Pairwise monotonically normal spaces

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Abstract. We introduce and study the notion of pairwise monotonically normal space as a bitopological extension of the monotonically normal spaces of Heath, Lutzer and Zenor. In particular, we characterize those spaces by using a mixed condition of insertion and extension of real-valued functions. This result generalizes, at the same time improves, a well-known theorem of Heath, Lutzer and Zenor. We also obtain some solutions to the quasi-metrization problem in terms of the pairwise monotone normality.

Keywords: pairwise monotonically normal space, quasi-metrizable bitopological space, pairwise stratifiable space, pairwise compact space

Classification: 54E55, 54D15, 54C30, 54E35, 54E20

1. Introduction.

Throughout this paper, all topologies are T_1 and the letter N will denote the set of positive integers. Terms and concepts which are not defined, are used as in [8].

The investigation of the metrization problem has motivated, in the last thirty years, the introduction and study of several types of topological spaces called generalized metric spaces (see [10, p. 425]). Stratifiable spaces [1], [4], form one of the more interesting classes of generalized metric spaces. This notion has been generalized to bitopological spaces [9], [12], [18], and several properties have been extended. Monotonically normal spaces are a useful generalization of stratifiable spaces. The property of monotone normality appears firstly in [1]. Later, Heath, Lutzer and Zenor [14] presented a systematized study of monotonically normal spaces, obtaining excellent results. Other contributions to the research of these spaces may be found in [2], [10], [11], [23], [25], etc.

In this paper, we introduce and study the notion of monotonically normal bitopological space. We will show that this class of spaces provides several satisfactory results and permits us to state appropriate generalizations of well-known theorems. In fact, in Section 2, we prove that a bitopological space is pairwise stratifiable if and only if it is pairwise monotonically normal and pairwise semi-stratifiable. In Section 3, we characterize pairwise monotonically normal spaces in terms of a mixed condition of insertion and extension of semi-continuous functions. This characterization provides an extension and, at the same time, an improvement of a well-known theorem of Heath, Lutzer and Zenor [14, Theorem 3.3]. Finally, in Section 4, we present some solutions to the quasi-metrization problem in terms of pairwise monotonically normal spaces.

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A quasi-metric on a set X is a non-negative real-valued function d on $X \times X$ such that, for all $x, y, z \in X$: (i) $d(x, y) = 0$ if and only if $x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Every quasi-metric on X induces a T_1 topology $T(d)$ on X which has as a base the family $\{B_d(x, r) : x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. A topological space (X, τ) is called quasi-metrizable, if there is a quasi-metric d on X such that $\tau = T(d)$. The Niemytzki plane, the Sorgenfrey line, the Michael line and the Kofner plane are relevant examples of quasi-metrizable topological spaces which are not metrizable. Every quasi-metric d on X induces a conjugate quasi-metric d^{-1} on X, defined by $d^{-1}(x, y) = d(y, x)$. Then the pair of topologies induced from a quasi-metric and its conjugate originate the following notion: a bitopological space is [17] and ordered triple (X, τ_1, τ_2) such that X is a nonempty set and τ_1 and τ_2 are topologies on X. The space (X, τ_1, τ_2) is said to be quasi-metrizable, if there is a quasi-metric d on X such that $\tau_1 = T(d)$ and $\tau_2 = T(d^{-1})$.

2. Definitions and basic properties.

In the rest of the paper, when we are concerned at the topologies τ_i and τ_j, we suppose $i, j = 1, 2$, and $i \neq j$. If τ_i is a topology for a set X and A is a subset of X, we write $\tau_i \operatorname{cl} A$ for the closure of A in the topological space (X, τ_i). Similarly, we write $\tau_i \operatorname{int} A$ for the interior of A in (X, τ_i).

Definition 1. A bitopological space (X, τ_1, τ_2) is called *pairwise monotonically normal*, if to each pair (H, K) of disjoint subsets of X such that H is τ_i-closed and K is τ_j-closed, one can assign a τ_i-open set $D(K, H)$ and a τ_j-open set $D(H, K)$ such that:

(i) $H \subset D(H, K) \subset \tau_i \operatorname{cl} D(H, K) \subset X - K$,
$K \subset D(K, H) \subset \tau_j \operatorname{cl} D(K, H) \subset X - H$

and

(ii) if the pairs (H, K) and (H', K') satisfy $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$ and $D(K', H') \subset D(K, H)$.

