Comment.Math.Univ.Carolin. 32,2 (1991)323–337

Forcing in the alternative set theory I

Jiří Sgall

Abstract. The technique of forcing is developed for the alternative set theory (AST) and similar weak theories, where it can be used to prove some new independence results. There are also introduced some new extensions of AST.

Keywords: alternative set theory, forcing, generic extension, symmetric extension, axiom of constructibility

Classification: Primary 03E70; Secondary 03E25, 03E35, 03E45

We develop the method of forcing in the alternative set theory (AST) and similar weak theories like the second order arithmetic to settle some questions on independence of some axioms not included in AST.

The material is divided into two parts. In this paper, the technique is developed and in its continuation (A. Sochor and J. Sgall: Forcing in the AST II, to appear in CMUC) concrete results will be proved. Most of them concern some forms of the axiom of choice not included in the basic axiomatics of AST. The main results are:

(1) The axiom of constructibility is independent of AST plus the strong scheme of choice plus the scheme of dependent choices.
(2) The scheme of choice is independent of A_2 (the second order arithmetic). This is already known, but our proof works in A_3, while the old one uses cardinals up to $\aleph_\omega$, which needs much stronger theory.
(3) The scheme of choice is independent of AST.

Let us sketch the main points different from the technique of forcing in the classical set theory:

We construct generic extensions such that sets are absolute (i.e. we add only classes). This saves many technical problems, as we have no troubles with sets like $\{G\}$, where G is a generic object. On the other side, we have only small cardinalities, so we must avoid such techniques as embedding orderings into complete boolean algebra or using groups of automorphisms. Another interesting difference is that there can be some new types of well-orderings in the extension, while in ZF, the class of ordinal numbers is preserved.

In the first section, we repeat briefly the axiomatization of AST and some basic facts. The second section gives basic notions and results concerning the syntactical aspects of forcing and the third section covers its model-theoretical aspects. In the last section, a special sort of symmetric extensions is studied.

1. The alternative set theory.

By AST, we mean the axiomatization of the alternative set theory as is formalized in [S 1979]. We list the axioms here, their discussion can be found in [S 1979] and [V].

There are two kinds of objects—classes and sets. From the formal viewpoint, $\mathrm{Set}\,(X)$ is a contraction for $(\exists\, Y)\,(X \in Y)$, small letters denote variables ranging over sets. A formula is called normal if there are only quantifiers with variables for sets. We suppose that our language is formalized with the basic symbols $=, \in, \neg, \&, (\forall\, x), (\forall\, X)$. By TC (theory of classes), we denote the theory (A1) + (A2) + (A3).

Axioms of the AST:

(A1) Axiom of extensionality. $(\forall\, X, Y)(X = Y \Leftrightarrow (\forall\, Z)(Z \in X \Leftrightarrow Z \in Y))$

(A2) Scheme of existence of classes (Morse's scheme).

Let a formula Φ be given. Then

$$(\forall\, X_1, \dots, X_m)(\exists\, Y)(\forall\, x)(x \in Y \Leftrightarrow \Phi(x, X_1, \dots, X_m)).$$

This axiom enables us to define usual operations with classes, the universal class V, the class of natural numbers N, the class of finite natural numbers $\mathrm{FN} = \{\alpha \in N; (\forall\, X \subseteq \alpha)\,\mathrm{Set}\,(X)\}$, the language of finite formulas FL, relation of satisfaction $\Vdash$ of formulas of FL in V, etc.

$\mathrm{We}(R)$ denotes that R is a well-ordering, $\mathrm{We}(A, R)$ means $\mathrm{We}(R)\,\&\,\mathrm{dom}\,(R) = A$. $Q \cong R$ denotes that Q and R are isomorphic well-orderings, $Q \overset{\sim}{\prec} R$ (resp. $Q \overset{\sim}{\preccurlyeq} R$) denotes that Q is a well-ordering of a smaller (resp. smaller or equal) type than R. The ordering is of the type Ω if it is an uncountable well-ordering of the smallest type, the class Ω is a fixed subclass of N such that $\langle \Omega, \leq \rangle$ is an ordering of the type Ω. The symbol $\mathcal{X}_X$ denotes the characteristic function of X, i.e. the class $(\{1\} \times X) \cup (\{0\} \times (V \setminus X))$.

(A3) Axiom of existence of sets. $\mathrm{Set}\,(\emptyset)\,\&\,(\forall\, x, y)\,\mathrm{Set}\,(x \cup \{y\})$

(A4) Axiom of induction for sets.

$$(\forall\, \varphi \in \mathrm{FL})(V \models (\varphi(\emptyset)\,\&\,(\forall\, x, y)(\varphi(x) \Rightarrow \varphi(x \cup \{y\}))) \Rightarrow (\forall\, x)\varphi(x)).$$

For us it is important that this axiom can be written in the form $(\exists\, X)\Phi(X)$, where Φ is a normal formula (see [S 1979]).

(A5) Prolongation axiom.

$$(\forall\, F)((\mathrm{Fnc}(F)\,\&\,\mathrm{dom}\,(F) = \mathrm{FN}) \Rightarrow (\exists\, f)(\mathrm{Fnc}(f)\,\&\,F \subseteq f)).$$

(A6) Axiom of choice. $(\exists\, R)\,\mathrm{We}(V, R)$

(A7) Axiom of cardinalities. $(\forall\, X)(X \preccurlyeq \mathrm{FN} \vee X \approx V)$

(A8) Scheme of regularity.

Let φ be a set-theoretical formula. Then

$$(\exists\, x)\varphi(x) \Rightarrow (\exists\, x)(\varphi(x)\,\&\,(\forall\, y \in x)\neg\varphi(y)).$$

System of classes in the theory TC means a system given by a formula with class parameters, i.e. a system of the form $\{X; \varphi(X, X_1, \ldots, X_k)\}$. A system $\mathcal{A}$ is codable if there exists a class S such that $\mathcal{A} = \{S''\{x\}; x \in V\}$.

We will refer also to some schemes and axioms not included into the axiomatization of the AST.:

(ADC) Axiom of dependent choices.

$$(\forall X)((\forall x)(\exists y)(\langle x, y\rangle \in X) \Rightarrow$$
$$\Rightarrow (\forall x)(\exists F)(\mathrm{dom}\,(F) = \mathrm{FN} \,\&\, F(0) = x \,\&\, (\forall n)(\langle F(n), F(n+1)\rangle \in X))).$$

This is a weaker form of the axiom of choice, but it will be sufficient in some proofs.

(SC) Scheme of choice (weak). Let a formula Φ be given. Then

$$(\forall n)(\exists X)\Phi(n, X) \Rightarrow (\exists Y)(\forall n)\Phi(n, Y''\{n\}).$$

(SSC) Strong scheme of choice. Let a formula Φ be given. Then

$$(\forall x)(\exists X)\Phi(x, X) \Rightarrow (\exists Y)(\forall x)\Phi(x, Y''\{x\}).$$

(SDC) Scheme of dependent choices. Let Φ be a formula such that $(\forall X)(\exists Y)$ $\Phi(X, Y)$. Then

$$(\forall X)(\exists Y)(\mathrm{dom}\,(Y) \subseteq \mathrm{FN} \,\&\, Y''\{0\} = X \,\&\, (\forall n \in \mathrm{FN})\Phi(Y''\{n\}, Y''\{n+1\})).$$

(Q) Axiom of constructibility.

