Existence and bifurcation results for a class of nonlinear boundary value problems in $(0,\infty)$

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Abstract. We consider the nonlinear Dirichlet problem

$$-u'' - r(x)|u|^\sigma u = \lambda u \text{ in } (0,\infty),\ u(0) = 0 \text{ and } \lim_{x\to\infty} u(x) = 0,$$

and develop conditions for the function r such that the considered problem has a positive classical solution. Moreover, we present some results showing that $\lambda = 0$ is a bifurcation point in $W^{1,2}(0,\infty)$ and in $L^p(0,\infty)$ $(2 \le p \le \infty)$.

Keywords: nonlinear Dirichlet problem, classical solution, bifurcation point, ordinary differential equation

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The aim of this paper is to prove some existence and bifurcation results for the nonlinear Dirichlet problem

$$-u'' - r(x)|u|^\sigma u = \lambda u \text{ in } (0,\infty) \tag{1}$$

with the boundary conditions $u(0) = 0$ and $\lim_{x\to\infty} u(x) = 0$, where $\sigma > 0$ and $\lambda < 0$ are given constants. In particular, we will generalize and complement some results of M.S. Berger (see [2, Theorem 4]) and C.A. Stuart (see [6, Theorem 7.4]).

In the following, the function r is always assumed to satisfy

(A) The function $r : (0,\infty) \to \mathbb{R}$ is measurable and satisfies $r > 0$ a.e. on a subinterval (δ_1,δ_2) $(0 < \delta_1 < \delta_2)$ of $(0,\infty)$. The negative part $r_- = \min(r,0)$ of r satisfies $\int_{x_1}^{x_2} |r_-(x)|\,dx < \infty$ for all constants $0 < x_1 < x_2 < \infty$; and from the positive part $r_+ = \max(r,0)$ we require that it can be written as

$$r_+ = r_1 + r_2 + r_3 + r_4, \quad \text{where}$$

(i) $0 \le r_1(x) \le f(x)\cdot x^{-2-\sigma/2}$ holds for almost all $x > 0$ and a function $f \in L^\infty(0,\infty)$ satisfying $f(x) \to 0$ as $x \to 0$,
(ii) the function r_2 fulfils $0 \le r_2 \in L^\infty(0,\infty)$ and $r_2(x) \to 0$ as $x \to \infty$,
(iii) $0 \le r_3 \in L^{p_0}(0,\infty)$ holds for some $p_0 \in (1,\infty)$,
(iv) and r_4 satisfies $0 \le r_4 \in L^1(0,\infty)$.

Then we will prove the following existence results:

Theorem 1. *Suppose that the function r satisfies* (A)*. Then, for each $\lambda < 0$, there exists a nonnegative, bounded function $u_\lambda \in W_0^{1,2}(0,\infty) \cap C^{0,1/2}([0,\infty))$ such that $u_\lambda \not\equiv 0$, $u_\lambda(0) = 0$, $\lim_{x\to\infty} u_\lambda(x) = 0$ and the equation* (1) *holds in the sense of distributions.*

Corollary 1. *Assume in addition to* (A) *that $r_3 \equiv r_4 \equiv 0$. Then, for each $\alpha \in (0, |\lambda|^{1/2})$, there exists a constant C_α such that $u_\lambda(x) \le C_\alpha \cdot e^{-\alpha\cdot x}$ holds for all $x \ge 0$.*

Corollary 2. *Suppose in addition to* (A) *that the function r is continuous in $(0,\infty)$. Then u_λ is positive in $(0,\infty)$, satisfies $u_\lambda \in C^2(0,\infty)$ and solves the equation* (1) *in the classical sense.*

In order to formulate our bifurcation results, we have to introduce some further notations and assumptions.

The constants δ_1 and δ_2 may be defined as in (A), and I may denote the interval $I = (\delta_1, \delta_2)$. Moreover, $(t_n)_n$ may be a sequence of real numbers satisfying $1 = t_1 < t_2 < \dots < t_n < t_{n+1} < \dots$ and $t_n \to \infty$ as $n \to \infty$.

By I_n, we denote the interval $I_n = t_n \cdot I$. Then, for $k > 0$, we introduce the following condition:

(A_k) There exists a nonnegative, measurable function h on $(0,\infty)$ such that $r(x) \ge h(x)\cdot |x|^{-k}$ holds a.e. in $\bigcup_{n=1}^\infty I_n$ and $\beta_n = \operatorname*{ess\,inf}_{y\in I_n} h(y) \to \infty$ as $n \to \infty$.

