Weak uniform rotundity of Musielak–Orlicz spaces

Małgorzata Doman

Abstract. We give necessary and sufficient conditions for weak uniform rotundity of Musielak–Orlicz spaces L_φ with the Luxemburg norm. The result is a generalization of a theorem by Kamińska and Kurc.

Keywords: Musielak–Orlicz space, rotundity

Classification: 46B20, 46B25

Introduction.

Let T be a set, $\sum$ a σ-algebra of subsets of T, μ a non-negative atomless σ-finite complete measure on $\sum$. A function $\varphi : R_+ \times T \to R_+$, where $R_+ = [0, +\infty)$, is said to be a Musielak–Orlicz function if $\varphi(0,t) = 0$ for μ-almost every $t \in T$, $\varphi(\ ,t)$ is a convex function on R_+ for μ-almost every $t \in T$ and $\varphi(u,\ )$ is a $\sum$-measurable function on T for every $u \geq 0$. The complementary function to a function φ is defined by $\varphi^*(v,t) = \sup_{u>0}(vu - \varphi(u,t))$ for $v \geq 0, t \in T$. We denote by M the set of all $\sum$-measurable functions $x : T \to R$. The functions which are different only on a null-set are considered as identical. The Musielak–Orlicz space L_φ is a subset of M such that $I_\varphi(\lambda x) = \int_T \varphi(\lambda |x(t)|, t)\, d\mu < +\infty$ for some $\lambda > 0$ dependent on x. The functionals $\|x\|_\varphi = \inf\{r > 0 : I_\varphi(\frac{x}{r}) \leq 1\}$ and $\|x\|_\varphi^0 = \sup\{\int_T x(t)y(t)\, d\mu : y \in L_{\varphi^*},\ I_{\varphi^*}(y) \leq 1\}$ are norms in this space, called the Luxemburg and the Orlicz norm, respectively. We say that a function φ satisfies the condition Δ_α, for some $\alpha > 1$, if there are a constant $K_\alpha > 0$ and a function $h_\alpha : T \to R_+$, such that $\int_T h_\alpha(t)\, d\mu < +\infty$ and $\varphi(\alpha u, t) \leq K_\alpha \varphi(u,t) + h_\alpha(t)$ for almost every $t \in T$ and for $u \geq u_0$ (u_0-some positive constant), when $\mu(T) < +\infty$, or for all $u \in R_+$, when $\mu(T) = +\infty$. Recall that a function φ is called strictly convex, if for all $u, v \in R_+, u \neq v, \alpha, \beta \in R_+ \setminus \{0\}, \alpha + \beta = 1$, we have $\varphi(\alpha u + \beta v, t) < \alpha\varphi(u,t) + \beta\varphi(v,t)$ outside of some null-set. For further details concerning Musielak–Orlicz spaces see [7].

We say that a Banach space $(X, \|\ \|)$ is weakly uniformly rotund (WUR), if for every $x^* \in X, x^* \neq 0$ and $\varepsilon > 0$ there exists $\delta(x^*, \varepsilon) > 0$, such that if $\|x\| = \|y\| = 1$ and $x^*(x - y) \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta(x^*, \varepsilon)$ (cf. [1]). If for all $x, y \in X$ such that $\|x\| = \|y\| = 1$ we have $\|\frac{x+y}{2}\| < 1$, then we say that $(X, \|\ \|)$ is rotund.

The aim of this paper is to give necessary and sufficient conditions for WUR of Musielak–Orlicz spaces. The result is a generalization of a theorem by Kamińska and Kurc ([6, Theorem 2.8]).

I am greatly indebted to Professor Anna Kamińska for suggesting the problem discussed here to me. I wish to thank also Professor Henryk Hudzik for his help in preparing this paper

Results.

For the proof of the main theorem, we need some lemmas.

Lemma 1 (cf. [6]). *If an arbitrary Banach space contains an isomorphic copy of l_1, then X is not WUR.*

Lemma 2. *If φ is a strictly convex Musielak–Orlicz function, then for every $\varepsilon > 0$ and every $\sum$-measurable functions $p, q : T \to (0, +\infty), p(t) < q(t)$ for μ-almost every $t \in T$, there exists a $\sum$-measurable function $r : T \to (0, 1)$ such that*

$$\varphi(\frac{u+v}{2}, t) \leq \frac{1 - r(t)}{2}\{\varphi(u, t) + \varphi(v, t)\}$$

for μ-almost every $t \in T$ whenever $|u - v| \geq \varepsilon \max\{|u|, |v|\}$ and $\max\{|u|, |v|\} \in [p(t), q(t)]$.