$(\exists Q)(\forall X)X \in \mathcal{L}(Q)$, where $\mathcal{L}(Q)$ denotes the smallest system of classes containing Q, FN and all sets which is closed under the Morse's scheme of existence of classes and in which well-orderings are absolute.

These schemes are studied in [S 1985], where it is also proved that the axiom of constructibility implies (in (TC)+(A6)) all three schemes of choice. There is also constructed an interpretation of AST+(Q) in AST.

For some of the results, we need to develop the technique of forcing with uncodable systems of conditions. In these cases we will use systems of classes and stronger theories than those listed before. Systems of classes will be denoted by $\mathcal{X}, \mathcal{D}, \mathcal{E}, \mathcal{P}$, their relation to classes is the same as the relation of classes to sets. $\overline{\mathrm{TC}}$ (resp. $\overline{\mathrm{AST}}$) denotes the theory TC (resp. AST) extended by the axiom of extensionality for the systems of classes and the Morse's scheme of existence of the systems of classes for formulas with variables for the systems of classes; thus even in the formulas defining classes we may quantify systems of classes and use them as parameters. We will need also the following modification of (Q) : $\overline{\boldsymbol{(Q)}}$: $(\exists Q)(\forall \mathcal{X})\mathcal{X} \in \mathfrak{L}(Q)$, where $\mathfrak{L}(Q)$ is the smallest collection of classes and systems of classes containing Q, FN and all sets which is closed under Morse's scheme and in which well-orderings are absolute (for a detailed analysis of the higher order constructible processes see [S 1991]). Note that $\mathfrak{L}(Q)$ contains all sets, but only some classes and only these selected classes are elements of systems in $\mathfrak{L}(Q)$.

The metamathematical strength of these extensions will be studied in the Section 3. Here we only note that $\overline{(Q)}$ need not imply (Q), but it implies all its important consequences, as e.g. all schemes of choice.

All theorems are proved in the theory TC and its extension, or in the theory $\overline{\mathrm{TC}}$ and its extensions, if we work with an uncodable system of conditions.

2. Forcing.

We will suppose that the ordering of the system of condition is given by inclusion. But even in this case, we may work with a class of conditions with some other ordering $\leq$. We may code conditions of such a system by an isomorphic codable system of classes ordered by $\subseteq$.

In the first part of this section, we give basic definitions and lemmas on the relation of forcing. In the rest of the section, we will study under what assumptions the axioms of AST hold in the extension. The most important result is that the axiom of prolongation holds in the generic extension, if and only if the system of conditions is closed under countable monotonous intersections. This restricts our choice of systems of conditions for the concrete proofs.

Notation. (i) By the letters π, ϱ, σ, we denote variables for conditions (see Definition 2.1).
(ii) The formula $(\forall\exists\, \varrho \subseteq \pi)\varphi(\varrho)$ is a contraction for $(\forall\, \sigma \subseteq \pi)(\exists\, \varrho \subseteq \sigma)\varphi(\varrho)$ (i.e. φ holds almost everywhere under π).

Definition 2.1. A system of conditions $\mathcal{P}$ is a system of subclasses of a class P such that $P \in \mathcal{P}$ and $(\forall\, \pi \in \mathcal{P})(\forall\, x \in \pi)\neg(\forall\exists\, \sigma \subseteq \pi)(x \notin \sigma)$.

Remark. The last condition says that all elements of any condition are in some sense substantial; if, in any system, we delete all not substantial elements of all conditions, then we get an isomorphic system such that this requirement holds.

Definition 2.2. (i) We say that a system $\mathcal{D}$ is a name, if $\mathcal{D} \subseteq V \times \mathcal{P}$. The letters $\mathcal{D}, \mathcal{E}$ denote variables for names.

- **(ii)** $\check{X} =_{\mathrm{df}} X \times \mathcal{P}$,
- **(iii)** $\Gamma =_{\mathrm{df}} \{\langle x, \pi\rangle; (\forall\, \varrho \subseteq \pi)(x \in \varrho)\}$,
- **(iv)** $\mathfrak{N} =_{\mathrm{df}} \{\mathcal{D}; \mathcal{D} \subseteq V \times \mathcal{P}\}$,
- **(v)** $\mathfrak{N}_0 =_{\mathrm{df}} \{\check{X}; X \subseteq V\}$.

Remarks. $\check{X}$ is a name of the class X, Γ is a name of the generic class, $\mathfrak{N}$ is the collection of all names and $\mathfrak{N}_0$ is the collection of the names of all classes of the ground model.

By collection of names, we always mean some collection given by a predicate (otherwise it would not be a legal object in our theory). But we use for them the set-theoretical notation, since it is more convenient.

Definition 2.3. Given a system of conditions $\mathcal{P}$ and a collection of names $\mathfrak{G} \supseteq \mathfrak{N}_0$, we define the relation

$$\pi[\mathcal{P}, \mathfrak{G}] \Vdash \varphi(x_1, \ldots, x_k, \mathcal{D}_1, \ldots, \mathcal{D}_m)$$

for $\varphi \in \mathrm{FL}, \mathcal{D}_1, \ldots, \mathcal{D}_m \in \mathfrak{G}$ by induction on the length of φ:

(a) $\pi \Vdash x_1 \in x_2 \Leftrightarrow_{\mathrm{df}} x_1 \in x_2$,
$\pi \Vdash x \in \mathcal{D} \Leftrightarrow_{\mathrm{df}} (\forall\exists\, \varrho \subseteq \pi)(x \in \mathcal{D}''\{\varrho\})$,
$\pi \Vdash \mathcal{D}_1 \in \mathcal{D}_2 \Leftrightarrow_{\mathrm{df}} \pi \Vdash (\exists\, x)(x = \mathcal{D}_1 \,\&\, x \in \mathcal{D}_2)$,

(b) $\pi \Vdash \neg\varphi \Leftrightarrow_{\mathrm{df}} (\forall\, \varrho \subseteq \pi)\neg(\varrho \Vdash \psi)$,
(c) $\pi \Vdash \varphi \,\&\, \psi \Leftrightarrow_{\mathrm{df}} \pi \Vdash \varphi \,\&\, \pi \Vdash \psi$,
(d) $\pi \Vdash (\forall\, x)\varphi(x) \Leftrightarrow_{\mathrm{df}} (\forall\, x)(\pi \Vdash \varphi(x))$,
(e) $\pi \Vdash (\forall\, \mathcal{D})(\varphi(\mathcal{D}) \Leftrightarrow_{\mathrm{df}} (\forall \mathcal{D} \in \mathfrak{G})(\pi \Vdash \varphi(\mathcal{D}))$,
(f) $\pi \Vdash x_1 = x_2 \Leftrightarrow_{\mathrm{df}} x_1 = x_2$,
$\pi \Vdash x = \mathcal{D} \Leftrightarrow_{\mathrm{df}} \pi \Vdash (\forall\, y)(y \in x \Leftrightarrow y \in \mathcal{D})$,
$\pi \Vdash \mathcal{D}_1 = \mathcal{D}_2 \Leftrightarrow_{\mathrm{df}} \pi \Vdash (\forall\, y)(y \in \mathcal{D}_1 \Leftrightarrow y \in \mathcal{D}_2)$.

Notation. We write only $\pi[\mathfrak{G}] \Vdash \varphi$ or $\pi \Vdash \varphi$ instead of $\pi[\mathcal{P}, \mathfrak{G}] \Vdash \varphi$, if it cannot lead to any misunderstanding.