Theorem 2. *Suppose that the assumption* (A) *is fulfilled and that λ_n is defined by $\lambda_n = -t_n^{-2}$ for all n. Then we have the following results:*

(a) *If in addition* (A_k) *is satisfied for $k = 2 + \frac{\sigma}{2}$, then $\|u'_{\lambda_n}\|_2 \to 0$ and $u_{\lambda_n} \to 0$ in $L^\infty_{\text{loc}}([0,\infty))$ as $n \to \infty$.*

(b) *If in addition* (A_k) *is satisfied for $k = 2$, then $\|u_{\lambda_n}\|_\infty \to 0$ as $n \to \infty$.*

(c) *Let $p \in (2,\infty)$, $0 < \sigma < 2\cdot p$ and assume additionally that* (A_k) *holds for $k = 2 - \frac{\sigma}{p}$. Then $\|u_{\lambda_n}\|_p \to 0$ as $n \to \infty$.*

(d) *Suppose additionally that $0 < \sigma < 4$ and* (A_k) *holds for $k = 2 - \frac{\sigma}{2}$. Then we have $\|u_{\lambda_n}\|_{W^{1,2}} \to 0$ as $n \to \infty$.*

Remark 1. Part (d) of Theorem 2 shows that $\lambda = 0$ is a bifurcation point for the equation (1) in $W^{1,2}$. A similar result was obtained by C.A. Stuart [6, Theorem 7.4]. But in the contrast to the part (d) of Theorem 2, in [6], it is assumed that r is nonnegative in $(0,\infty)$.

For the special case that $0 < \sigma < 4$ and $r(x) = c_0 \cdot x^{-\sigma}$ (c_0 is a positive constant), the existence of a nontrivial, nonnegative solution of the equation (1) already has been proved in [2] (see Lemma 1 and Theorem 4).

1. Some preliminaries.

By $W^{1,2}(0,\infty)$, we denote the Hilbert space of functions u defined on the interval $(0,\infty)$ such that u and its derivative u' are in $L^2(0,\infty)$. The inner product of two functions $u, v \in W^{1,2}(0,\infty)$ is given by $\langle u, v\rangle = \int_0^\infty (u \cdot v + u' \cdot v')\,dx$. Moreover, by $W_0^{1,2}(0,\infty)$ we denote the closure of $C_0^\infty(0,\infty)$ in $W^{1,2}(0,\infty)$.

The following lemma plays a crucial role in our proofs. The essential parts of it can be found in [6, p. 188].

Lemma 1. *Each function $u \in W_0^{1,2}(0,\infty)$ can be identified with a continuous function on $[0,\infty)$, still denoted by u, such that*

(a) $u(0) = 0$, $\lim_{x\to\infty} u(x) = 0$,
(b) $|u(x)| \le \sqrt{2} \cdot \|u\|_2^{1/2} \cdot \|u'\|_2^{1/2}$ *holds for* $x \ge 0$,
(c) $|u(x_1) - u(x_2)| \le \|u'\|_2 \cdot |x_1 - x_2|^{1/2}$ *holds for all* $x_1, x_2 \ge 0$ *and*
(d) $\int_0^\infty x^{-2-\sigma/2} \cdot |u(x)|^{2+\sigma}\,dx \le 4 \cdot \|u'\|_2^{2+\sigma}$.

PROOF: Let $\varphi \in C_0^\infty(0,\infty)$. Then we see that

$$\varphi^2(x) = 2 \cdot \int_0^x \varphi(s) \cdot \varphi'(s)\,ds, \quad \varphi(x_1) - \varphi(x_2) = \int_{x_2}^{x_1} \varphi'(s)\,ds$$

and, by Hardy's inequality, that $\int_0^\infty x^{-2} \cdot \varphi^2(x)\,dx \le 4 \cdot \|\varphi'\|_2^2$. Hence, by Hölder's inequality, it follows that (b) and (c) hold for all $\varphi \in C_0^\infty(0,\infty)$. Moreover, the part (c) implies

$$|\varphi(x)| \le \|\varphi'\|_2 \cdot x^{1/2} \quad \text{for} \quad x \ge 0$$

and

$$\int_0^\infty x^{-2-\sigma/2} \cdot |\varphi(x)|^{2+\sigma}\,dx \le 4 \cdot \|\varphi'\|_2^{2+\sigma}.$$