The function D defined in this way is called a pairwise monotone normality operator for (X, τ_1, τ_2). Note that (X, τ_1, τ_2) is pairwise monotonically normal if and only if (X, τ_2, τ_1) is pairwise monotonically normal.

Remark 1. It is not a restriction to assume that $D(H, K) \cap D(K, H) = \emptyset$. In fact, if D does not satisfy this condition, just let $D'(H, K) = D(H, K) \cap (X - \tau_j \operatorname{cl} D(K, H))$ and $D'(K, H) = D(K, H) \cap (X - \tau_i \operatorname{cl} D(H, K))$. Thus, we will suppose in the following that the operator D satisfies the condition in remark 1.

Remark 2. Let (X, τ_1, τ_2) be a space such that τ_1 (or τ_2) is the discrete topology on X. Then it is immediate to show that (X, τ_1, τ_2) is pairwise monotonically normal.

Definition 2. Given a space (X, τ_1, τ_2), we say that a pair (H, K) of subsets of X is (1, 2)-*separated*, if $K \cap \tau_1 \operatorname{cl} H = \emptyset$ and $H \cap \tau_2 \operatorname{cl} K = \emptyset$. Similarly we define the notion of a (2, 1)-separated pair.

The next result is useful in Section 4. We omit the proof because it is similar to [14, Lemma 2.2].

Lemma 1. *A space (X,τ_1,τ_2) is pairwise monotonically normal if and only if there is a function D which assigns to each (i,j)-separated pair (H,K) a τ_j-open set $D(H,K)$ such that*

(a) $H \subset D(H,K) \subset \tau_i \operatorname{cl} D(H,K) \subset X - K$,
(b) *if the (i,j)-separated pairs (H,K) and (H',K') satisfy $H \subset H'$ and $K' \subset K$, then $D(H,K) \subset D(H',K')$.*

A space (X,τ_1,τ_2) is called τ_1-semi-stratifiable with respect to τ_2, if to each τ_1-open set $U \subset X$, one can assign a sequence $(U_n)_{n\in N}$ of τ_2-closed sets such that: (i) $U = \bigcup_{n=1}^{\infty} U_n$ and (ii) if $U \subset V$ then $U_n \subset V_n$ for all $n \in N$, where $(V_n)_{n\in N}$ is the sequence assigned to the τ_1-open set V. If also: (iii) $U = \bigcup_{n=1}^{\infty} \tau_1 \operatorname{int} U_n$, then (X,τ_1,τ_2) is called τ_1-stratifiable with respect to τ_2. A space (X,τ_1,τ_2) is called pairwise (semi-)stratifiable ([19]), [9], [12], [18], if it is τ_1-(semi-)stratifiable with respect to τ_2 and τ_2-(semi-)stratifiable with respect to τ_1.

It immediately follows from the preceding definitions that a space (X,τ_1,τ_2) is pairwise semi-stratifiable if and only if to each τ_i-closed set $H \subset X$, one can assign a sequence $(H_n)_{n\in N}$ of τ_j-open sets such that: (i) $H = \bigcap_{n=1}^{\infty} H_n$ and (ii) if $H \subset K$, then $H_n \subset K_n$ for all $n \in N$, where $(K_n)_{n\in N}$ is the sequence assigned to the τ_i-closed set K. Similarly, (X,τ_1,τ_2) is pairwise stratifiable if and only if to each τ_i-closed set $H \subset X$, one can assign a sequence $(H_n)_{n\in N}$ of τ_j-open sets satisfying the above conditions (i) and (ii) and (iii): $H = \bigcap_{n=1}^{\infty} \tau_i \operatorname{cl} H_n$.

Our next result provides a relation between pairwise stratifiable and pairwise monotonically normal spaces (compare [14, Theorem 2.5]).

Proposition 1. *A space (X,τ_1,τ_2) is pairwise stratifiable if and only if it is a pairwise monotonically normal pairwise semi-stratifiable space.*

Proof: Suppose that (X,τ_1,τ_2) is pairwise stratifiable. Let (H,K) be a pair of disjoint subsets of X such that H is τ_i-closed and K is τ_j-closed. Then there exists a decreasing sequence $(H_n)_{n\in N}$ of τ_j-open sets such that $H = \bigcap_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} \tau_i \operatorname{cl} H_n$. Similarly, there exists a decreasing sequence $(K_n)_{n\in N}$ of τ_i-open sets such that $K = \bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} \tau_j \operatorname{cl} K_n$. Put

$$D(H,K) = \bigcup_{n=1}^{\infty} (H_n - \tau_j \operatorname{cl} K_n)$$

and

$$D(K,H) = \bigcup_{n=1}^{\infty} (K_n - \tau_i \operatorname{cl} H_n).$$

Then, $D(H,K)$ is a τ_j-open set which contains H because $H \cap K = \emptyset$. It is easy to see that $\tau_i \operatorname{cl} D(H,K) \subset X - K$. Furthermore, if the pair (H',K') (with H' τ_i-closed,