Remarks. In a general case, we can speak about $\mathcal{P}$ and $\mathfrak{G}$ only in the theory $\overline{\mathrm{TC}}$. But if the system $\mathcal{P}$ is codable, we can code the names by classes and use the $\mathcal{P}$ and $\mathfrak{G}$ even in the theory TC. In the corresponding theory, we may use the relation $\Vdash$ in the definitions of classes. In the sequel, we will work in the theory $\overline{\mathrm{TC}}$ or in its extension, but if the system $\mathcal{P}$ is codable, then all proofs can be done in the corresponding extension of the theory TC.

The forcing with $\mathfrak{R}$ corresponds to the construction of the generic extension, while the forcing with some symmetric collection $\mathfrak{G}$ (see Definition 2.8) corresponds to the construction of some symmetric extension.

We leave next four lemmas without proofs, because all of them are only simple calculations. They give us some rules for counting with $\Vdash$.

Lemma 2.4. *The following is equivalent:*

(i) $\pi \Vdash \varphi$,
(ii) $(\forall\, \varrho \subseteq \pi)(\varrho \Vdash \varphi)$,
(iii) $(\forall \exists\, \varrho \subseteq \pi)(\varrho \Vdash \varphi)$.

Lemma 2.5.

(i) $\pi \Vdash \varphi \vee \psi \Leftrightarrow (\forall \exists\, \varrho \subseteq \pi)(\varrho \Vdash \varphi \vee \varrho \Vdash \psi)$,
(ii) $\pi \Vdash \varphi \Rightarrow \psi \Leftrightarrow (\forall\, \varrho \subseteq \pi)(\varrho \Vdash \varphi \Rightarrow (\exists\, \sigma \subseteq \varrho)(\sigma \Vdash \psi))$,
$\pi \Vdash \varphi \Rightarrow \psi \Leftrightarrow (\forall\, \varrho \subseteq \pi)(\varrho \Vdash \varphi \Rightarrow \varrho \Vdash \psi)$,
(iii) $\pi \Vdash \varphi \Leftrightarrow \psi \Leftrightarrow (\forall\, \varrho \subseteq \pi)(\varrho \Vdash \varphi \Leftrightarrow \varrho \Vdash \psi)$,
(iv) $\pi \Vdash (\exists\, x)\varphi(x) \Leftrightarrow (\forall \exists\, \varrho \subseteq \pi)(\exists\, x)(\varrho \Vdash \varphi(x))$,
$\pi \Vdash (\exists\, \mathcal{D})\varphi(\mathcal{D}) \Leftrightarrow (\forall \exists\, \varrho \subseteq \pi)(\exists\, \mathcal{D} \in \mathfrak{G})(\varrho \Vdash \varphi(\mathcal{D}))$.

Lemma 2.6. *Let* $\Gamma \in \mathfrak{G}$. *Then*

(i) $\pi \Vdash x \in \Gamma \Leftrightarrow (\forall\, \varrho \subseteq \pi)(x \in \varrho)$,
(ii) $\pi \Vdash x \notin \Gamma \Leftrightarrow x \notin \pi$.

Lemma 2.7.

(i) $\pi[\mathfrak{R}_0] \Vdash \varphi(\check{X}_1, \ldots, \check{X}_m)) \Leftrightarrow \varphi(X_1, \ldots, X_m)$.
(ii) *Let* φ *be a normal formula,* $\mathcal{D}_1, \ldots, \mathcal{D}_m \in \mathfrak{G}_1 \cap \mathfrak{G}_2$. *Then*

$$(\pi[\mathfrak{G}_1] \Vdash \varphi(\mathcal{D}_1, \ldots, \mathcal{D}_m)) \Leftrightarrow (\pi[\mathfrak{G}_2] \Vdash \varphi(\mathcal{D}_1, \ldots, \mathcal{D}_m)).$$

(iii) *Let φ be a normal formula, $\mathfrak{G}_1 \subseteq \mathfrak{G}_2$. Then*

$$(\pi[\mathfrak{G}_1] \Vdash (\exists\,\mathcal{D}_1)\dots(\exists\,\mathcal{D}_m)\varphi(\mathcal{D}_1,\dots,\mathcal{D}_m)) \Rightarrow \\ \Rightarrow (\pi[\mathfrak{G}_2] \Vdash (\exists\,\mathcal{D}_1)\dots(\exists\,\mathcal{D}_m)\varphi(\mathcal{D}_1,\dots,\mathcal{D}_m)).$$

(iv) *Let φ be a normal formula and let $\varphi(X)$ holds. Then $(\forall\,\pi)(\pi[\mathfrak{G}] \Vdash \varphi(\check{X}))$.*

In the next part, we are going to prove that under suitable assumptions the axioms of AST hold in the extension.

Definition 2.8. A collection of names $\mathfrak{G}$ is symmetric, iff

$$(\forall\,\varphi \in \mathrm{FL})(\forall\,\mathcal{D}_1,\dots,\mathcal{D}_m \in \mathfrak{G})(\{\langle x,\pi\rangle; (\pi \Vdash \varphi(x,\mathcal{D}_1,\dots,\mathcal{D}_m))\} \in \mathfrak{G}).$$

Observation. The collections $\mathfrak{N}$ and $\mathfrak{N}_0$ are symmetric.

In the sequel, we suppose that $\mathfrak{G}$ is a symmetric collection of names. This ensures that in the extension the Morse's scheme holds.

Theorem 2.9.

(i) $P \models \mathrm{TC}$.

(ii) *If* (A4) *holds, then* $P \Vdash$ (A4).

(iii) *If* (A8) *holds, then* $P \Vdash$ (A8).

Proof: **P** $\Vdash$ **A1** follows from Definition 2.3 (f).
P $\Vdash$ **A3**, **(A4)**, **(A8)** follows from (A3), (A4), (A8) and Lemma 2.7 (iv).
P $\Vdash$ **(A2)**. The proof is similar to the classical proof of the scheme of comprehension in generic extensions. Let $\varphi \in \mathrm{FL}, \mathcal{D}_1,\dots,\mathcal{D}_m \in \mathfrak{G}$. Let

$$\mathcal{E} = \{\langle x,\pi\rangle; \pi \Vdash \varphi(x,\mathcal{D}_1,\dots,\mathcal{D}_m)\}.$$

By Definition 2.8, we have $\mathcal{E} \in \mathfrak{G}$. It is easy to verify that $P \Vdash (x \in \mathcal{E} \Leftrightarrow \varphi(x,\mathcal{D}_1,\dots,\mathcal{D}_m))$ and so $P \Vdash$ (A2). □

The next lemma says that well-orderings, finite sets and FN are preserved.

Lemma 2.10.

(i) $\mathrm{We}(R) \Leftrightarrow P \Vdash \mathrm{We}(\check{R})$.

(ii) $\mathrm{Fin}(X) \Leftrightarrow P \Vdash \mathrm{Fin}(\check{X})$.

(iii) $P \Vdash \mathrm{FN} = \check{\mathrm{FN}}$.

Proof: **(i)** $\Leftarrow$ is trivial.
$\Rightarrow$: Let $\mathrm{We}(R)$ and let $\mathcal{D}$ and π be such that $\pi \Vdash \mathcal{D} \neq \emptyset$. Let y be the R-minimum of the class $\{y; (\exists\,\varrho \subseteq \pi)(\varrho \Vdash y \in \mathcal{D})\}$ which is nonempty; let $\varrho \subseteq \pi$ be such that $\varrho \Vdash y \in \mathcal{D}$. It follows that

$$\varrho \Vdash'' \text{ is the } \check{R}\text{-minimum of } \mathcal{D}''.$$

We have proved $P \Vdash \mathrm{We}(\check{R})$.