Now let $u \in W_0^{1,2}(0,\infty)$ and $(\varphi_n)_n$ be a sequence of functions $\varphi_n \in C_0^\infty(0,\infty)$ such that $\varphi_n \to u$ in $W_0^{1,2}(0,\infty)$ as $n \to \infty$. Then, according to part (b), $(\varphi_n)_n$ is a Cauchy sequence in $L^\infty([0,\infty))$. Hence, there exists a function Φ, continuous on $[0,\infty)$, such that

$$\varphi_n \to \Phi \quad \text{in} \quad L^\infty([0,\infty)) \quad \text{as} \quad n \to \infty.$$

Clearly, we have $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x) = 0$ and $\Phi(x) = u(x)$ a.e. in $(0,\infty)$. Furthermore, it is not difficult to show that (b)–(d) even hold for the function Φ. □

2. Proof of the existence results.

For $\lambda < 0$, we define

$$D_\lambda = \{u \in W_0^{1,2}(0,\infty) \mid \int_0^\infty |r_-| \cdot |u|^{2+\sigma}\,dx < \infty \quad \text{and} \quad |u|_\lambda := (\|u'\|_2^2 + |\lambda|\,\|u\|_2^2)^{1/2} \le 1\}.$$

Then, from (A) and Lemma 1, one easily concludes

Lemma 2. *There exist constants $c_0, c_1, \dots, c_5$, independent of $u \in D_\lambda, R > 0$ and $S > 0$, such that*

(a) $\int_0^\infty r_+ \cdot |u|^{2+\sigma}\,dx \le c_0$,

(b) $\int_R^\infty r_1 \cdot |u|^{2+\sigma}\,dx \le c_1 \cdot R^{-2-\sigma/2}$,

(c) $\int_R^\infty r_2 \cdot |u|^{2+\sigma}\,dx \le c_2 \cdot \sup_{y\ge R} r_2(y)$,

(d) $\int_R^\infty r_3 \cdot |u|^{2+\sigma}\,dx \le c_3 \cdot \left(\int_R^\infty r_3^{p_0}\,dx\right)^{1/p_0}$,

(e) $\int_R^\infty r_4 \cdot |u|^{2+\sigma}\,dx \le c_4 \cdot \int_R^\infty r_4\,dx$

and

(f) $\int_0^S r_1 \cdot |u|^{2+\sigma}\,dx \le c_5 \cdot \sup_{0<y\le S} f(y)$.

The nonlinear functional ζ will be defined by

$$\zeta(u) = -\frac{1}{2+\sigma} \cdot \int_0^\infty r(x)|u(x)|^{2+\sigma}\,dx.$$

Then, the part (a) of Lemma 2 shows that ζ is well defined on D_λ and that

$$M_\lambda = \inf_{u\in D_\lambda} \zeta(u)$$

is a well defined real number.

The interval (δ_1, δ_2) may be defined as in (A) and the function $\varphi_0 \in C_0^\infty(0,\infty)$ may be chosen such that $\operatorname{supp}\varphi_0 \subset (\delta_1,\delta_2)$ and $|\varphi_0|_\lambda = 1$. Then

$$\zeta(\varphi_0) < 0 \quad \text{implies} \quad M_\lambda < 0. \tag{2}$$

Lemma 3. *There exists a function $u_\infty \in D_\lambda$ such that $|u_\infty|_\lambda = 1$, $u_\infty \ge 0$ and $\zeta(u_\infty) = M_\lambda$.*

Proof: Let $(u_n)_n \subset D_\lambda$ be a sequence such that $\zeta(u_n) \to M_\lambda$ as $n \to \infty$. Then, according to (2), we can assume without restrictions that $\zeta(u_n) \le 0$ holds for all n. Furthermore, since $\||u|'\|_2 = \|u'\|_2$ (see [4, Lemma 7.6]), we may assume that $u_n \ge 0$.

