New properties of the concentric circle space and its applications to cardinal inequalities

Shu-Hao Sun, Koo-Guan Choo

Abstract. It is well-known that the concentric circle space has no G_δ-diagonal nor any countable point-separating open cover. In this paper, we reveal two new properties of the concentric circle space, which are the weak versions of G_δ-diagonal and countable point-separating open cover. Then we introduce two new cardinal functions and sharpen some known cardinal inequalities.

Keywords: concentric circle space, weak G_δ-diagonal, point-separating *-open cover, cardinal function

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1. Concentric circle space and its new properties.

Let us first recall the definition of the concentric circle space or the Alexandroff double circle space. Let

$$C_i = \{(x,y) \mid x^2 + y^2 = i\}, \quad (i = 1,2),$$

and let $P : C_1 \to C_2$ be the projection of C_1 onto C_2 from the origin $(0,0)$. Let $X = C_1 \cup C_2$ and we define the neighbourhood system $\{\mathcal{B}(z)\}$ of X as follows: let

$$\{\mathcal{B}(z)\} = \begin{cases} \{\{z\}\}, & \text{for } z \in C_2, \\ \{U_j(z)\}_{j=1}^{\infty} & \text{for } z \in C_1, \end{cases}$$

where

$$U_j(z) = V_j(z) \cup P(V_j(z) - \{z\}),$$

and $V_j(z)$ is the arc of C_1 with center at z and length $1/j$. Then such X (with the defined neighbourhood system) is called the *concentric circle space* or *Alexandroff double circle space*. It is well-known that the concentric circle space X is a compact T_2 space (in fact, T_5 space) (cf. [2]).

Next, we recall that a topological space Y has a *G_δ-diagonal*, iff there exists a sequence of open covers $\{\mathcal{U}_n\}$ of Y with

$$\bigcap_n \mathrm{St}\,(y, \mathcal{U}_n) = \{y\}$$

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for each $y \in Y$, where

$$\operatorname{St}(y, \mathcal{U}) = \bigcap \{B \in \mathcal{U} \mid y \in B\}.$$

A cover $\mathcal{U}$ of Y is called *point-separating*, if for each $y \in Y$,

$$\bigcap \{U \in \mathcal{U} \mid y \in U\} = \{y\}.$$

It is also well-known that the concentric circle space X is not metrizable, and so it has no G_δ-diagonal nor any countable point-separating open cover. Although X has no G_δ-diagonal, we will show that it has a weak G_δ-diagonal as defined below. We will also show that X has a countable point-separating *-open cover as defined below.

Definition. Let Y be any topological space. Then a collection $\mathcal{U}$ of subsets of Y is called a **-open collection*, if for each $y \in Y, \operatorname{St}(y, \mathcal{U})$ is an open set. Moreover, if for each $y \in Y, \operatorname{St}(y, \mathcal{U})$ is a non-empty open set, then $\mathcal{U}$ is called a **-open cover*.

A space Y is said to have a *weak G_δ-diagonal*, if there is a sequence $\{\mathcal{U}_n\}$ of *-open covers such that

$$\bigcap_n \operatorname{St}(y, \mathcal{U}_n) = \{y\},$$

for each $y \in Y$.

Remark. A collection of open sets is clearly a *-open collection. But the converse is not true. For example

$$\mathcal{U} = \{\{y\}\} \cup \{Y\}$$

is a *-open cover of Y, but it is not an open cover, if Y is not discrete. On the other hand, if for each $\mathcal{V} \subseteq \mathcal{U}$, $\mathcal{V}$ is a *-open collection, then it is easy to check that $\mathcal{U}$ has to be an open collection.

Lemma 1. *A topological space Y has a weak G_δ-diagonal, if there is a mapping $g : Y \times \mathbb{N} \to \tau$, where τ is the topology of Y, such that for each $y \in Y$,*

$$\bigcap_{n \in \mathbb{N}} g(y, n) = \{y\},$$

and for each $n \in \mathbb{N}, x, y, \in Y, y \in g(x, n)$ implies $x \in g(y, n)$.

Proof: Suppose that Y has a weak G_δ-diagonal; i.e., suppose that Y has a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of *-open covers such that $\bigcap_n \operatorname{St}(y, \mathcal{U}_n) = \{y\}$ for each $y \in Y$. Define $g : Y \times \mathbb{N} \to \tau$ by

$$g(y, n) = \operatorname{St}(y, \mathcal{U}_n).$$

Then clearly g has the required properties.