(ii) and **(iii)** are simple consequences. □

Theorem 2.11 ($\overline{\mathbf{AST}}$–(A5)). $P \Vdash$ AST–(A5).

Proof: Let R be a well-ordering of V of the type Ω.
P $\Vdash$ (A6) follows directly from the Lemma 2.10 (i).
P $\Vdash$ (A7). From the Lemma 2.10 (i), it follows that $P \Vdash \mathrm{We}(\check{V}, \check{R})$. Because $P \Vdash$ TC, we can prove basic theorems on well-orderings, and because $P \Vdash \mathrm{FN} = \check{\mathrm{FN}}$, we have

$$P \Vdash \check{R} \text{ is of the type } \Omega.$$

$P \Vdash$ (A7) is an easy consequence. □

Till now we did not study the axiom of prolongation. This axiom does not hold in all cases. Lemma 2.12 says that it holds, iff there are no new countable classes in the extension. Lemma 2.13 says that this can be achieved, if the system $\mathcal{P}$ is closed under countable monotonous intersections. The converse also holds for generic extensions, but for symmetric extensions, this condition is only sufficient, not necessary.

Lemma 2.12 ($\overline{\mathbf{TC}}$+(A5)).

$$\pi \Vdash (\mathrm{A5}) \Leftrightarrow (\forall \mathcal{D} \in \mathfrak{G})((\pi \Vdash \mathrm{Fnc}(\mathcal{D}) \,\&\, \mathrm{dom}\,(\mathcal{D}) = \mathrm{FN}) \Rightarrow$$
$$\Rightarrow (\forall \exists\, \varrho \subseteq \pi)(\exists\, X) \varrho \Vdash \check{X} = \mathcal{D}).$$

Proof: $\Rightarrow$: Let $\pi \Vdash \mathrm{Fnc}(\mathcal{D}) \,\&\, \mathrm{dom}\,(\mathcal{D}) = \mathrm{FN}$. By $\pi \Vdash$ (A5), we have $\pi \Vdash (\exists\, f)(\mathcal{D} \subseteq f)$, thus $(\forall \exists\, \varrho \subseteq \pi)(\exists\, f)(\varrho \Vdash \mathcal{D} \subseteq f)$ and it is enough to take $X = f \restriction \mathrm{FN}$.
$\Leftarrow$: Let $\varrho \subseteq \pi, \varrho \Vdash \mathrm{Fnc}(\mathcal{D}) \,\&\, \mathrm{dom}\,(\mathcal{D}) = \mathrm{FN}$. By the assumption, we have $(\forall \exists\, \sigma \subseteq \varrho)(\exists\, X)(\sigma \Vdash \check{X} = \mathcal{D})$. Let us take $\sigma \subseteq \varrho$ and X such that $\sigma \Vdash \check{X} = \mathcal{D}$. Thus $\sigma \Vdash \mathrm{Fnc}(\check{X}) \,\&\, \mathrm{dom}\,(\check{X}) = \mathrm{FN}$ and by Lemma 2.7 we have $\mathrm{Fnc}(X) \,\&\, \mathrm{dom}\,(X) = \mathrm{FN}$. From (A5) we have f such that $X \subseteq f$, by Lemma 2.7 it holds that $\sigma \Vdash (\exists\, f)\mathcal{D} \subseteq f$. We proved that

$$(\forall\, \varrho \subseteq \pi)((\varrho \Vdash \mathrm{Fnc}(\mathcal{D}) \,\&\, \mathrm{dom}\,(\mathcal{D}) = \mathrm{FN}) \Rightarrow (\exists\, \sigma \subseteq \varrho)(\sigma \Vdash (\exists\, f)(\mathcal{D} \subseteq f)))$$

and by Lemma 2.5 (ii) $\pi \Vdash$ (A5). □

Lemma 2.13 ($\overline{\mathbf{AST}}$+ (SDC)). *Let $\mathcal{P}$ be closed under countable monotonous intersections. Then* $P \Vdash$ (A5).

Proof: We are going to prove the condition from Lemma 2.12. Let $\mathcal{D}$ be such that $\pi \Vdash \mathrm{Fnc}(\mathcal{D}) \,\&\, \mathrm{dom}\,(\mathcal{D}) = \mathrm{FN}, \varrho \subseteq \pi$. Now it is possible to construct a sequence of conditions $\varrho = \sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_n \supseteq \dots$ such that

$$(\forall\, n)(\exists\,!\, x)(\sigma_{n+1} \Vdash \langle x, n\rangle \in \mathcal{D}).$$

By (SDC) and the assumption, the class $\sigma = \bigcap\{\sigma_n; n \in \mathrm{FN}\}$ is a condition, we have

$$(\forall\, n)(\exists\,!\, x)(\sigma \Vdash \langle x, n\rangle \in \mathcal{D}).$$

Let $X = \{\langle x, n\rangle; \sigma \Vdash \langle x, n\rangle \in \mathcal{D}\}$. We have $\sigma \Vdash \check{X} = \mathcal{D}$ and by Lemma 2.12 $\pi \Vdash$ (A5). □

Remark. If $\mathcal{P}$ is codable, it is enough to use the axiom of dependent choices instead of (SDC).

The next theorem summarizes the results of this paragraph:

Theorem 2.14 ($\overline{\mathbf{AST}}$ + (SDC)). *Let the system of conditions $\mathcal{P}$ be closed under countable monotonous intersections. Then $P \Vdash \mathrm{AST}$.*

3. Constructibility, generic extensions.

In this section, we have concentrated all model-theoretical results.

At first, we are going to prove two metatheorems on the relative consistency of theories containing $\overline{(Q)}$. We use the technique of constructive processes in higher order arithmetic, see [S 1991].

Metatheorem 3.1. $\overline{\mathrm{TC}} + \mathrm{We}(N, \leq) + (\mathrm{A8}) + \overline{(Q)}$ *is consistent relatively to A_3.*

Demonstration: Let us construct an interpretation using the constructive process. We start from hereditary finite sets HF and use the interpretation corresponding to $\mathfrak{L}(\emptyset)$ (i.e. we add all constructible subsets of HF as classes and all constructible subclasses of them as systems of classes). This is an interpretation of the theory $\overline{\mathrm{TC}} + \mathrm{We}(N, \leq) + (\mathrm{A8}) \ + (\forall \mathcal{X})\mathcal{X} \in \mathcal{L}(\emptyset)$. □

Metatheorem 3.2. $\overline{\mathrm{AST}} + \overline{(Q)}$ *is consistent relatively to A_4.*

Demonstration: First, we use the interpretation $\mathbf{L}$ given by the higher order constructive process, which interprets schemes of choice. Now we take (under $\mathbf{L}$) the ultraproduct model $\mathfrak{N}$ of AST, i.e. sets are all sets in the ultraproduct $U = \mathrm{HF}^V / Z$, where Z is some nontrivial ultrafilter on natural numbers, and classes are all subsets of U, for details see [S 1982] (note that from the validity of schemes of choice and continuum hypothesis under $\mathbf{L}$ we get the axioms of choice and cardinalities in $\mathfrak{N}$). Let Q be some well-ordering of $V^{\mathfrak{N}}$ of the type Ω (in the sense of $\mathfrak{N}$). Now we take the interpretation corresponding to the constructive process $\mathfrak{L}(Q)$. This interpretation composed with $\mathbf{L}$ gives an interpretation of $\overline{\mathrm{AST}} + \overline{(Q)}$. □

The next part covers model-theoretical aspects of forcing. It is very close to the classical case, so we omit the proofs. Note that our definition of a generic class is a little different from the classical case. We define it not as a filter of conditions, but as its intersection—this ensures that it is still a class.