K' τ_j-closed and $H' \cap K' = \emptyset$) satisfies $H \subset H'$ and $K' \subset K$, then $D(H,K) \subset D(H',K')$. We deduce in a similar way that $K \subset D(K,H) \subset \tau_j \text{ cl } D(K,H) \subset X - H$ and $D(K',H') \subset D(K,H)$. Therefore, (X,τ_1,τ_2) is pairwise monotonically normal. Conversely, suppose that D is a pairwise monotone normality operator for (X,τ_1,τ_2) and let H be a τ_i-closed set. Then there exists a sequence $(H_n)_{n\in N}$ of τ_j-open sets such that $H = \bigcap_{n=1}^{\infty} H_n$. Since, for each $n \in N$, $H \cap (X - H_n) = \emptyset$ we have $H \subset D(H, X - H_n) \subset \tau_i \text{ cl } D(H, X - H_n) \subset H_n$. Define $H'_n = D(H, X - H_n)$. Thus, $H = \bigcap_{n=1}^{\infty} H'_n = \bigcap_{n=1}^{\infty} \tau_i \text{ cl } H'_n$. Finally, if $H \subset G$ (with H and G τ_i-closed sets), it follows $D(H, X - H_n) \subset D(G, X - G_n)$ and, hence, $H'_n \subset G'_n$ for all $n \in N$. This completes the proof. □

Given two bitopological spaces $(X,\tau_1,\tau_2), (Y,\tau'_1,\tau'_2)$ and a mapping f from X onto Y, we say that f is continuous (closed) from (X,τ_1,τ_2) onto (Y,τ'_1,τ'_2), if f is a continuous (closed) mapping from (X,τ_i) onto (Y,τ'_i), $i = 1,2$.

Proposition 2. *Let f be a continuous and closed mapping from the pairwise monotonically normal space (X,τ_1,τ_2) onto the space (Y,τ'_1,τ'_2). Then (Y,τ'_1,τ'_2) is pairwise monotonically normal.*

Proof: Let D be a pairwise monotone normality operator for (X,τ_1,τ_2). For each pair (H',K') of disjoint subsets of Y such that H' is τ'_i-closed and K' is τ'_j-closed, define

$$D'(H',K') = Y - f(X - D(f^{-1}(H'), f^{-1}(K'))).$$

Then D' is a pairwise monotone normality operator for (Y,τ'_1,τ'_2). □

It is proved in [19] that pairwise semi-stratifiable spaces are preserved by continuous closed mappings. From this result and Propositions 1 and 2, we derive the following result.

Corollary [12], [18]**.** *Let f be a continuous and closed mapping from the pairwise stratifiable space (X,τ_1,τ_2) onto the space (Y,τ'_1,τ'_2). Then, (Y,τ'_1,τ'_2) is pairwise stratifiable.*

In [4], Ceder defined the class of Nagata spaces and showed that a topological space is a Nagata space if and only if it is stratifiable and first countable. Later, Borges [3] obtained a characterization of stratifiable spaces which generalizes the notion of a Nagata space. A similar characterization for monotonically normal spaces is also proved by Borges in [2, Theorem 1.2]. Our next results provide the bitopological counterpart of these characterizations.

Proposition 3. *A space (X,τ_1,τ_2) is pairwise stratifiable if and only if for each $x \in X$ there exist two bases $\{U_i(\alpha_i,n,x) : \alpha_i \in D_i(x), n \in N\}$ and $\{S_i(\alpha_i,n,x) : \alpha_i \in D_i(x), n \in N\}$ of τ_i-neighbourhoods of x such that if $S_i(\alpha_i,n,x) \cap S_j(\alpha_j,n,y) \neq \emptyset$, then $x \in U_j(\alpha_j,n,y)$ and $y \in U_i(\alpha_i,n,x)$.*