Conversely, suppose that the mapping g with the required property is given. For each $y \in Y$ and $n \in \mathbb{N}$, let

$$R_n(y) = \{\{y, x\} \mid x \in g(y, n)\}$$

and

$$\mathcal{U}_n = \bigcup_{y \in Y} R_n(y).$$

Then $\{\mathcal{U}_n\}_{n=1}^{\infty}$ is the required sequence of *-open covers such that

$$\bigcap_n \mathrm{St}\,(y, \mathcal{U}_n) = \{y\}$$

for each $y \in Y$. Firstly, for each $n \in n$, $\mathcal{U}_n$ is a cover of Y. Next, for each $y \in Y$ and $n \in \mathbb{N}, \mathrm{St}\,(y, \mathcal{U}_n) = g(y, n)$. Clearly $g(y, n) \subseteq \mathrm{St}\,(y, \mathcal{U}_n)$. Now, if $x \in \mathrm{St}\,(y, \mathcal{U}_n)$, then

$$y \in \bigcup_{x \in Y} R_n(x),$$

i.e., $y \in g(x, n)$ so that $x \in g(y, n)$. Thus $\mathrm{St}\,(y, \mathcal{U}_n) \subseteq g(y, n)$. This completes the proof. □

Proposition 1. *The concentric circle space X has a weak G_δ-diagonal.*

Proof: Define $g : X \times \mathbb{N} \to \tau$ by

$$g(x, n) = \begin{cases} U_n(x), & \text{if } x = z \in C_1, \\ (U_n(z) - \{z\}) \cup \{x\}, & \text{if } x = P(z) \in C_2, z \in C_1. \end{cases}$$

Then clearly for each $n \in \mathbb{N}$,

$$\bigcup_{x \in X} g(x, n) = X,$$

for each $x \in X$,

$$\bigcap_{n \in \mathbb{N}} g(x, n) = \{x\},$$

and for each $x \in X, n \in \mathbb{N}, g(x, n)$ is open.

By Lemma 1, it remains to show that for each $n \in \mathbb{N}$, and for any $x, y \in X$, $x \in g(y, n)$ implies $y \in g(x, n)$. We divide this into four cases.

(i) Both $x, y \in C_1$. If

$$y \in g(x, n) = U_n(x) = V_n(x) \cup P(V_n(x) - \{x\}),$$

then $y \in V_n(x)$ so that $x \in V_n(y) \subseteq U_n(y) = g(y, n)$.

(ii) $x \in C_1$ and $y \in C_2$. Let $y = P(z)$, where $z \in C_1$. If

$$y \in g(x,n) = V_n(x) \cup P(V_n(x) - \{x\}),$$

then $y \in P(V_n(x) - \{x\})$ so that $z \in V_n(x) - \{x\}$ and thus

$$x \in V_n(z) - \{z\} \subseteq (U_n(z) - \{z\}) \cup \{y\} = g(y,n).$$

(iii) $x \in C_2$ and $y \in C_1$. Let $x = P(w)$, where $w \in C_1$. If

$$y \in g(x,n) = (U_n(w) - \{w\}) \cup \{x\},$$

then $y \in V_n(w) - \{w\}$ so that $w \in V_n(y) - \{y\}$ and hence

$$x = P(w) \in P(V_n(y) - \{y\}) \subseteq U_n(y) = g(y,n).$$

(iv) Both $x, y \in C_2$. Let $x = P(w)$ and $y = P(z)$, where $w, z \in C_1$. If

$$y \in g(x,n) = (U_n(a) - \{a\}) \cup \{x\},$$

then $y \in P(V_n(w))$ so that $z \in V_n(w)$. Thus $w \in V_n(z)$ and therefore

$$x = P(w) \in P(V_n(z)) \subseteq g(y,n).$$

This completes the proof. □

For convenience, we now modify slightly the basic sets in the Alexandroff double circle space as follows: let

$$X_i = [0,1] \times \{i\}$$

replace C_i for $i = 1, 2$, and transform the projection P onto a mapping which maps $(a, 1)$ into $(a, 2)$ for each $a \in [0, 1]$. Since a circle is obtained by identifying the end points of $[0, 1]$, this is consistent with the previous definition.