The proof of this lemma is analogous to that of Lemma 1 in [5], so it is omitted here. □

Lemma 3. *Assume that φ is a Musielak–Orlicz function satisfying the Δ_2-condition. Then for every $\alpha > 1$ and $\varepsilon > 0$, there is a set T_0 of measure 0, a constant $K_{\alpha,\varepsilon} > 0$ and a $\sum$-measurable function $h_{\alpha,\varepsilon} : T \to [0, +\infty)$ such that $\int_T h_{\alpha,\varepsilon}(t)\, d\mu \leq \varepsilon$ and $\varphi(\alpha u, t) \leq K_{\alpha,\varepsilon}\varphi(u, t) + h_{\alpha,\varepsilon}(t)$ for any $t \in T \setminus T_0$ and any $u \in R$.*

The proof for $\alpha = 2$ is given in [4]. The proof for an arbitrary $\alpha > 1$ can proceed in the same way, if we notice that φ satisfies the Δ_2-condition if and only if it satisfies the Δ_α-condition for every $\alpha > 1$. □

Lemma 4 (cf. [4]). *Let φ be a Musielak–Orlicz function that satisfies the Δ_2-condition. Then*

(i) *there is a function $\beta : (0, 1) \to (0, 1)$ such that $\|x\|_\varphi \leq 1 - \beta(\varepsilon)$ whenever $I_\varphi(x) \leq 1 - \varepsilon$.*
(ii) *$\|x\|_\varphi = 1$ if and only if $I_\varphi(x) = 1$.*

Lemma 5. *Assume that φ is a Musielak–Orlicz function vanishing only at 0 and that φ and φ^* satisfy the Δ_2-condition. Let $x^* \in (L_\varphi)^*$ be regular and nontrivial (i.e. there exists $z \in L_{\varphi^*}, z \neq 0$ such that $x^*(x) = \int_T x(t)z(t)\, d\mu$ for every $x \in L_\varphi$). Let $(B_n)_{n=1}^\infty$ be an increasing sequence of sets with finite and positive measures such that $\bigcup_n B_n = \operatorname{supp} z$. Denote $C_n = \{t \in T : \frac{1}{n} \leq |z(t)| \leq n\}$ and put $D_n = C_n \cap B_n$. Then $(D_n)_{n=1}^\infty$ is increasing, $\bigcup_n D_n = \operatorname{supp} z$ and*

$$\int_{D_n} y(t)z(t)\, d\mu \to \int_T y(t)z(t)\, d\mu$$

uniformly with respect to y in every bounded set in L_φ.

PROOF: In virtue of B. Levi theorem and the Δ_2-condition for φ^*, we have $\|z - z_n\|^0_{\varphi^*} \to 0$ as $n \to +\infty$, where $z_n = z\chi_{D_n}$. Then

$$0 \le \left| \int_T y(t)z(t)\,d\mu - \int_{D_m} y(t)z(t)\,d\mu \right| = \\ = \left| \int_T y(t)z(t)\,d\mu - \int_T y(t)z_m(t)\,d\mu \right| \le \\ \le \|y\|_\varphi \,\|z - z_m\|^0_{\varphi^*} \le C\|z - z_m\|^0_{\varphi^*}\,.$$

Hence the desired result follows. □

The next two lemmas are analogs of Lemma 2.5 and Lemma 2.6 of [6].

Lemma 6. *Let $\mu(T) < +\infty$ and φ be a Musielak–Orlicz function such that for every $t \in T$ $\frac{\varphi(u,t)}{u} \to +\infty$ as $u \to +\infty$. Then for every $\varepsilon > 0$, there exist $\sum$-measurable functions $p, q : T \to (0, +\infty)$ such that for every $x, y \in L_\varphi$ satisfying $I_\varphi(x) = I_\varphi(y) = 1$ and $\int_T |x(t) - y(t)|\,d\mu \ge \varepsilon$, we have $\int_A |x(t) - y(t)|\,d\mu \ge \frac{\varepsilon}{4}$ whenever*