The sequence $(u_n)_n$ is bounded in $W_0^{1,2}(0,\infty)$. Hence, using Lemma 1, the Arzela–Ascoli theorem, the reflexivity of $W_0^{1,2}(0,\infty)$, and a standard diagonal process, we see that there exists a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, such that

$$u_n \underset{w}{\longmapsto} u_\infty \ \text{ in } \ W_0^{1,2}(0,\infty) \ \text{ as } \ n \to \infty,$$

and

$$\sup_{0\le x\le d} |u_\infty(x) - u_n(x)| \underset{n\to\infty}{\to} 0 \tag{3}$$

holds for all constants $0 \le d < \infty$.

As an immediate consequence of these results, we obtain

$$|u_\infty|_\lambda \le 1 \quad \text{and} \quad u_\infty \ge 0.$$

Since $\zeta(u_n) \le 0$ holds for all n, we conclude from the part (a) of Lemma 2:

$$\int_0^\infty |r_-|\,|u_n|^{2+\sigma}\,dx \le c_0 \quad \text{for all } n. \tag{4}$$

But (4) and Fatou's lemma imply $\int_0^\infty |r_-|\,|u_\infty|^{2+\sigma}\,dx < \infty$.

Furthermore, it follows by Lemma 2 that for each $\varepsilon > 0$ there exist constants $R_\varepsilon > 0$ and $S_\varepsilon > 0$ such that

$$\int_{R_\varepsilon}^\infty r_+ \cdot |u_n|^{2+\sigma}\,dx \le \varepsilon \tag{5}$$

and

$$\int_0^{S_\varepsilon} r_1 \cdot |u_n|^{2+\sigma}\,dx \le \varepsilon \quad \text{hold for all } \; n \in \mathbb{N} \cup \{\infty\}. \tag{6}$$

From (3)–(6), we conclude that

$$\lim_{n\to\infty} \int_0^\infty r_+(x) \cdot |u_n(x)|^{2+\sigma}\,dx = \int_0^\infty r_+(x) \cdot |u_\infty(x)|^{2+\sigma}\,dx\,. \tag{7}$$

Moreover, Fatou's lemma and (7) imply

$$M_\lambda \le \zeta(u_\infty) \le \liminf \zeta(u_n) = M_\lambda\,.$$

Since $\zeta(u_\infty) = M_\lambda$, the inequality (2) shows that $|u_\infty|_\lambda > 0$.

Finally, $M_\lambda < 0$ and $M_\lambda \le \zeta(|u_\infty|_\lambda^{-1} \cdot u_\infty) = |u_\infty|_\lambda^{-2-\sigma} \cdot M_\lambda$ prove that $|u_\infty|_\lambda = 1$. □

Proof of Theorem 1: The function u_∞ may be chosen as in Lemma 3. Then, for each $\varphi \in C_0^\infty(0,\infty)$, there exists an $\varepsilon_0 = \varepsilon_0(\varphi) \in (0,1]$ such that $|u_\infty + \varepsilon\cdot\varphi|_\lambda > 0$ holds for all $|\varepsilon| \le \varepsilon_0(\varphi)$.

For $|\varepsilon| < \varepsilon_0(\varphi)$, we define

$$\eta(\varepsilon) = \zeta((u_\infty + \varepsilon\cdot\varphi)\cdot|u_\infty + \varepsilon\cdot\varphi|_\lambda^{-1}) = \zeta(u_\infty + \varepsilon\cdot\varphi)\cdot|u_\infty + \varepsilon\cdot\varphi|_\lambda^{-2-\sigma}\,,$$

and $\psi(\varepsilon) = \zeta(u_\infty + \varepsilon\cdot\varphi)$. Then, using the inequality

$$\big|\,|b|^{2+\sigma} - |a|^{2+\sigma}\big| \le (2+\sigma)\cdot 2^{1+\sigma}\cdot|b-a|\cdot(|a|^{1+\sigma} + |b|^{1+\sigma}) \quad (a,b\in\mathbb{R}),$$

it is not difficult to show that there exists a constant $C = C(\sigma)$ such that

$$\begin{aligned}
&|r(x)|\cdot\big|\,|u_\infty(x) + \varepsilon\cdot\varphi(x)|^{2+\sigma} - |u_\infty(x)|^{2+\sigma}\big|\cdot|\varepsilon|^{-1}\\
&\quad\le C\cdot|r(x)|\cdot|\varphi(x)|\cdot(|u_\infty(x)|^{1+\sigma} + |\varphi(x)|^{1+\sigma})\\
&\quad\le C\cdot(\|u_\infty\|_\infty^{1+\sigma} + \|\varphi\|_\infty^{1+\sigma})\cdot r(x)\cdot\varphi(x)
\end{aligned}$$

holds for almost all $x \geq 0$.

