Coloring digraphs by iterated antichains

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Abstract. We show that the minimum chromatic number of a product of two n-chromatic graphs is either bounded by 9, or tends to infinity. The result is obtained by the study of coloring iterated adjoints of a digraph by iterated antichains of a poset.

Keywords: graph product, chromatic number, antichain

Classification: 05C15, 06A10

This note is motivated by a conjecture by Hedetniemi on the chromatic number of the product of two graphs. (The *product* $G \times H$ of two unoriented graphs G and H is the graph on the vertex set $V(G) \times V(H)$ and with the edges $((u_1, u_2), (v_1, v_2))$ for $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.) Hedetniemi [H] conjectured that

$$\chi(G \times H) = \min(\chi G, \chi H)$$

for any pair G and H of graphs. The conjecture is also sometimes called the *Lovász–Hedetniemi* conjecture. The inequality '$\leq$' in the conjecture is obvious, and it is also easy to see that the conjecture is valid for 1-, 2-, and 3-chromatic graphs. The validity for 4-chromatic graphs has been proved in [ES]. On the other hand, no lower bound on $\chi(G \times H)$ is known. It is even not known whether the function $f(n)$ defined by $f(n) = \min\{\chi(G \times H) \mid \chi G = \chi H = n\}$ tends to infinity for $n \to \infty$. However, it has been proved in [PR] that if the function is bounded, then $f(n) \leq 16$ for all n. The purpose of this note is to decrease the bound from 16 to 9.

A survey of other known results on Hedetniemi's conjecture can be found in [DSW], and some further related results have been published in [HHMN]. A special case was proved also in [T].

The result here is obtained by extending the technique of coloring digraphs by antichains (see [HE] and [PR]) to coloring iterated adjoints of digraphs by iterated antichains.

Let L be a poset and let $A(L)$ be the set of all (not necessarily maximal) antichains of L. We introduce a partial order on $A(L)$ as follows. For $a, a' \in A(L)$, we write $a < a'$, if for every $x \in a$ there is some $y \in a'$ such that $x < y$. It is easy to check that if $a < a'$ and $a' < a$, then $a = a'$, and $a < a'$ and $a' < a''$ give $a < a''$. For $i > 0$, we define $A^i(L) = A(A^{i-1}(L))$. (Note that our construction of a poset on antichains slightly differs from that of Dilworth [D], where only maximum sized antichains were considered.)

Let $G = (V, E)$ be a digraph. We say that a mapping f from V to a poset L is a *homomorphism*, if $f(u) < f(v)$ for every edge $uv \in E$.

The *adjoint* δG of a digraph G is the digraph whose vertex set is $E(G)$, and edges of δG are the pairs of consecutive edges of G, i.e. $E(\delta G) = \{(uv, vw) \mid uv, vw \in E(G)\}$. For $i > 0$, we define the i-th adjoint $\delta^i G = \delta(\delta^{i-1} G)$.

Lemma 1. *Let G be a digraph and L be a poset. Then there is a homomorphism f from G to $A(L)$, if and only if there is a homomorphism ϕ from δG to L.*

PROOF: Let f be a homomorphism from G to $A(L)$. We define ϕ as follows. Given $e = uv \in V(\delta G)$, where e is an edge of G, choose an arbitrary $x \in f(u)$ for which $\{x\} < f(v)$, and set $\phi(e) = x$. (A suitable x must exist since $f(u) < f(v)$.) We check that the mapping ϕ is a homomorphism from δG to L. Let $ee' \in E(\delta G)$, where $e = uv$ and $e' = vw$ are edges of G. We have $\phi(e) < \phi(e')$ since $\phi(e) \in f(u), \phi(e') \in f(v)$ and $\{\phi(e)\} < f(v)$.

Conversely, let ϕ be a homomorphism from δG to L. We define a homomorphism f as follows. Given $u \in V(G)$, let $S(u) = \{\phi(uv) \mid uv \in E(G)\}$. Since $S(u)$ is not necessarily an antichain, we define $f(u)$ as the set of the maximal elements of $S(u)$. It is straightforward to check that $f(u) < f(v)$ for $uv \in E(G)$. □

The *chromatic number* χG of a digraph G is the chromatic number of the graph obtained from G after forgetting the orientation of the edges. Equivalently, it is the minimum k for which there is a homomorphism from G to D_k, where D_k denotes the *discrete poset* on k elements. A digraph $G = (V, E)$ is said to be *symmetric*, if for every edge uv it contains also the reversed edge vu. For a poset L, $\alpha(L)$ denotes the size of the maximum antichain in L.