$$A = \{t \in T : p(t) \le \max(|x(t)|, |y(t)|) \le q(t)\}\,.$$

PROOF: Define for any $n \in N$ $p_n(t) = \inf\{u > 0 : \frac{\varphi(u,t)}{u} \ge n\}$. Then p_n is a $\sum$-measurable function and $\varphi(u,t) \ge nu$ for every $u \ge p_n(t)$. Define $A_n = \{t \in T : |x(t)| \le p_n(t)\}$, $A_n^1 = \{t \in T : |y(t)| \le p_n(t)\}$. We have

$$\int_{T\setminus A_n} |x(t)|\,d\mu \le \frac{1}{n} \int_{T\setminus A_n} \varphi(|x(t)|, t)\,d\mu \le \frac{1}{n}\,.$$

In the same way, we can obtain $\int_{T\setminus A_n^1} |y(t)|\,d\mu \le \frac{1}{n}$. Moreover,

$$\int_{T\setminus A_n} |y(t)|\,d\mu = \int_{(T\setminus A_n)\cap(T\setminus A_n^1)} |y(t)|\,d\mu + \int_{A_n^1\setminus A_n} |y(t)|\,d\mu \le \\ \le \int_{T\setminus A_n^1} |y(t)|\,d\mu + \int_{T\setminus A_n} |x(t)|\,d\mu \le \frac{2}{n}\,.$$

Similarly $\int_{T\setminus A_n^1} |x(t)|\,d\mu \le \frac{2}{n}$. Hence $\int_{T\setminus(A_n\cap A_n^1)} |x(t) - y(t)|\,d\mu \le \int_{T\setminus A_n} |x(t)|\,d\mu + \int_{T\setminus A_n} |y(t)|\,d\mu + \int_{T\setminus A_n^1} |x(t)|\,d\mu + \int_{T\setminus A_n^1} |y(t)|\,d\mu \le \frac{6}{n}$. Since $\int_T |x(t) - y(t)|\,d\mu \ge \varepsilon$ by the assumptions, we have $\int_{A_n\cap A_n^1} |x(t) - y(t)|\,d\mu \ge \varepsilon - \frac{6}{n} \ge \frac{\varepsilon}{2}$ if n is such that $\frac{6}{n} \le \frac{\varepsilon}{2}$. Define $A_n^2 = \{t \in T : \frac{\varepsilon}{8\mu(T)} \le \max(|x(t)|, |y(t)|)\}$. If $t \notin A_n^2$, then $|x(t)| < \frac{\varepsilon}{8\mu(T)}$ and $|y(t)| < \frac{\varepsilon}{8\mu(T)}$. Therefore $\int_{(A_n\cap A_n^1)\setminus A_n^2} |x(t) - y(t)|\,d\mu \le \frac{\varepsilon}{8\mu(T)}\mu(T\setminus A_n^2) + \frac{\varepsilon}{8\mu(T)}\mu(T\setminus A_n^2) \le \frac{\varepsilon}{4}$. Thus $\int_{A_n\cap A_n^1\cap A_n^2} |x(t) - y(t)|\,d\mu \ge \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$. Putting $A = A_n \cap A_n^1 \cap A_n^2, p(t) = \frac{\varepsilon}{8\mu(T)}$ and $q(t) = p_n(t)$, we get the desired inequality. □

Lemma 7. *Let φ be a Musielak–Orlicz function satisfying the Δ_2-condition and let $B \in \sum, \varepsilon > 0$ and $\sigma \in (0,1)$ be such that $I_\varphi((x-y)\chi_B) \geq \varepsilon$ and $I_\varphi(\frac{x+y}{2}) \leq 1 - \frac{\sigma}{2}(I_\varphi(x\chi_B) + I_\varphi(y\chi_B))$, where x, y are arbitrary measurable functions with $I_\varphi(x) = I_\varphi(y) = 1$. Then there exists a constant $q \in (0,1)$, such that $I_\varphi(\frac{x+y}{2}) \leq 1 - q$.*

Proof: Let $K = K_{2,\frac{\varepsilon}{2}}$, where $K_{2,\frac{\varepsilon}{2}}$ is the constant from Lemma 3. Then

$$\varepsilon \leq I_\varphi((x-y)\chi_B) \leq \frac{K}{2}(I_\varphi(x\chi_B) + I_\varphi(y\chi_B)) + \frac{\varepsilon}{2}\,.$$