Hence, we can apply Lebesgue's convergence theorem and obtain

$$\frac{d\psi}{d\varepsilon}(0) = -\int_0^\infty r \cdot |u_\infty|^\sigma \cdot u_\infty \cdot \varphi \, dx.$$

Furthermore, $\frac{d\eta}{d\varepsilon}(0) = 0$ implies

$$\mu(\lambda) \cdot \Big(\int_0^\infty u_\infty' \cdot \varphi' \, dx + |\lambda| \cdot \int_0^\infty u_\infty \cdot \varphi \, dx\Big) = \int_0^\infty r \cdot |u_\infty|^\sigma \cdot u_\infty \cdot \varphi \, dx,$$

where $\mu(\lambda) = \int_0^\infty r(x) \cdot |u_\infty(x)|^{2+\sigma} \, dx = -(2+\sigma) \cdot M_\lambda > 0$.

Now we define $u_\lambda = \mu(\lambda)^{-1/\sigma} \cdot u_\infty$ and conclude that

$$\int_0^\infty u_\lambda' \cdot \varphi' \, dx - \int_0^\infty r(x) |u_\lambda|^\sigma u_\lambda \cdot \varphi \, dx = \lambda \cdot \int_0^\infty u_\lambda \cdot \varphi \, dx \tag{8}$$

holds for all $\varphi \in C_0^\infty(0,\infty)$. The remaining assertions follow from Lemma 1. □

Proof of Corollary 1: From (8), we conclude for all nonnegative functions

$$\varphi \in C_0^\infty(0,\infty) : \int_0^\infty u_\lambda' \cdot \varphi' \, dx \leq \lambda \cdot \int_0^\infty u_\lambda \cdot \varphi \, dx + \int_0^\infty r_+(x) u_\lambda^{1+\sigma} \cdot \varphi \, dx.$$

For functions $v \in W_0^{1,2}(0,\infty)$ satisfying $v \geq 0$ there exist sequences $(\varphi_n)_n$ of nonnegative functions $\varphi_n \in C_0^\infty(0,\infty)$ such that $\varphi_n \to v$ in $W_0^{1,2}(0,\infty)$ as $n \to \infty$ (see [3, p. 147]). Hence, we obtain

$$\int_0^\infty u_\lambda' \cdot v' \, dx \leq \lambda \cdot \int_0^\infty u_\lambda \cdot v \, dx + \int_0^\infty r_+(x) \cdot u_\lambda^{1+\sigma} \cdot v \, dx \tag{9}$$

for all functions $v \in W_0^{1,2}(0,\infty)$ satisfying $v \geq 0$.

The constant $\varepsilon_1 > 0$ may be chosen such that $\varepsilon_1 \leq |\lambda| - \alpha^2$. Then it follows from the assumptions and Lemma 1 that there exists a constant $R_1 > 0$ such that

$$r_+(x) \cdot u_\lambda^\sigma(x) \leq \varepsilon_1 \quad \text{holds for all } \ x \geq R_1 . \tag{10}$$

Since u_λ is bounded, we can find a constant $C_\alpha > 0$ such that

$$u_\lambda(x) \leq C_\alpha \cdot e^{-\alpha \cdot x} \quad \text{holds for all } \ x \in [0, R_1 + 1].$$

The function ψ_α may be defined by $\psi_\alpha(x) = C_\alpha \cdot e^{-\alpha \cdot x}$ for $x \geq 0$. Then one easily verifies that $\psi_\alpha \in W^{1,2}(0,\infty)$ and

$$\int_0^\infty \psi_\alpha' \cdot v' \, dx = -\alpha^2 \cdot \int_0^\infty \psi_\alpha \cdot v \, dx \quad \text{holds for all } \ v \in W_0^{1,2}(0,\infty). \tag{11}$$