Let $\mathfrak{M}$ be a countable model of $\overline{\mathrm{TC}}$, let $\mathcal{P}$ and $\mathfrak{G}$ be given in $\mathfrak{M}$.

Definition 3.3. **(i)** G is a generic class, if it is an intersection of a generic filter on the system of condition (i.e. of a filter $\mathcal{F} \subseteq \mathcal{P}$ such that every dense subsystem $\mathcal{R} \subseteq \mathcal{P}, \mathcal{R} \in \mathfrak{M}$ has a nonempty intersection with $\mathcal{F}$).

(ii) Let $\mathcal{D} \in \mathfrak{G}$. We define $\mathcal{D}_G = \{x; (\exists\, \pi \supseteq G)\pi \Vdash x \in \mathcal{D}\}$.

(iii) $\mathfrak{M}[G, \mathfrak{G}] = \{\mathcal{D}_G; \mathcal{D} \in \mathfrak{G}\}$.

As in the classical case, we can prove the existence of a generic class for any countable $\mathfrak{M}$.

Metatheorem 3.4. *Let $\mathcal{D}_1, \dots, \mathcal{D}_m \in \mathfrak{G}$. Then*

(i) $\pi \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m)$, *iff for all generic classes $G \subseteq \pi$ there holds $\mathfrak{M}[G, \mathfrak{G}] \models \varphi(x, (\mathcal{D}_1)_G, \dots, (\mathcal{D}_m)_G)$.*

(ii) *Let G be a generic class. Then*

$$\mathfrak{M}[G, \mathfrak{G}] \models \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m) \Leftrightarrow (\exists\, \pi \supseteq G)(\pi \models \varphi(x, (\mathcal{D}_1)_G, \dots, (\mathcal{D}_m)_G)).$$

Now we are going to study the systems $\mathcal{L}(X)$. We reformulate the absoluteness of $\mathcal{L}$ in the language of forcing, which will be useful in the Section 4. Then we prove that in the case of a codable system of conditions, the generic extension satisfies the axiom of constructibility. For an uncodable system of conditions, the situation with constructibility is more complicated. We cannot ensure that we start in a model which satisfies (Q). But even so we can prove that the model $\mathfrak{M}$ (restricted to classes) is definable in $\mathfrak{M}[\mathfrak{G}, G]$, if $\overline{(Q)}$ holds in $\mathfrak{M}$.

Lemma 3.5. *Let $\mathcal{D} \in \mathfrak{G}$ and $\mathcal{E} \in \mathfrak{R}$. Then*

$$\pi[\mathfrak{R}] \Vdash \mathcal{E} \in \mathcal{L}(\mathcal{D}) \Leftrightarrow (\forall \exists\, \varrho \subseteq \pi)(\exists\, \mathcal{E}_0 \in \mathfrak{G})(\varrho[\mathfrak{R}] \Vdash \mathcal{E} = \mathcal{E}_0 \,\&\, \varrho[\mathfrak{G}] \Vdash \mathcal{E}_0 \in \mathcal{L}(\mathcal{D})).$$

Proof: We will prove that the assertion holds in any countable model of $\overline{\mathrm{TC}}$— this implies that it is provable in $\overline{\mathrm{TC}}$. Let $\mathfrak{M}$ be any countable model of $\overline{\mathrm{TC}}$. By Metatheorem 3.4, the assertion of the lemma in $\mathfrak{M}$ is a consequence of:

$$\text{For all } G \text{ generic, } D \in \mathfrak{M}[G, \mathfrak{G}] \text{ and } E \in \mathfrak{M}[G, \mathfrak{R}]$$
$$\mathfrak{M}[G, \mathfrak{R}] \models E \in \mathcal{L}(\mathcal{D}) \Leftrightarrow (E \in \mathfrak{M}[G, \mathfrak{G}] \,\&\, \mathfrak{M}[G, \mathfrak{G}] \models E \in \mathcal{L}(\mathcal{D})).$$

But this is true for any $\mathfrak{M}$, since $\mathfrak{M}[G, \mathfrak{G}]$ is closed under Morse's scheme and $\mathcal{L}$ is absolute (because sets, $\in$ and well-orderings are absolute in the models in question). □

Metatheorem 3.6. *Let $\mathfrak{M} \models$ TC$+(Q)$, let $\mathcal{P}$ be codable. Then $\mathfrak{M}[G, \mathfrak{R}] \models$ TC$+(Q)$, namely $\mathfrak{M}[G, \mathfrak{R}] \models (\forall X) X \in \mathcal{L}(\langle Q, G \rangle)$, where Q is such that $\mathfrak{M} \models (\forall X) X \in \mathcal{L}(Q)$.*

Proof: Each element of $\mathfrak{M}[G, \mathfrak{R}]$ is definable from G and some $\mathcal{D} \in \mathfrak{M} = \mathcal{L}(Q)$, thus $\mathfrak{M}[G, \mathfrak{R}] = \mathcal{L}(\langle Q, G \rangle)$. □

Corollary (TC+(Q)). *Let $\mathcal{P}$ be codable, let $\mathfrak{G} = \mathfrak{R}$. Then $P \Vdash (Q)$.*

Metatheorem 3.7. *Let $\mathfrak{M} \models \overline{\mathrm{TC}} + \overline{(Q)}$, Let G generic be given. Then there exists a formula $\Phi(X)$ in the language of* TC *such that $X \in \mathfrak{M}$, iff $\mathfrak{M}[G, \mathfrak{G}] \models \Phi(X)$.*

Proof: Let us fix Q such that $\mathfrak{M} \models (\forall \mathcal{X}) \mathcal{X} \in \mathfrak{L}(Q)$ and let us study the constructive process $\mathfrak{L}(Q)$ in $\mathfrak{M}$. By the standard argument (used in the proof of the continuum hypothesis in classical L), each class is added at some codable step of this process. But each such fragment of the process can be coded by some class constructible without mentioning system of classes, and thus $\mathfrak{M} \models (\forall X)(\exists R)(\mathrm{We}(R) \,\&\, X \in$

$\mathcal{L}(R,Q))$ (where $\mathcal{L}(R,Q)$ denotes the system of all classes constructed up to the step R of the constructive process starting from Q).

Because the systems $\mathcal{L}(R,Q)$ are absolute and depend only on the type of the ordering R, it is now sufficient to define in $\mathfrak{M}[G,\mathfrak{G}]$ the system of all well-orderings isomorphic to some well-ordering in $\mathfrak{M}$. We distinguish two cases.

(A) Suppose that in $\mathfrak{M}[G,\mathfrak{G}]$ there are no well-orderings longer than in $\mathfrak{M}$. Then we have

$$X \in \mathfrak{M} \text{ iff } \mathfrak{M}[G,\mathfrak{G}] \models (\exists\, R)(\mathrm{We}(R) \,\&\, X \in \mathcal{L}(R,Q))$$

and we are done.

(B) Let S be a well-ordering in $\mathfrak{M}[G,\mathfrak{G}]$ such that each of its segments except for S itself is isomorphic to some well-ordering in $\mathfrak{M}$. S must be limit, hence $\mathcal{L}(S,Q) = \bigcup\{\mathcal{L}(R,Q); R \in \mathfrak{M} \,\&\, \mathrm{We}(R)\}$ and

$$X \in \mathfrak{M} \text{ iff } \mathfrak{M}[G,\mathfrak{G}] \models X \in \mathcal{L}(S,Q).$$

□

The last theorem of this section characterizes metamathematical consequences of forcing. It states the model-theoretical consequence of forcing, the relative consistency of corresponding theories easily follows.

