim that there exist $c \geq 0$ and $C = \{w_k\}_1^\infty \in \mathfrak{L}_\uparrow$ such that $c < \|w_k\|_\varrho \to c$ and $w_k \uparrow h$ on A. The proof of this claim is exactly the same as in Step 2 of the proof of Theorem 1.5 and will be omitted.

Step 3: Proceeding as in Step 3 of the proof of Theorem 1.5, using Lemma 1.11 in place of Proposition 2.3.5 in [4], we can show that $C \in \mathfrak{L}_n$ and that $P_{\|\cdot\|_\varrho}(0, C) = \Phi$, contradicting (e) and proving that $L_\varrho = E_\varrho$. This completes the proof. □

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