The following proposition shows that although the Alexandroff double circle space X does not have any countable point-separating open cover, it does have a pointwise countable point-separating *-open cover.

Proposition 2. *For the Alexandroff double circle space X, there is a cover $\mathcal{U}$ such that*

$$\bigcap \mathcal{U}_x = \bigcap \{B \in \mathcal{U} \mid x \in B\} = \{x\},$$

$|\mathcal{U}_x| \le \omega_0$ and $\mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x$ is a *-open collection, for each $x \in X$, where $|\mathcal{U}_x|$ denotes the cardinality of $\mathcal{U}_x$ and ω_0 is the least infinite cardinality.

Proof: Let $\mathcal{Q}_i$ be the family of all non-empty open intervals with rational end points in X_i, for $i = 1, 2$. Then let $\mathcal{U}$ be the collection

$$\mathcal{U} = \{\{x\}\}_{x \in X} \cup \{Q_1 \cup Q_2 \mid Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}.$$

Then $\mathcal{U}$ is the cover having the desired properties.

Clearly, $\mathcal{U}$ is a point-separating cover of X and we have

$$\bigcap \mathcal{U}_x = \bigcap \{B \in \mathcal{U} \mid x \in B\} = \{x\},$$

and $|\mathcal{U}_x| \leq \omega_0$, for each $x \in X$ (i.e., $\mathcal{U}$ is a pointwise countable cover).

We now show that for each $x \in X, \mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x$ is a *-open collection. It suffices to show that for each $B \in \mathcal{V}_x$, if $w \in B$, then there exists a sequence $\{B_j\} \subseteq \mathcal{V}_x$ such that $w \in B_j$, for each j, and $\bigcup_j B_j$ is open. Clearly, we can take

$$B = Q_1 \cup Q_2, \qquad (Q_i \in \mathcal{Q}_i,\ i = 1, 2).$$

Let $w \in B = Q_1 \cup Q_2 \in \mathcal{V}_x$. Then there are two cases:

(i) $w \in Q_1$ and $x \in P(Q_1)$. Let $x = (a, 2)$, where $a \in [0, 1]$. Let ℓ_1 (resp. r_1) denote the left (resp. right) end point of Q_1. Then there is an increasing sequence $\{\ell_n\}$ of rational numbers and a decreasing sequence $\{r_n\}$ such that $\sup\{\ell_n\} = a$ and $\inf\{r_n\} = a$. Now let

$$D_j = (\ell_1, \ell_j) \times \{2\}, \quad E_j = (r_j, r_1) \times \{2\}, \quad (j = 1, 2, \dots).$$

Then $D_j \cup Q_1$ and $E_j \cup Q_1$ are in $\mathcal{V}_x$, for $j = 2, 3, \dots$, and

$$\bigcup_{j=2}^{\infty} (D_j \cup E_j) \cup Q_1$$

is an open set. Hence $\mathrm{St}\,(w, \mathcal{V}_x)$ is open.

Similarly, if $w \in Q_2$ and $x \in P^{-1}(Q_2)$, then $\mathrm{St}\,(w, \mathcal{V}_x)$ is again open.

(ii) $w \in Q_1$ and $x \notin P(Q_1)$. Since $B = Q_1 \cup Q_2 \in \mathcal{V}_x$ (i.e., $B \in \mathcal{U}, x \notin B$), we see that $x \notin Q_1$ and so $x \notin Q_1 \cup P(Q_1)$ and $Q_1 \cup P(Q_1) \in \mathcal{V}_x$. The facts that $Q_1 \cup P(Q_1)$ is open and $w \in Q_1 \cup P(Q_1) \subseteq \mathrm{St}\,(w, \mathcal{V}_x)$ are clear. The same conclusion remains valid, if $w \in Q_2$ and $x \notin P^{-1}(Q_2)$. This completes the proof. □

2. Two new cardinal inequalities.

Let X be a T_1 space. Then we have the following known cardinal inequalities:

$|X| \leq 2^{e(X)\,\mathrm{psw}(X)}$, (D.K. Burke and R. Hodel [1]),

$|X| \leq 2^{e(X)\Delta(X)}$, (J. Ginsburg and G. Wood [3]),

where

$\mathrm{psw}(X) = \min\{\kappa \mid$ there is an open cover $\mathcal{U}$ of X such that

$$\bigcap \mathcal{U}_x = \{x\},\ |\mathcal{U}_x| \leq \kappa, \quad \text{for each } x \in X\},$$