Metatheorem 3.8. *Let T, S but such theories that for some $\mathcal{P}, \mathfrak{G}$ and $\pi \in \mathcal{P}$ there holds $T \vdash (\pi \Vdash S)$, let $\mathfrak{M}$ be a countable model of T. Then there exists a countable model $\mathfrak{N} \supseteq \mathfrak{M}$ such that sets and the relation of $\in$ are absolute relatively to $\mathfrak{M}$ and $\mathfrak{N} \models S$.*

Proof: As in the classical case, we can find a G generic such that $G \subseteq \pi$. Now take $\mathfrak{N} = \mathfrak{M}[G,\mathfrak{G}]$ and use Metatheorem 3.4. □

4. Iterated forcing.

In this paragraph, we are going to study symmetric extensions of a special type—extensions given as a union of generic extensions with the system of conditions $\mathcal{P}_Z$ over an ideal of sets $\mathcal{A}$, where $\mathcal{P}_Z$ is the cartesian power of $\mathcal{P}$ to Z. We will show that this extension can be defined as a symmetric extension.

We will study the system $\mathcal{L}(X)$ carefully. We will show that these extensions do not satisfy the axiom of constructibility (except for the case of the trivial system $\mathcal{A}$—in this case, the extension is a generic extension and the axiom of constructibility does hold at least in the case $\mathcal{P}$ codable). Then we will show that in some cases it is possible to define the smallest extension (from our system) which includes a new class.

Let $\mathcal{A}$ be a codable system such that for $X, Y \in \mathcal{A}$ there exists $Z \in \mathcal{A}$ such that $X \cup Y \subseteq Z$. Let us denote $A = \bigcup \mathcal{A}$. Let $Z \in \mathcal{A}$.

Definition 4.1.

(i) $\mathcal{P}_Z =_{\mathrm{df}} \{\pi \subseteq P \times Z; (\forall\, a \in Z)(\pi''\{a\} \in \mathcal{P})\}$.

(ii) $\pi/Z =_{\mathrm{df}} \pi \cup (P \times (A \setminus Z))$.

(iii) $P_{\mathcal{A}} =_{\mathrm{df}} \{\pi/Z; Z \in \mathcal{A}, \pi \in \mathcal{P}_Z\}$

(π/Z has two meanings: if $\pi \in \mathcal{P}_Z$, then π/Z is its image in $\mathcal{P}_{\mathcal{A}}$ ($\restriction Z$ is in this case the inverse operation to $/Z$); if $\pi \in \mathcal{P}_{\mathcal{A}}$, then π/Z is the greatest condition ϱ such that $\varrho \restriction Z \subseteq \pi$).

Observation. (i) The system $\mathcal{P}_Z$ is closed under countable monotonous intersections, iff the system $\mathcal{P}$ is closed under countable monotonous intersections.

(ii) If the system $\mathcal{P}$ is a system of conditions (in sense of Definition 2.1), then also the systems $\mathcal{P}_Z$ and $\mathcal{P}_{\mathcal{A}}$ are systems of conditions.

Both the maximal conditions in $\mathcal{P}_Z$ and $\mathcal{P}_{\mathcal{A}}$ are denoted by P. The collections $\mathfrak{N}$ for the systems $\mathcal{P}_Z$ and $\mathcal{P}_{\mathcal{A}}$ are denoted by $\mathfrak{N}_Z$ and $\mathfrak{N}_{\mathcal{A}}$.

Now we will define the collections $\mathfrak{G}_Z \subseteq \mathfrak{N}_{\mathcal{A}}$ such that forcing with $\mathcal{P}_{\mathcal{A}}$ and $\mathfrak{G}_Z$ is equivalent to forcing with $\mathcal{P}_Z$ and $\mathfrak{N}_Z$. The union of these collections will be the main object of our interest.

Definition 4.2. Let us define the collections $\mathfrak{G}_Z, \mathfrak{G}_{\mathcal{A}} \subseteq \mathfrak{N}_{\mathcal{A}}$:

- **(i)** $\mathfrak{G}_Z =_{\text{df}} \{\mathcal{D} \in \mathfrak{N}_{\mathcal{A}}; (\forall \pi \in \mathcal{P}_{\mathcal{A}})(\mathcal{D}''\{\pi\} = \mathcal{D}''\{\pi/Z\})\}$ (i.e. $\mathcal{D}$ depends only on the part of π over Z),
- **(ii)** $\mathfrak{G}_{\mathcal{A}} =_{\text{df}} \bigcup\{\mathfrak{G}_Z; Z \in \mathcal{A}\}$.

Observation. $(\forall T \in \mathcal{A})(T \subseteq Z \Rightarrow \mathfrak{G}_T \subseteq \mathfrak{G}_Z)$.

Notation. For each name $\mathcal{D} \in \mathfrak{N}_Z$ there exists a unique corresponding name in $\mathfrak{G}_Z$; we will not distinguish these names. Thus Γ_Z denotes the name Γ for the collection $\mathfrak{N}_Z$ and the unique corresponding name from $\mathfrak{G}_Z$ as well.

By an easy induction on the length of the formula φ, we can prove the following lemma, which formulates the equivalence between forcing with $[\mathcal{P}_{\mathcal{A}}, \mathfrak{G}_Z]$ and $[\mathcal{P}_Z, \mathfrak{N}_Z]$. The corollary summarizes the results from the paragraphs 2 and 3 for $\mathfrak{G}_Z$.

Lemma 4.3. *Let* $\varphi \in \text{FL}, \mathcal{D}_1, \ldots, \mathcal{D}_m \in \mathfrak{G}_Z, \pi \in \mathcal{P}_{\mathcal{A}}$. *Then*

$$\pi[\mathcal{P}_{\mathcal{A}}, \mathfrak{G}_Z] \Vdash \varphi(x_1, \ldots, x_k, \mathcal{D}_1, \ldots, \mathcal{D}_m) \Leftrightarrow$$
$$\Leftrightarrow \pi \restriction Z[\mathcal{P}_Z, \mathfrak{N}_Z] \Vdash \varphi(x_1, \ldots, x_k, \mathcal{D}_1, \ldots, \mathcal{D}_m).$$

Corollary.

- **(i)** $\mathfrak{G}_Z$ *is a symmetric collection of names.*
- **(ii)** $P[\mathfrak{G}_Z] \Vdash \text{TC}$.
- **(iii)** *Let the system* $\mathcal{P}$ *be closed under countable monotonous intersections. Then* $\overline{\text{AST}} + (\text{SDC}) \vdash P[\mathfrak{G}_Z] \Vdash \text{AST}$.
- **(iv)** *Let* $\mathcal{P}$ *be codable. Then* $\text{TC} + (Q) \vdash P[\mathfrak{G}_Z] \Vdash \text{TC} + (Q)$.

The symbol $\Vdash$ in the rest of this paragraph means $[\mathcal{P}_{\mathcal{A}}, \mathfrak{G}_{\mathcal{A}}] \Vdash$. We are going to study this forcing. Our first goal is to prove that $\mathfrak{G}_{\mathcal{A}}$ is a symmetric collection of names. This will be done by a technique similar to the one used in the construction of symmetric extensions.

Definition 4.4.