The function $(u_\lambda - \psi_\alpha)_+$ satisfies $(u_\lambda - \psi_\alpha)_+ \in W_0^{1,2}(0,\infty)$, $(u_\lambda - \psi_\alpha)_+(x) = 0$ for $x \in [0, R_1 + 1]$, $(u_\lambda - \psi_\alpha)'_+ = (u_\lambda - \psi_\alpha)'$ on $\{u_\lambda > \psi_\alpha\}$ and $(u_\lambda - \psi_\alpha)'_+ = 0$ on $\{u_\lambda \le \psi_\alpha\}$.
Hence, we obtain from (9)–(11):

$$\int_0^\infty ((u_\lambda - \psi_\alpha)'_+)^2\, dx \le \lambda \cdot \int_0^\infty u_\lambda \cdot (u_\lambda - \psi_\alpha)_+\, dx + \varepsilon_1 \cdot \int_0^\infty u_\lambda \cdot (u_\lambda - \psi_\alpha)_+\, dx + \\ + \alpha^2 \cdot \int_0^\infty \psi_\alpha \cdot (u_\lambda - \psi_\alpha)_+\, dx \le -\alpha^2 \cdot \int_0^\infty (u_\lambda - \psi_\alpha)_+^2\, dx \le 0.$$

Thus, Lemma 1 implies $(u_\lambda - \psi_\alpha)_+ \equiv 0$ and $u_\lambda(x) \le \psi_\alpha(x)$ for all $x \ge 0$. □

Proof of Corollary 2: For $x \in (0,\infty)$, we define

$$l(x) = -r(x) \cdot u_\lambda^{1+\sigma}(x) - \lambda \cdot u_\lambda(x).$$

Then, from the assumptions and Theorem 1, it follows that l is continuous in $(0,\infty)$. The function U may be defined by

$$U(x) = \int_1^x \int_1^y l(s)\; dsdy \qquad \text{for} \quad x > 0.$$

Then we see that $U \in C^2(0,\infty)$ and $U''(x) = l(x)$ holds for $x > 0$. Moreover, for all functions $\varphi \in C_0^\infty(0,\infty)$, we obtain

$$\int_0^\infty (u'_\lambda - U') \cdot \varphi'\, dx = 0. \tag{12}$$

Corollary 3.27 in [1] and (12) imply the existence of a constant K such that

$$u'_\lambda = U' + K \qquad \text{holds in} \;\; \mathcal{D}'(0,\infty). \tag{13}$$

Then, according to Theorem 1.4.2 in [5], we see that (13) holds even in the classical sense and that $u_\lambda \in C^2(0,\infty)$.

To prove that the function u_λ is positive in $(0,\infty)$, we assume that there exists an $x_0 \in (0,\infty)$ such that $u_\lambda(x_0) = 0$. Since $u_\lambda(x) \ge 0$ holds for all $x \ge 0$, we see that $u'_\lambda(x_0) = 0$. Hence the vectorvalued function $(y_1, y_2) = (u_\lambda, u'_\lambda)$ solves the initial value problem

$$(y'_1, y'_2) = F(x, y_1, y_2) = (y_2, -\lambda \cdot y_1 - r(x) \cdot |y_1|^\sigma \cdot y_1), \\ (y_1(x_0), y_2(x_0)) = (0,0).$$

The function F is continuous in $(0,\infty) \times \mathbb{R}^2$ and the partial derivatives $\partial_{y_1} F$ and $\partial_{y_2} F$ of F are also continuous in $(0,\infty) \times \mathbb{R}^2$. Then, it follows by a standard result from the theory of ordinary differential equations that $u_\lambda \equiv 0$ in $(0,\infty)$. □

3. Proof of the bifurcation results.

The function u_∞ may be chosen as in Lemma 3. Then we have $u_\lambda = \mu(\lambda)^{-1/\sigma} \cdot u_\infty$, where $\mu(\lambda) = -(2+\sigma) \cdot M_\lambda$. Since $|u_\infty|_\lambda = 1$, it follows that

$$\|u'_\lambda\|_2 \le \mu(\lambda)^{-1/\sigma} \quad \text{and} \quad \|u_\lambda\|_2 \le \mu(\lambda)^{-1/\sigma} \cdot |\lambda|^{-1/2}. \tag{14}$$

The function $\varphi_1 \in C_0^\infty(0,\infty)$ may be chosen such that $\operatorname{supp}\varphi_1 \subset I = (\delta_1, \delta_2)$ and $\|\varphi'_1\|_2^2 + \|\varphi_1\|_2^2 = 1$. The functions φ_n may be defined by $\varphi_n(x) = t_n^{1/2} \cdot \varphi_1(t_n^{-1} \cdot x)$. Then, it follows that $\operatorname{supp}\varphi_n \subset I_n$ and

$$\|\varphi'_n\|_2^2 + t_n^{-2} \cdot \|\varphi_n\|_2^2 = \|\varphi'_1\|_2^2 + \|\varphi_1\|_2^2 = 1. \tag{15}$$

