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# Measurable cardinals and category bases

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*Abstract.* We show that the existence of a non-trivial category base on a set of regular cardinality with each subset being Baire is equiconsistent to the existence of a measurable cardinal.

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The existence of a non-trivial measure $m$ on a set (of regular cardinality) such that each its subset is $m$-measurable, is equiconsistent to the existence of a measurable cardinal [S].

The existence of a non-trivial topology on a set of regular cardinality such that each its subset has the Baire property, is equiconsistent to the existence of a measurable cardinal [KT].

Category bases constitute a common generalization of measure and topological structures. In this case the family of Baire sets plays the role of the $\sigma$-field of measurable sets or the $\sigma$-field of sets with the Baire property. Therefore it is natural to ask about the existence of a non-trivial category base on a set of regular cardinality such that each its subset would be a Baire set. We show that the answer is the same as for the measure or topology, namely, it is equiconsistent to the existence of a measurable cardinal.

Following J.C. Morgan [M1], we say that a pair (X,C) is a category base, if the elements in $C$, called regions, satisfy the following axioms:

(1) Every point of $X$ belongs to some region, i.e., $X = \bigcup C$.
(2) Let $A$ be a region and let $D$ be any non-empty family of disjoint regions which has the power less than the power of $C$.
(a) If $A \cap (\bigcup D)$ contains a region, then there is a region $B \in D$ such that $A \cap B$ contains a region.
(b) If $A \cap (\bigcup D)$ contains no region, then there is a subregion of $A$ which is disjoint from every region in $D$.

As is readily verified, every topology or all subsets of positive measure with respect to a complete $\sigma$-finite measure is a category base. With respect to a given category base, J.C. Morgan has defined abstract versions of the concepts of nowhere dense sets or null sets, sets with the Baire property, and so on.

A set is singular, if every region contains a subregion which is disjoint from it. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is abundant everywhere in a region, if every subregion of the region intersects the set on an abundant set. A set is a Baire set, if every region contains a subregion, whose intersection with either the set or its complement is meager.

For a given category base $(X, C)$ let $M(C)$ and $B(C)$ denote the family of all meager subsets of $X$ and Baire subsets of $X$, respectively. In the case that $C$ is a topology on $X$, the singular sets coincide with nowhere dense sets, $M(C)$ coincides with the family of first category sets and $B(C)$ coincides with the family of sets with the Baire property. In the case that $C$ is the family of all subsets of positive measure with respect to a $\sigma$-finite complete measure, the singular and meager sets coincide with the null-sets and $B(C)$ coincides with the $\sigma$-field of all measurable sets. We are going to utilize also the following basic facts from the theory of category bases.

**Fact 1.** Let $(X, C)$ be a category base. If $A$ is a region and $B$ is a Baire set such that $A \cap B$ is abundant, then there exists an abundant everywhere in itself subregion $D$ of $A$ such that $D - B$ is meager.

**Fact 2.** The intersection of two regions is either a singular set or contains a region.

For proofs of these two particular facts as well as for a more extensive treatment of category bases, we refer to [M1]. The lemma below is new and it constitutes the basic ingredient in the proof of our main theorem.

Let $(X, C)$ be a category base. We say that $R$ is a category decomposition of $X$, if $R$ is a disjoint family of regions such that each region intersects a member of $R$ on a non-singular set. Notice few elementary facts about category decompositions.

**Fact 3.** Let $R$ be a category decomposition of $X$. Then:

(i) $X - (\bigcup R)$ is singular;
(ii) if $E \subset X$ and $E \cap A$ is singular for every $A \in R$, then $E$ is singular;
(iii) if $F \subset X$ and $F \cap A$ is meager for every $A \in R$, then $F$ is meager.

Let $S$ be a Baire set. An $M(C)$-partition of $S$ is a maximal collection $W$ of abundant Baire subsets of $S$ such that $E \cap F \in M(C)$ for any distinct $E, F \in W$.