$\Delta(X) = \min\{\kappa \mid$ there is a collection of open covers $\{\mathcal{U}_\alpha\}_{\alpha<\kappa}$

$$\text{of } X \text{ such that } \bigcap \mathrm{St}\,(x, \mathcal{U}_\alpha) = \{x\} \text{ for each } x \in X\},$$

$e(X) = \sup\{\kappa \mid A$ is a closed discrete subspace of X with $|A| \leq \kappa\}$.

Here κ denotes cardinality and $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$.

We will sharpen these inequalities. For this purpose, we define the following cardinal functions:

$$\mathrm{wpsw}(X) = \min\{\kappa \mid \text{there is a cover } \mathcal{U} \text{ of } X \text{ such that } \bigcap \mathcal{U}_x = \{x\},$$

$$|\mathcal{U}_x| \le \kappa \text{ and } \mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x \text{ is } {}^*\text{-open}, \quad \text{for each } x \in X\},$$

$$\overline{\Delta}(X) = \min\{\kappa \mid \text{there is a collection of } {}^*\text{-open covers } \{\mathcal{U}_\alpha\}_{\alpha<\kappa}$$

$$\text{of } X \text{ such that } \bigcap_{\alpha<\kappa} \mathrm{St}\,(x, \mathcal{U}_\alpha) = \{x\}, \quad \text{for each } x \in X\}.$$

Then we have:

Theorem 1. *For any T_1 space, $|X| \le 2^{e(X)\,\mathrm{wpsw}(X)\psi(X)}$.*

Theorem 2. *For any T_1 space, $|X| \le^{e(X)\overline{\Delta}(X)}$.*

To prove our theorems, we need the following results, the first one is easy to prove and the second is due to D.K. Burke.

Lemma 1. *If $\mathcal{U}$ is a *-open cover of a T_1 space, then there exists a maximal subset D such that $x, y \in D$ and $x \ne y$ imply $x \notin \mathrm{St}\,(y, \mathcal{U})$; and that D is a discrete closed subspace of X with*

$$\bigcup_{d\in D} \mathrm{St}\,(d, \mathcal{U}) = X.$$

Lemma 2 (D.K. Burke)**.** *If $\{A_\alpha \mid \alpha \in \Lambda\}$ is an indexed collection of sets in which every member has cardinality less than or equal to λ, where $|\Lambda| > 2^\lambda$, and each A_α is a disjoint union of two subsets A'_α, A''_α, then there is a set $\Lambda' \subseteq \Lambda$ such that $|\Lambda'| > 2^\lambda$ and $A'_\alpha \cap A''_\beta = \emptyset$ whenever $\alpha, \beta \in \Lambda'$.*

Proof of Theorem 1: Let $e(X)\,\mathrm{wpsw}(X)\psi(X) = \kappa$. Then there is a *-open cover $\mathcal{U}$ of X such that $\bigcap \mathcal{U}_x = \{x\}$ and $|\mathcal{U}_x| \le \kappa$ for each $x \in X$, and a collection of open sets $\{U_\alpha(x)\}_{\alpha<\kappa}$ such that $\{x\} = \bigcap_{\alpha<\kappa} U_\alpha(x)$.

For each $x_0 \in X$, we will construct a set

$$A_{x_0} = A'_{x_0} \cup A''_{x_0}$$

satisfying the assumption of Lemma 2. Firstly, since $|\mathcal{U}_{x_0}| = |\{B \in \mathcal{U} \mid x_0 \in B\}| \le \kappa$, we let

$$A'_{x_0} = \mathcal{U}_{x_0}.$$

Then $\mathcal{V}_{x_0} = \mathcal{U} \setminus \mathcal{U}_{x_0}$ and $\bigcup \mathcal{V}_{x_0} = X \setminus \{x_0\}$. For each $\alpha < \kappa$, let

$$\mathcal{U}_\alpha = \mathcal{V}_{x_0} \cup \{U_\alpha(x_0)\}.$$

Then $\mathcal{U}_\alpha$ is a cover of X such that $\mathrm{St}\,(x,\mathcal{U}_\alpha)$ is an open set for each $x \in X$. By Lemma 1, there exists a closed subset $D_\alpha(x_0)$ such that

$$\bigcup\{\mathrm{St}\,(d,\mathcal{U}_\alpha) \mid d \in D_\alpha(x_0)\} = X;$$

i.e.,

$$\bigcup\{\mathrm{St}\,(d,\mathcal{V}_{x_0}) \mid d \in D_\alpha(x_0)\} \cup U_\alpha(x_0) = X.$$