Hence $I_\varphi(x\chi_B) + I_\varphi(y\chi_B) \geq \frac{\varepsilon}{2} \cdot \frac{2}{K} = \frac{\varepsilon}{K}$. Therefore $I_\varphi(\frac{x+y}{2}) \leq 1 - \frac{\sigma\varepsilon}{2K}$, and it suffices to put $q = \frac{\sigma\varepsilon}{2K}$ $\square$

Theorem 1. *A Musielak–Orlicz space L_φ is WUR if and only if*

(i) *φ is strictly convex,*
(ii) *φ satisfies the Δ_2-condition,*
(iii) *φ^* satisfies the Δ_2-condition.*

Proof: Sufficiency. Assume that the conditions (i), (ii), (iii) are satisfied. Let $x, y \in L_\varphi, \|x\|_\varphi = \|y\|_\varphi = 1$, $x^* \in (Ł_\varphi)^*$ and $x^*(x-y) \geq \varepsilon$, where $\varepsilon \in (0,1)$. In virtue of the representation of x^*, we have $\int_T (x(t) - y(t))z(t)\,d\mu \geq \varepsilon$ for some $z \in L_{\varphi^*}$. Define z_n as in the proof of Lemma 5. Then in view of this lemma, there is $n_0 \in N$ (n_0 independent of x and y) such that $\int_T (x(t) - y(t))z_{n_0}(t)\,d\mu \geq \frac{\varepsilon}{2}$. Since $|z_{n_0}(t)| < n_0$, denoting $T_0 = \operatorname{supp} z_{n_0}$, we get $\int_{T_0} |x(t) - y(t)|\,d\mu \geq \frac{\varepsilon}{2n_0}$. Since, according to Lemma 2.4 of [6], (iii) implies $\varphi(u,t)/u \to +\infty$ when $u \to +\infty$ for every $t \in T$, it follows from Lemma 6 that there exist two $\sum$-measurable functions $p, q : T_0 \to (0, +\infty)$, such that denoting

$$A = \{t \in T_0 : p(t) \leq \max(|x(t), y(t)|) \leq q(t)\}, \text{ we have}$$
$$\int_A |x(t) - y(t)|\,d\mu \geq \frac{\varepsilon}{8n_0}\,.$$

Define $B = \{t \in A : |x(t) - y(t)| \geq \frac{\varepsilon}{8n_0 K}\max(|x(t)|, |y(t)|)\}$, where $K = K_{\alpha,\frac{1}{2}}$ is the constant from Lemma 3 corresponding to $\alpha = \max\{\frac{64n_0}{\varepsilon}\|\chi_{T_0}\|_{\varphi^*}, 1\}$. In virtue of Lemma 2 there is a function $r : B \to (0,1)$ such that

$$\varphi(\frac{|x(t) + y(t)|}{2}, t) \leq \frac{1 - r(t)}{2}\{\varphi(|x(t)|, t) + \varphi(|y(t)|, t)\}.$$

Define $B_m = \{t \in B : r(t) \geq \frac{1}{m}\}$. We have $B_m \nearrow$ and $\bigcup_{n=1}^\infty B_m = B$. Thus, defining $C_m = (A \setminus B) \cup B_m$, we obtain the increasing sequence of sets such that $\bigcup_{n=1}^\infty C_n = A$. By Lemma 5 there is $s \in N$ (s independent of x and y) such that

$$\int_{C_s} |x(t) - y(t)|\,d\mu \geq \int_A |x(t) - y(t)|\,d\mu - \frac{1}{4} \cdot \frac{\varepsilon}{8n_0}\,.$$

i.e.

$$\int_{C_s} |x(t) - y(t)|\, d\mu \geq \frac{\varepsilon}{32n_0}\,. \tag{1}$$

For $t \in B_s$, we have

$$\varphi(\frac{|x(t) + y(t)|}{2}, t) \leq \frac{1 - \frac{1}{s}}{2}\{\varphi(|x(t)|, t) + \varphi(|y(t)|, t)\}.$$

Hence, using the convexity of φ and the fact that $I_\varphi(x) = I_\varphi(y) = 1$, we get

$$I_\varphi(\frac{x+y}{2}) \leq 1 - \frac{1}{2s}\{I_\varphi(x)\chi_{B_s} + I_\varphi(y\chi_{B_s})\}. \tag{2}$$