**Lemma.** *Let $S$ be an abundant Baire set. Then for any category decomposition $R$ of $X$ and for any $M(C)$-partition $W$ of $S$, there exists a category decomposition $Q$ of $X$ satisfying the following:*

(a) *every region in $Q$ is a subregion of a region in $R$;*
(b) *if $A \in Q$ and $A \cap S$ is abundant, then $A$ is abundant everywhere in itself and there exists $B \in W$ such that $A - B$ is meager.*

Proof: Let $C = \{C_\alpha : \alpha < \kappa\}$, where $\kappa = |C|$. For every $\beta < \kappa$ we will define a region $A_\beta$ so that the following are satisfied:

(1) the family $\{A_\alpha : \alpha \leq \beta\}$ is a disjoint family for every $\beta < \kappa$;
(2) $C_\beta$ intersects some $A_\alpha$ with $\alpha \leq \beta$ on a non-singular set;
(3) $A_\beta$ is a subregion of some region from $R$;
(4) if $A_\beta \cap S$ is abundant, then $A_\beta$ is abundant everywhere in itself and there exists $B \in W$ such that $A_\beta - B$ is meager.

We proceed to define $A_0$. Since $R$ is a category decomposition of $X$, there exists $D \in R$ such that $C_0 \cap D$ is not singular. By virtue of Fact 2, there exists a region $A \subset C_0 \cap D$. If $A \cap S$ is meager, then we set $A_0 = A$. If $A \cap S$ is abundant then, since $W$ is an $M(C)$-partition of $S$, there exists $B \in W$ such that $A \cap B$ is abundant. According to Fact 1, there exists a subregion $A'$ of $A$ which is abundant everywhere in itself and yet $A' - B$ is meager. We then set $A_0$ to be $A'$.

Let $\beta < \kappa$ and suppose $A_\alpha$ has been well defined for every $\alpha < \beta$. If $C_\beta \cap A_\alpha$ is non-singular for some $\alpha < \beta$, then we set $A_\beta = A_0$. If $C_\beta \cap A_\alpha$ is singular for each $\alpha < \beta$, then there exists a subregion $A$ of $C_\beta$ disjoint with every $A_\alpha, \alpha < \beta$. We may assume that $A$ is also a subregion of some region from $R$. If $A \cap S$ is meager, then we set $A_\beta = A$. If $A \cap S$ is abundant, then, arguing as in the first part, there exist $B \in W$ and a subregion $A'$ of $A$ such that $A'$ is abundant everywhere in itself and $A' - B$ is meager. Then we set $A_\beta = A'$.

It follows immediately from the conditions (1)–(4) that the family $Q = \{A_\alpha : \alpha < \kappa\}$ is a category decomposition of $X$ satisfying the conditions (a) and (b). □

Now we will need some facts about ideals. Let $\kappa$ be an infinite cardinal. A set $I$ of subsets of $\kappa$ is an ideal over $\kappa$, if

(i) $\{\alpha\} \in I$ for each $\alpha \in \kappa$ and $\kappa \notin I$.
(ii) If $X \in I$ and $Y \subset X$, then $Y \in I$.
(iii) If $X \in I$ and $Y \in I$, then $X \cup Y \in I$.

A $\sigma$-complete ideal is an ideal which is closed under countable unions. If $(X, C)$ is a non-trivial category base in the sense that one-point sets are singular and the set $X$ is abundant, then $M(C)$ is a $\sigma$-ideal over $|X|$. Customarily it is convenient to adopt the following terminology about ideals. Let $I$ be an ideal over $\kappa$. Elements of $I$ are called sets of $I$-measure zero; elements of $I^+ = P(\kappa) - I$ are called sets of positive $I$-measure; elements of $I^* = \{\kappa - x : x \in I\}$ are called sets of $I$-measure one; when the context is clear, we drop the prefix "$I$-" in all above phrases.

Let $S$ be a set of positive measure. An $I$-partition of $S$ is a maximal collection $W$ of subsets of $S$ of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An $I$-partition $W_1$ of $S$ is a refinement of an $I$-partition $W_2$ of $S$, $W_1 \leq W_2$, if every element of $W_1$ is a subset of an element of $W_2$.

The ideal $I$ is weakly precipitous, if $I$ is $\sigma$-complete and whenever $S$ is a set of positive measure and $\{W_n : n \in \omega\}$ are $I$-partitions of $S$ such that $W_0 \geq W_1 \geq \cdots \geq W_n \geq \ldots$, then there exists a sequence $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \ldots$ such that $X_n \in W_n$ for each $n$, and $\bigcap\{X_n : n \in \omega\} \neq \emptyset$. The following is of a basic importance for us.

**Fact 4.** If a regular uncountable cardinal $\kappa$ carries a weakly precipitous ideal, then $\kappa$ is measurable in some transitive model of ZFC.