Since $e(X) \leq \kappa$, it follows that $|D_\alpha(x_0)| \leq e(X) \leq \kappa$. Therefore

$$\bigcup_{\alpha<\kappa} \bigcup_{d \in D_\alpha(x_0)} \mathrm{St}\,(d,\mathcal{V}_{x_0}) \supset \bigcup_{\alpha<\kappa} (X \setminus U_\alpha(x_0)) = X \setminus \{x_0\}.$$

On the other hand,

$$x_0 \notin \bigcup_{\alpha<\kappa} \bigcup_{d \in D_\alpha(x_0)} \mathrm{St}\,(d,\mathcal{V}_{x_0}).$$

Let $D_{x_0} = \bigcup_{\alpha<\kappa} D_\alpha(x_0)$. Then we see that

$$\bigcup_{d \in D_\alpha(x_0)} \mathrm{St}\,(d,\mathcal{V}_{x_0}) = X \setminus \{x_0\} \ \text{ and } \ |D_{x_0}| \leq \kappa \cdot \kappa = \kappa.$$

Now let

$$A''_{x_0} = \bigcup_{d \in D_\alpha(x_0)} \{B \in \mathcal{V}_{x_0} \mid d \in B\}.$$

Then $|A''_{x_0}| \leq \kappa \cdot \kappa = \kappa$. Clearly $A'_{x_0} \cap A''_{x_0} = \emptyset$.

If $|X| > 2^\kappa$, then by Lemma 2, there is a set $X' \subseteq X$ such that $|X'| > 2^\kappa$ and $A'_x \cap A''_y = \emptyset$ for each pair $x, y \in X'$. But this is impossible. Since

$$y \in X \setminus \{x\} = \bigcup_{d \in D_x} \mathrm{St}\,(d,\mathcal{V}_x),$$

there is a $B \in \mathcal{V}_x$ and $d' \in D_x$ such that $y, d' \in B$ and so $B \in A'_y \cap A''_x$; i.e., $A'_y \cap A''_x \neq \emptyset$, for each distinct pair $x, y \in X'$.

Hence $|X| \leq 2^\kappa$ and the proof is complete. □

Remark. We use the technique of Burke in the proof of Theorem 1.

Proof of Theorem 2: Let $e(X)\overline{\Delta}(X) = \kappa$. Let $\{\mathcal{W}_\alpha\}_{\alpha<\kappa}$ be a collection of *-open covers of X such that $\bigcap_{\alpha<\kappa} \mathrm{St}\,(x,\mathcal{W}_\alpha) = \{x\}$. We will construct an increasing sequence $\{B_\alpha \mid 0 \leq \alpha < \kappa^+\}$ of subsets in X and a sequence $\{\mathcal{U}_\alpha \mid 0 < \alpha < \kappa^+\}$ of open collections in X such that

(i) $|B_\alpha| \leq 2^\kappa, 0 \leq \alpha < \kappa^+$;

(ii) $\mathcal{U}_\alpha - \bigcup_x\{\{\mathrm{St}\,(x,\mathcal{W}_{\alpha'}) \mid \alpha' < \kappa\}\}$, where x runs over the set $\bigcup_{\beta<\alpha} B_\beta$, for $0 < \alpha < \kappa^+$;

(iii) if $X \setminus (\bigcup \mathcal{U}) \neq \emptyset$, then $B_\alpha \setminus (\bigcup \mathcal{U}) \neq \emptyset$, for each $\mathcal{U} \in [\mathcal{U}_\alpha]^{\leq\kappa}$, where

$$[\mathcal{U}_\alpha]^{\leq\kappa} = \{\mathcal{V} \subseteq \mathcal{U}_\alpha \mid |\mathcal{V}| \leq \kappa\}.$$