If $t \in A \setminus B$, then

$$|x(t) - y(t)| < \frac{\varepsilon}{8n_0 K}\max(|x(t)|, |y(t)|).$$

Hence

$$I_\varphi((x-y)\chi_{A\setminus B}) \leq \frac{\varepsilon}{8n_0 K}\{I_\varphi(x\chi_{A\setminus B}) + I_\varphi(y\chi_{A\setminus B})\} \leq \frac{\varepsilon}{4n_0 K}\,. \tag{3}$$

Applying the inequality (1) and the Hölder inequality, we get

$$2\|(x-y)\chi_{C_s}\|_\varphi \|\chi_{T_0}\|_{\varphi^*} \geq \int_{C_s} |x(t) - y(t)|\, d\mu \geq \frac{\varepsilon}{32n_0}\,,$$

i.e.

$$\frac{64n_0}{\varepsilon}\|\chi_{T_0}\|_{\varphi^*}\|(x-y)\chi_{C_s}\|_\varphi \geq 1,$$

hence denoting $\alpha_1 = \frac{64n_0}{\varepsilon}\|\chi_{T_0}\|_{\varphi^*}$, we have $\alpha_1 \leq \alpha$, and

$$1 \leq I_\varphi(\alpha(x-y)\chi_{C_s}) \leq K I_\varphi((x-y)\chi_{C_s}) + \frac{1}{2}\,.$$

Thus

$$I_\varphi((x-y)\chi_{C_s}) \geq \frac{1}{2K}\,.$$

Combining this with the inequality (3), we get

$$I_\varphi((x-y)\chi_{B_s}) \geq I_\varphi((x-y)\chi_{C_s}) - I_\varphi((x-y)\chi_{A\setminus B}) \geq \frac{1}{2K} - \frac{\varepsilon}{4n_0 K} = \beta.$$

Applying Lemma 7, the inequality (2) and the last inequality, we get

$$I_\varphi(\frac{x+y}{2}) \leq 1 - q.$$

Now, in view of Lemma 4, we have

$$\left\| \frac{x+y}{2} \right\|_\varphi \le 1 - \beta(q),$$

where $\beta(q) \in (0,1)$, and depends only on x^*, ε and φ.

Necessity. If φ does not satisfy the condition (i) or the condition (ii), then L_φ is not rotund (cf. [5]). Since WUR implies rotundity, L_φ is not WUR as well. Assume now that φ satisfies the condition (i) and it does not satisfy the condition (iii). Then $(L_\varphi)^* = L_{\varphi^*}$, where L_{φ^*} is equipped with the Orlicz norm. Since φ^* does not satisfy the Δ_2-condition, L_{φ^*} contains an isomorphic copy of l_∞. Hence it follows that L_φ contains an isomorphic copy of l_1. Therefore, in view of Lemma 1, L_φ is not WUR. The proof is finished. □

Theorem 1.2 of [3] and Theorem 1.2 of [2] imply the following version of our result.

Theorem 2. *A Musielak–Orlicz space L_φ is WUR if and only if it is rotund and reflexive.*

References

[1] Diestel J., *Geometry of Banach spaces—selected topics*, Springer Lecture Notes in Mathematics, vol. 485, 1983.
[2] Hudzik H., *On some equivalent conditions in Musielak–Orlicz spaces*, Comment. Math. **24** (1984), 57–64.
[3] Hudzik H., *Strict convexity of Musielak–Orlicz spaces with Luxemburg's norm*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astronom. et Phys. **29** (1981), 235–247.
[4] Hudzik H., *Uniform convexity of Musielak–Orlicz spaces with Luxemburg's norm*, Comment. Math. **23** (1983), 21–32.
[5] Kamińska A., *On some convexity properties of Musielak–Orlicz spaces*, Supplemento ai Rendicoti del Circolo Matematico di Palermo, The 12th Winter School on Abstract Analysis, Srní 1984.
[6] Kamińska A., *Weak uniform rotundity in Orlicz spaces*, Comment. Math. Univ. Carolinae **27** (1986), 651–664.
[7] Musielak J., *Orlicz spaces and modular spaces*, Springer Lecture Notes in Mathematics, vol. 1034, 1983.

Academy of economics, Department of Mathematics, Al. Niepodległości 10,
60–967 Poznań, Poland

(*Received* November 28, 1990, *revised* February 26, 1991)