(i) $\mathcal{F}$ is an automorphism of the system $\mathcal{P}$, if $\mathcal{F}$ is a one to one map from $\mathcal{P}$ onto $\mathcal{P}$ and there holds

$$\varrho \subseteq \pi \Leftrightarrow \mathcal{F}(\varrho) \subseteq \mathcal{F}(\pi).$$

(ii) $\mathcal{F}$ is symmetric automorphism of the system $\mathcal{P}_{\mathcal{A}}$, if $\mathcal{F}$ is an automorphism of $\mathcal{P}_{\mathcal{A}}$ and

$$(\forall Z \in \mathcal{A})(\exists Y \in \mathcal{A})(Z \subseteq Y \,\&\, \mathcal{F} \text{ is an automorphism of } \mathcal{P}_Y).$$

(iii) Let $\mathcal{F}$ be an automorphism of $\mathcal{P}$. Let $\mathcal{D} \in \mathfrak{N}$. We define the name $\overline{\mathcal{F}}(\mathcal{D}) \in \mathfrak{N}$ by

$$\langle x, \pi\rangle \in \overline{\mathcal{F}}(\mathcal{D}) \Leftrightarrow_{\text{df}} \langle x, \mathcal{F}^{-1}(\pi)\rangle \in \mathcal{D}.$$

Following two simple lemmas assert that the extension of the automorphism $\mathcal{F}$ on names is well-defined and that it has a good relation to the forcing.

Lemma 4.5. **(i)** *Let $\mathcal{F}$ be an automorphism of $\mathcal{P}$. Then $\overline{\mathcal{F}}$ is one to one map from $\mathfrak{N}$ onto $\mathfrak{N}$.*

(ii) *Let $\mathcal{F}$ be a symmetric automorphism of $\mathcal{P}_{\mathcal{A}}$. Then $\overline{\mathcal{F}}$ is a one to one map from $\mathfrak{S}_{\mathcal{A}}$ onto $\mathfrak{S}_{\mathcal{A}}$.*

Lemma 4.6. *Let $\varphi \in \mathrm{FL}, \mathcal{D}_1, \dots, \mathcal{D}_m \in \mathfrak{S}_{\mathcal{A}}$. Let $\mathcal{F}$ be a symmetric automorphism of $\mathcal{P}_{\mathcal{A}}$. Then*

$$\pi \Vdash \varphi(x_1, \dots, x_k, \mathcal{D}_1, \dots, \mathcal{D}_m) \Leftrightarrow$$
$$\Leftrightarrow \mathcal{F}(\pi) \Vdash \varphi(x_1, \dots, x_k, \overline{\mathcal{F}}(\mathcal{D}_1), \dots, \overline{\mathcal{F}}(\mathcal{D}_m)).$$

For the proof that $\mathfrak{S}_{\mathcal{A}}$ is a symmetric collection of names, we need some concrete automorphisms. Let G be a permutation of A such that $(\exists T \in \mathcal{A})(G \restriction (A \setminus T) = \mathrm{Id} \restriction (A \setminus T))$. Let us denote

$$F_G =_{\text{df}} \{\langle\langle p, G(a)\rangle, \langle p, a\rangle\rangle; p \in P, a \in A\},$$
$$\mathcal{F}_G(\pi) =_{\text{df}} F_G'' \pi$$

(i.e. $\mathcal{F}$ rotates the components of π by the permutation G). Then the map $\mathcal{F}_G$ is a symmetric automorphism of $\mathcal{P}_{\mathcal{A}}$.

Theorem 4.7 ($\overline{\mathbf{TC}}$ + (A6)). *$\mathfrak{S}_{\mathcal{A}}$ is a symmetric collection of names.*

PROOF: In the theory TC + (A6) it is possible to define cardinalities of classes and these will be well-ordered. Thus we may fix $T \in \mathcal{A}$ such that

$$(\forall Y \in \mathcal{A}, Y \supseteq T)(\exists U \in \mathcal{A})(Y \subseteq U \,\&\, Y \setminus T \preccurlyeq U \setminus Y)$$

(if for some Y this does not hold, we may take it as the next candidate on T, but due to the well-ordering of cardinalities, this process must stop after a finite number of steps).

Let $\varphi \in \mathrm{FL}, \mathcal{D}_1, \dots, \mathcal{D}_m \in \mathfrak{G}_{\mathcal{A}}$. Let us take $Z \in \mathcal{A}$ such that $\mathcal{D}_1, \dots, \mathcal{D}_m \in \mathfrak{G}_Z$ and $T \subseteq Z$. We need to prove that

$$\pi \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m) \Leftrightarrow \pi/Z \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m).$$

$\Leftarrow$ is trivial.

$\Rightarrow$: Let $\varrho \subseteq \pi/Z$. We will find $\sigma \subseteq \varrho$ such that

$$\sigma \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m).$$

Let us take $Y \in \mathcal{A}$ such that $Z \subseteq Y$ and $\varrho \in \mathcal{P}_Y$. By the definition of T there exists $U \in \mathcal{A}$ such that $Y \setminus Z \preccurlyeq Y \setminus T \preccurlyeq U \setminus Y$. Let H be one to one map of $Y \setminus Z$ into $U \setminus Y$. Let G be as follows:

$$G = H \cup H^{-1} \cup \mathrm{Id} \restriction (A \setminus \mathrm{dom}\,(H \cup H^{-1})).$$

Trivially $G \restriction Z = \mathrm{Id} \restriction Z$ and $G \restriction (A \setminus U) = \mathrm{Id} \restriction (A \setminus U)$. Let $\sigma = \varrho \cap \mathcal{F}_G(\pi)$. We will verify that $\sigma \in \mathcal{P}_U$:

If $a \in Z$, then $\varrho''\{a\} \subseteq \pi''\{a\} = (\mathcal{F}_G(\pi))''\{a\}$, thus $\sigma''\{a\} = \varrho''\{a\}$.
If $a \in Y \setminus Z$, then $(\mathcal{F}_G(\pi))''\{a\} = P$, thus $\sigma''\{a\} = \varrho''\{a\}$.
If $a \in A \setminus Y$, then $\varrho''\{a\} = P$, thus $\sigma''\{a\} = (\mathcal{F}_G(\pi))''\{a\}$.

$\mathcal{F}_G$ is a symmetric automorphism of $\mathcal{P}_{\mathcal{A}}$. Because $\mathcal{F}_G \restriction \mathcal{P}_Z = \mathrm{Id}$, we have

$$\overline{\mathcal{F}}_G(\mathcal{D}_1) = \mathcal{D}_1, \dots, \overline{\mathcal{F}}_G(\mathcal{D}_m) = \mathcal{D}_m.$$

By Lemma 4.6 $\mathcal{F}_G(\pi) \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m)$, and thus $\sigma \Vdash \varphi(x, \mathcal{D}_1, \dots, \mathcal{D}_m)$. □

The following theorem summarizes the results from the paragraph 2 for the forcing with $\mathcal{P}_{\mathcal{A}}$ and $\mathfrak{G}_{\mathcal{A}}$.

Observation. Let Φ be a normal formula. Then

$$\pi \Vdash (\exists\, \mathcal{D})\Phi(\mathcal{D}) \Leftrightarrow (\forall \exists\, \varrho \subseteq \pi)(\exists\, U \in \mathcal{A})(\varrho[\mathfrak{G}_U] \Vdash (\exists\, \mathcal{D})\Phi(\mathcal{D})).$$

Theorem 4.8.

(i) $P \Vdash \mathrm{TC}$.
(ii) *Let $\mathcal{P}$ be closed under countable monotonous intersections. Then* AST $\vdash$ $P \Vdash$ AST.