The construction goes by transfinite induction. Let $0 < \alpha < \kappa^+$ and assume that $\{B_\beta \mid \beta < \alpha\}$ have already been constructed. Note that $\mathcal{U}_\alpha$ is defined by (ii) and $|\mathcal{U}_\alpha| \leq 2^\kappa$. For each $\mathcal{U} \in [\mathcal{U}_\alpha]^{\leq\kappa}$ with $X \setminus (\bigcup \mathcal{U}) \neq \emptyset$, choose one point in $X \setminus (\bigcup \mathcal{U})$. Let A_α be the set of all the points chosen in this way. Since $|\mathcal{U}_\alpha| \leq 2^\kappa$, it follows that $|A_\alpha| \leq (2^\kappa)^\kappa = 2^\kappa$. Now let

$$B_\alpha = A_\alpha \cup \bigcup_{\beta<\alpha} B_\beta.$$

Clearly, $B_\beta \subseteq B_\alpha$ for all $\beta < \alpha$, and $|B_\alpha| \leq 2^\kappa$. This completes the construction of the increasing sequence $\{B_\alpha \mid 0 \leq \alpha < \kappa^+\}$.

Next, let

$$B = \bigcup_{\alpha<\kappa^+} B_\alpha\,.$$

Then $|B| \leq 2^\kappa$. The proof is complete, if $X = B$. Suppose $X \neq B$ and choose $p \in X \setminus B$. For each $\alpha < \kappa$, let $F_\alpha = X \setminus \mathrm{St}\,(p,\mathcal{W}_\alpha)$. Then F_α is closed and

$$\bigcup_{\alpha<\kappa} F_\alpha = X \setminus \bigcap_{\alpha<\kappa} \mathrm{St}\,(p,\mathcal{W}_\alpha) = X \setminus \{p\} \supseteq B.$$

Let $\mathcal{V}_\alpha = \{W \in \mathcal{W}_\alpha \mid W \cap (F_\alpha \cap B) \neq \emptyset\}$. Then we claim that $\bigcup \mathcal{V}_\alpha \supseteq \overline{F_\alpha \cap B}$. In fact, if $y \in \overline{F_\alpha \cap B}$, then there exists $b \in \mathrm{St}\,(y,\mathcal{W}_\alpha) \cap (\overline{F_\alpha \cap B})$, and so $y \in \mathrm{St}\,(b,\mathcal{W}_\alpha) \subseteq \bigcup \mathcal{V}_\alpha$.

Since $e(X) \leq \kappa$, we have $e(\overline{F_\alpha \cap B}) \leq \kappa$, so that there is a set $C_\alpha \subseteq F_\alpha \cap B$ such that C_α is closed discrete with $|C_\alpha| \leq \kappa$ and

$$\bigcup_{b\in C_\alpha} \mathrm{St}\,(b,\mathcal{W}_\alpha) = \bigcup_{b\in C_\alpha} \mathrm{St}\,(b,\mathcal{V}_\alpha) \subseteq F_\alpha \cap B.$$

It is sufficient to take the maximal $C_\alpha \subseteq F_\alpha \cap B$ such that $d_1 \notin \mathrm{St}\,(d_2,\mathcal{V}_\alpha)$ for each distinct pair $d_1, d_2 \in X$. Let $C = \bigcup_{\alpha<\kappa} C_\alpha \subseteq B$. Then $|C| \leq \kappa$ and

$$\bigcup_{\alpha<\kappa} \bigcup_{d\in C_\alpha} \mathrm{St}\,(d,\mathcal{W}_\alpha) \subseteq \bigcup_{\alpha<\kappa} (F_\alpha \cap B) = B.$$

Therefore there exists $\alpha_0 < \kappa^+$ such that $C \subseteq B_{\alpha_0}$. Finally, let

$$\mathcal{U} = \bigcup_{\alpha<\kappa} \{\mathrm{St}\,(d,\mathcal{W}_\alpha) \mid d \in C_\alpha\}.$$

Then $\mathcal{U} \in [\mathcal{U}_{\alpha_0}]^{\leq\kappa}$ and hence

$$B_{\alpha_0+1} \setminus (\bigcup \mathcal{U}) \neq \emptyset,$$

by (iii), which is a contradiction. This completes the proof. □

Remark. The results in Section 1 on the concentric circle space show that the above extensions are not trivial.

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School of Mathematics and Statistics, University of Sydney, Sydney, N.S.W. 2006, Australia

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