This strong theorem is due to T. Jech, M. Magidor, W. Mitchell and K. Prikry [JP]. Originally, the theorem was formulated for precipitous ideals but the proof works for weakly precipitous ideals as well. For more extensive treatment about ideals, we refer to Jech's book [J].

**Theorem.** CON (ZFC + *"there exists a non-trivial category base* $(X, C)$ *such that* $|X|$ *is a regular cardinal and* $B(C) = P(X)$*"), if and only if* CON (ZFC + *"there exists a measurable cardinal"*).

PROOF: Assume first the consistency of the existence of a measurable cardinal. According to a theorem of Solovay [S], it implies the consistency of the existence of a set $X$ of regular cardinality not exceeding the power of the continuum and a probabilistic non-trivial measure $m$ on $P(X)$. Hence the family $C$ of all sets of positive $m$-measure is a non-trivial category base on $X$ such that $B(C) = P(X)$.

To prove the converse implication, we rely on Fact 4 and we will try to discover a weakly precipitous ideal. In fact, if $(X, C)$ is a non-trivial category base such that $|X|$ is a regular cardinal and $B(C) = P(X)$, then $M(C)$ is weakly precipitous.

Clearly, $M(C)$ is $\sigma$-complete. Let $S \subset X$ be a set of positive measure and let $\{W_n : n \in \omega\}$ be an appropriate sequence of partitions of $S$. We define inductively a sequence $\{Q_n : n \in \omega\}$ of category decompositions of $X$. We set $Q_0$ as a category decomposition of $X$ obtained by an application of Lemma for $R = \{X\}$ and $W = W_0$. We set $Q_{n+1}$ as a category decomposition of $X$ obtained by an application of Lemma for $R = Q_n$ and $W = W_{n+1}$. Let $Q_{n^*} = \{A \in Q_n : A \cap S \text{ is abundant }\}$ and let $B(A)$ denote a unique element in $W_n$ such that $A - B(A)$ is meager whenever $A \in Q_{n^*}$.

Observe that both sets $E_n = S - (\bigcup Q_{n^*})$ and $F_n = \bigcup\{A - B(A) : A \in Q_{n^*}\}$ are meager for each $n \in \omega$. To see this, notice that both $E_n$ and $F_n$ intersect every member of the family $Q_n$ on a meager set. Since $Q_n$ is a category decomposition of $X$, they are meager, according to Fact 3 (iii).

Consider now the set $S^* = S - (\bigcup\{E_n \cup F_n : n \in \omega\})$. Since $S$ is abundant, $S^*$ is not empty and let $p$ be its arbitrary point. Since $S^* \subset S - (\bigcup\{E_n : n \in \omega\}) = S \cap \bigcap\{\bigcup Q_{n^*} : n \in \omega\}$, for each $n \in \omega$ there exists a unique element $A_n$ of $Q_{n^*}$ containing the point $p$. Let $B_n = B(A_n)$ be a unique element of the partition $W_n$ such that $A_n - B_n$ is meager. Since $A_n - B_n \subset F_n$, $p \in B_n$ for every $n \in \omega$. At the end let us prove that the selected sets $B_n$ form a decreasing sequence. Notice at first that this fact is true about the sequence $A_n$, since each $Q_n$ is a disjoint family being a refinement of the preceding one. Let $B \in W_{n-1}$ be such that $B_n \subset B$. If $B$ were different from $B_{n-1}$, then $B \cap B_{n-1}$ would be meager. Since $A_{n-1} - B_{n-1}$ is meager, $B \cap A_{n-1}$ is meager and so is $B \cap A_n$. From the other side, every $A_n$ is abundant and $A_n - B_n$ is meager. Since $B_n \subset B, B \cap A_n$ would be abundant; a contradiction. □

## References


[J] Jech T., *Set Theory*, Academic Press, 1978.

[JP] Jech T., Magidor M., Mitchell W., Prikry K., *Precipitous ideals*, Journal of Symbolic Logic **45**, 1–8.

[KT] Kunen K., Szymański A., Tall F., *Baire irresolvable spaces and ideal theory*, Acta Math. Silesiana, No. 14 (1986), 98–107.

[M1] Morgan J.C.II, *Point Set Theory*, Marcel Dekker, Inc., 1990.
[S] Solovay R., *Real valued measurable cardinals*, in Axiomatic Set Theory, Proc. Symp. Pure Math. 13.1 (D. Scott ed.), 397–428, Amer. Math. Soc., 1971.

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