Theorem 2. *Let G be a symmetric digraph, and i a nonnegative positive integer. Then $\chi(\delta^i G)$ is equal to the minimum k for which χG is less or equals $\alpha(A^i(D_k))$.*

PROOF: Let $\chi(\delta^i G) = k$. Then there is a homomorphism f from $\delta^i G$ to D_k, and hence also a homomorphism ϕ from G to $A^i(D_k)$ by the repeated use of Lemma 1. Since G is symmetric, $\phi(u)$ and $\phi(v)$ are incomparable elements of $A^i(D_k)$ for every edge uv of G. Let H be the complement of the comparability graph of $A^i(D_k)$. The existence of ϕ implies that $\chi(\delta^i G) \leq \chi H$. Since H is a perfect graph, χH equals the size of the maximum clique of H, which is the size of the maximum antichain in $A^i(D_k)$. Hence the inequality $\chi G \leq \alpha(A^i(D_k))$ is established.

Conversely, let $\chi G \leq \alpha(A^i(D_k))$. Then there is a homomorphism ϕ from $\delta^i G$ to $A^i(D_k)$. By a repeated use of Lemma 1, there is a homomorphism f from $\delta^i G$ to D_k. Clearly, f is a coloring of $\delta^i G$ since D_k is discrete. Hence $\chi(\delta^i G) \leq k$. □

We recall that D_k is a discrete poset. Then $A(D_k)$ is the set of all subsets of $\{1, 2, \ldots, k\}$ ordered by inclusion, and $A^2(D_k)$ is the set of the Sperner systems on the underlying k-element set.

Lemma 3. *We have* $\alpha(A^2(D_3)) = 4$.

PROOF: The following four sets $\{\{1,2\}\}, \{\{2,3\}\}, \{\{1,3\}\}$ and $\{\{1\},\{2\},\{3\}\}$ form an antichain in $A^2(D_3)$. It is easy to check that it is an antichain of the maximum size. □

Lemma 4 ([HE]). *We have* $\chi(\delta G) \geq \log_2 \chi G$.

The *product* $G_1 \times G_2$ of two digraph G_1 and G_2 is the digraph with the vertex set $V(G_1) \times V(G_2)$ and the edges $((u_1, u_2), (v_1, v_2))$ for $u_1 v_1 \in E(G_1)$ and $u_2 v_2 \in E(G_2)$.

We define $g(n)$ as the minimum chromatic number of the product of two n-chromatic digraphs. It has been proved in [PR] that the function g is either bounded by 4, or tends to infinity. Here we present an improvement of that result.

Theorem 5. *The function* $g(n)$ *is either bounded by* 3, *or tends to infinity.*

Proof: Assume that the function g is bounded by a constant c, i.e. for all n sufficiently large, say $n > n_0$, we have $g(n) = c$. It has been proved in [PR] that $c \leq 4$. For a contradiction, assume that $c = 4$. Let $n > 2^{2^{n_0}}$, and G_1 and G_2 be a pair of n-chromatic digraphs for which $\chi(G_1 \times G_2) = \chi H = 4$, where $H = G_1 \times G_2$. Since $\alpha A^2(D_3) = 4$ by Lemma 3, we have $\chi(\delta^2 H) \leq 3$ by Theorem 2.

On the other hand, we have $\chi(\delta^2 G_1), \chi(\delta^2 G_2) > n_0$ by Lemma 4, and hence $\chi(\delta^2 G_1 \times \delta^2 G_2) \geq 4$ by our assumption on g. Since $\delta^2 H = \delta^2(G_1 \times G_2) = \delta^2 G_1 \times \delta^2 G_2$ (the latter equality is easy to see, cf. Proposition 2.2 of [PR]), we get $\chi(\delta^2 H) = 4$, which is a contradiction. □

Let $h(n) = \min\{\max(\chi(G_1 \times G_2), \chi(G_1 \times G_2^{-1})) \mid G_1$ and G_2 are digraphs with $\chi G_1 = \chi G_2 = n\}$, where G_2^{-1} denotes the digraph obtained from G_2 by reversing the edges. Quite analogously as above, it is possible to show that $h(n)$ is either bounded by 3 or tends to infinity. However, it is not yet excluded that $g(n)$ is bounded, while $h(n)$ is not.

Theorem 6. *The minimum chromatic number of a product of two n-chromatic graphs is either bounded by* 9, *or tends to infinity.*

Proof: Let $f(n)$ be the minimum chromatic number of a product of two (undirected) n-chromatic graphs. The statement follows from the inequality $h(n) \leq f(n) \leq h^2(n)$ established in the proof of Theorem 3.6 of [PR]. □

I have been recently informed by V. Rödl that the possibility of improving the construction of [PR]was also observed by J. Schmerl.

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(*Received* August 23, 1990)