Proof: Nontrivial is only the proof of $P \Vdash$ (A5). This follows by the observation from the fact that $\neg$ (A5) is of the form $(\exists\, X)\varphi(X)$, (φ is a normal formula with the parameter FN) and from $P[\mathfrak{G}_Z] \Vdash$ (A5). □

Now we are going to study the questions connected with the systems $\mathcal{L}(X)$.

Lemma 4.9 ($\overline{\mathbf{TC}}$). *Let $\mathcal{D}, \mathcal{E} \in \mathfrak{G}_Z$. Then $\pi \Vdash \mathcal{E} \in \mathcal{L}(\mathcal{D}) \Leftrightarrow \pi/Z[\mathfrak{G}_Z] \Vdash \mathcal{E} \in \mathcal{L}(\mathcal{D})$.*

Proof: As in Lemma 3.5. □

In the next theorem, we prove that the axiom of the constructibility does not hold, if both $\mathcal{P}$ and $\mathcal{A}$ are nontrivial (i.e. $\mathcal{P}$ is atomless and $A \notin \mathcal{A}$).

Theorem 4.10 ($\overline{\mathbf{TC}}$ + (A6)). *Let* $(\forall\,\pi \in \mathcal{P})(\exists\,\varrho)(\varrho \subset \sigma)$ *and* $x \notin Z$*, let* $\mathcal{D} \in \mathfrak{G}_Z$*. Then* $P \Vdash \Gamma_{\{x\}} \notin \mathcal{L}(\mathcal{D})$.

Proof: By contradiction. In the opposite case, by Lemma 3.5 and Lemma 4.9 there must exists π and $\mathcal{E} \in \mathfrak{G}_Z$ such that

$$\pi \Vdash \mathcal{E} = \Gamma_{\{x\}}\,.$$

Let us take $\varrho \subset \pi''\{x\}$ from the assumption of the theorem, $a \in \pi''\{x\} \setminus \varrho$, the conditions ϱ_1 and π_1 such that

$$\varrho_1''\{x\} = \varrho, \pi_1''\{x\} = \pi''\{x\},$$
$$\varrho_1''\{y\} = \pi_1''\{y\} = \pi''\{y\} \text{ if } y \neq x.$$

By Lemma 2.6, we have

$$\varrho_1 \Vdash \langle a, x\rangle \notin \Gamma_{\{x\}}\,, \neg(\pi_1 \Vdash \langle a, x\rangle \notin \Gamma_{\{x\}}).$$

Since $\mathcal{E} \in \mathfrak{G}_Z$, we have by the definition of $\mathfrak{G}_Z$

$$\varrho_1 \Vdash \langle a, x\rangle \notin \mathcal{E} \Leftrightarrow \pi_1 \Vdash \langle a, x\rangle \notin \mathcal{E},$$

a contradiction with $\pi \Vdash \mathcal{E} = \Gamma_{\{x\}}$. □

Corollary. *In addition, let* $A \notin \mathcal{A}$*. Then* $P \Vdash (\forall\,\mathcal{D})(\exists\,\mathcal{E})(\mathcal{E} \notin \mathcal{L}(\mathcal{D}))$.

Now we will show that (under some assumptions) it is possible to define the least Z such that in $\mathfrak{G}_Z$ we can define a class equal to $\mathcal{D}$.

Notation. $\varrho =_a \sigma \Leftrightarrow (\forall\,x \neq a)(\varrho''\{x\} = \sigma''\{x\})$.

Definition 4.11. Let $\mathcal{D} \in \mathfrak{G}_{\mathcal{A}}$.

$$\operatorname{rank}(\mathcal{D}) =_{\mathrm{df}} \{\langle a, \pi\rangle; (\exists\,x)(\exists\,\sigma_1, \sigma_2 \subseteq \pi)(\sigma_1 =_a \sigma_2 \,\&\, \sigma_1 \Vdash x \in \mathcal{D} \,\&\, \sigma_2 \Vdash x \notin \mathcal{D})\}.$$

Observation. Let $\mathcal{D} \in \mathfrak{G}_Z$. Then $\operatorname{rank}(\mathcal{D}) \in \mathfrak{G}_Z$ and $P \Vdash \operatorname{rank}(\mathcal{D}) \subseteq \check{Z}$.

Theorem 4.12 ($\overline{\mathbf{TC}}$ + (*Q*)). *Let* Z *be finite,* $\mathcal{D} \in \mathfrak{G}_Z, T \subseteq Z$*, let* $\pi \Vdash \operatorname{rank}(\mathcal{D}) = \check{T}$*. Then* $(\forall\exists\,\varrho \subseteq \pi)(\exists\,\mathcal{E} \in \mathfrak{G}_T)(\varrho \Vdash \mathcal{D} = \mathcal{E})$.

Because of finiteness of Z, it is enough to find, for a given $y \in Z \setminus T$, some $\varrho \subseteq \pi$ and $\mathcal{E} \in \mathfrak{G}_{Z\setminus\{y\}}$ such that $\varrho \Vdash \mathcal{D} = \mathcal{E}$. Let $y \in Z \setminus T$. Then

$$\pi \Vdash y \notin \operatorname{rank}(\mathcal{D})$$

and thus there exists $\varrho \subseteq \pi$ such that

$$(\forall\,x)(\forall\;\sigma_1, \sigma_2 \subseteq \varrho)(\sigma_1 =_y \sigma_2 \Rightarrow \neg(\sigma_1 \Vdash x \in \mathcal{D} \,\&\, \sigma_2 \Vdash x \notin \mathcal{D})).$$

Let us take

$$\mathcal{E} = \{\langle x, \sigma\rangle; (\exists\,\sigma_1 \subseteq \varrho)(\sigma_1 =_y \sigma \,\&\, \sigma_1 \Vdash x \in \mathcal{D})\}.$$

Trivially $\mathcal{E} \in \mathfrak{G}_{Z\setminus\{y\}}$. We want to prove that $\varrho \Vdash \mathcal{E} = \mathcal{D}$. We will show that for $\sigma \subseteq \varrho$ there holds $\langle x, \sigma\rangle \in \mathcal{E} \Leftrightarrow \sigma \Vdash x \in \mathcal{D}$:
$\Leftarrow$ is trivial.
$\Rightarrow$: We have $(\exists\,\sigma_1 \subseteq \varrho)(\sigma_1 =_y \sigma \,\&\, \sigma_1 \Vdash x \in \mathcal{D})$. By assumption on ϱ, we have $(\forall\,\sigma_2 \subseteq \sigma)\neg(\sigma_2 \Vdash x \notin \mathcal{D})$ and thus $(\forall\exists\,\sigma_2 \subseteq \sigma)(\sigma_2 \Vdash x \in \mathcal{D})$ and $\sigma \Vdash x \in \mathcal{D}$. □

References

[S 1979] Sochor A., *Metamathematics of the Alternative set theory I*, Comment. Math. Univ. Carolinae **20** (1979), 697–722.
[S 1982] Sochor A., *Metamathematics of the Alternative set theory II*, Comment. Math. Univ. Carolinae **23** (1982), 55–79.
[S 1985] Sochor A., *Constructibility and shiftings of view*, Comment. Math. Univ. Carolinae **26** (1985), 477–498.
[S 1991] Sochor A., *Scheme of choice in higher order arithmetics*, to appear.
[V] Vopěnka P., *Mathematics in the Alternative Set Theory*, Leipzig, 1979.

Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia

(*Received* August 16, 1990)