Lemma 4. *Let $\lambda_n = -t_n^{-2}$ for all n and suppose that (A_k) holds for some $k > 0$. Then it follows that*

(a) $\|u'_{\lambda_n}\|_2 \le (\beta_n \cdot t_n^{2+\sigma/2-k} \cdot \gamma_0)^{-1/\sigma}$

and

(b) $\|u_{\lambda_n}\|_2 \le t_n \cdot (\beta_n \cdot t_n^{2+\sigma/2-k} \cdot \gamma_0)^{-1/\sigma}$

holds for all n, where $\gamma_0 = \int_I |x|^{-k} \cdot |\varphi_1(x)|^{2+\sigma}\, dx > 0$.

Proof: The identity (15) shows that $|\varphi_n|_{\lambda_n} = 1$. Hence, we obtain

$$\begin{aligned} M_{\lambda_n} \le \zeta(\varphi_n) &= -(2+\sigma)^{-1} \cdot t_n^{1+\sigma/2} \cdot \int_0^\infty r(x) \cdot |\varphi_1(t_n^{-1} \cdot x)|^{2+\sigma}\, dx \\ &= -(2+\sigma)^{-1} \cdot t_n^{1+\sigma/2} \cdot \int_I r(t_n \cdot x) \cdot |\varphi_1(x)|^{2+\sigma}\, dx \\ &\le -(2+\sigma)^{-1} \cdot t_n^{1+\sigma/2-k} \cdot \beta_n \cdot \int_I |x|^{-k} \cdot |\varphi_1(x)|^{2+\sigma}\, dx. \end{aligned} \tag{16}$$

Since $\mu(\lambda_n) = -(2+\sigma) \cdot M_{\lambda_n}$, the assertions follow from (14), (15) and (16). □

Proof of Theorem 2: Assume first that (A_k) is satisfied for $k = 2 + \sigma/2$. Since $\beta_n \to \infty$ as $n \to \infty$, we obtain from the part (a) of Lemma 4 that $\|u'_{\lambda_n}\|_2 \to 0$ as $n \to \infty$. The part (c) of Lemma 1 implies

$$|u_{\lambda_n}(x)| \le \|u'_{\lambda_n}\|_2 \cdot x^{1/2} \quad \text{for all } x \ge 0.$$

Hence, we see that $u_{\lambda_n} \to 0$ in $L^\infty_{\mathrm{loc}}([0,\infty))$ as $n \to \infty$.

From the part (b) of Lemma 1 it follows that

$$\|u_{\lambda_n}\|_\infty \le \sqrt{2} \cdot \|u_{\lambda_n}\|_2^{1/2} \cdot \|u'_{\lambda_n}\|_2^{1/2} \quad \text{holds for all } n. \tag{17}$$

Then, combining Lemma 4 and (17), we show that

$$\|u_{\lambda_n}\|_\infty \to 0 \ (n \to \infty), \quad \text{if } (\mathrm{A}_k) \text{ holds for } k = 2.$$

Now let $p \in [2,\infty)$ be a real number and suppose that $0 < \sigma < 2 \cdot p$. Since

$$\|u_{\lambda_n}\|_p \leq \|u_{\lambda_n}\|_\infty^{1-2/p} \cdot \|u_{\lambda_n}\|_2^{2/p} \leq 2^{1/2-1/p} \cdot \|u'_{\lambda_n}\|_2^{1/2-1/p} \cdot \|u_{\lambda_n}\|_2^{1/2-1/p}$$

holds for all n, we obtain from Lemma 4 that

$$\|u_{\lambda_n}\|_p \to 0 \ (n \to \infty) \quad \text{if } (\mathrm{A}_k) \text{ holds for } \ k = 2 - \sigma/p.$$

If (A_{k_1}) is satisfied for some $k_1 > 0$, then (A_k) holds for all $k \in [k_1, \infty)$. In particular, we see that $(\mathrm{A}_{2-\sigma/2})$ implies $(\mathrm{A}_{2+\sigma/2})$. Hence the part (d) of Theorem 2 follows from the above considerations. □

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