Proof: Necessary condition. Given $x \in X$, let $\mathcal{M}_i(x) = \{M_i(\alpha_i,x) : \alpha_i \in D_i(x)\}$ be a base of τ_i-neighbourhodds of x, $i = 1,2$. Now let U be a τ_i-open set. Then there exists a sequence $(U_n)_{n\in N}$ of τ_j-closed sets satisfying $U = \bigcup_{n=1}^{\infty} U_n =$

$\bigcup_{n=1}^{\infty} \tau_i \operatorname{int} U_n$ and $U_n \subset V_n$ whenever $U \subset V$, with U, V, τ_i-open sets. For each $\alpha_i \in D_i(x)$ and each $n \in N$ define

$$U_i(\alpha_i, n, x) = \bigcap \{U : U \text{ is } \tau_i\text{-open and } M_i(\alpha_i, x) \subset U_n\},$$

$$S_i'(\alpha_i, n, x) = \bigcap \{U_n : M_i(\alpha_i, x) \subset U_n\} - \bigcup \{V_n : x \notin V, V\, \tau_j\text{-open}\}$$

and

$$S_i(\alpha_i, n, x) = \bigcap_{k=1}^{n} S_i'(\alpha_i, k, x).$$

One can easily verify that the collections $\{U_i(\alpha_i, n, x) : \alpha_i \in D_i(x), n \in N\}$ and $\{S_i(\alpha_i, n, x) : \alpha_i \in D_i(x), n \in N\}$ satisfy the required conditions.

Sufficient condition. It is enough to define, for each τ_i-open set U and each $n \in N$,

$$U_n = \tau_j \operatorname{cl} [\bigcup \{\tau_i \operatorname{int} S_i(\alpha_i, n, x) : U_i(\alpha_i, n, x) \subset U\}].$$

□

Proposition 4. *A space (X, τ_1, τ_2) is pairwise monotonically normal if and only if for each τ_i-open set U and each $x \in U$ there exists a τ_i-open neighbourhood U_x of x such that if $U_x \cap V_y \neq \emptyset$, then $x \in V$ or $y \in U$, where V is a τ_j-open set with $y \in V$.*

Proof: Necessary condition. Let D be a pairwise monotone normality operator for (X, τ_1, τ_2). Given a τ_i-open set U and an $x \in U$, we have $\{x\} \subset D(\{x\}, X-U) \subset \tau_j \operatorname{cl} D(\{x\}, X-U) \subset U$. Define $U_x = D(\{x\}, X-U)$. Then, $x \in U_x \subset U$. Now let $U_x \cap V_y \neq \emptyset$, where U is a τ_i-open set with $x \in U$ and V is a τ_j-open set with $y \in V$. Assume $x \notin V$ and $y \notin U$. Then, $D(\{y\}, X-V) \subset D(\{y\}, \{x\})$ and $D(\{x\}, X-U) \subset D(\{x\}, \{y\})$. From Remark 2, it follows $D(\{y\}, X-V) \cap D(\{x\}, X-U) = \emptyset$, this is, $V_y \cap U_x = \emptyset$, a contradiction.

Sufficient condition. For each pair (H, K) of disjoint subsets of X such that H is τ_i-closed and K is τ_j-closed, define

$$D(H, K) = \bigcup_{x \in H} \{V_x : x \in V, \quad V \cap K = \emptyset, \quad V\, \tau_j\text{-open}\}$$

and

$$D(K, H) = \bigcup_{x \in K} \{U_x : x \in U, \quad U \cap H = \emptyset, \quad U\, \tau_i\text{-open}\}.$$

Then, the function D defined in this way is a pairwise monotone normality operator for (X, τ_1, τ_2). □

Corollary. *Pairwise monotone normality is a hereditary property.*

3. Pairwise monotone normality and real-valued functions.

In this section we mean by function a real-valued function. The upper(lower) semicontinuous functions are abbreviated to u.s.c. (l.s.c.) functions.

Lemma 2. *Let (X,τ_1,τ_2) be a pairwise monotonically normal space. Then to each pair of sequences $\{(F_n)_{n\in N},(G_n)_{n\in N}\}$ such that, for each $n\in N$, F_n is τ_i-closed, G_n is τ_j-open, $\tau_i\operatorname{cl}F\subset G$ and $F\subset\tau_j\operatorname{int}G$ (where $F=\bigcup_{n=1}^{\infty}F_n$ and $G=\bigcap_{n=1}^{\infty}G_n$), there is $H\subset X$ satisfying*

(a) *$F\subset\tau_j\operatorname{int}H\subset\tau_i\operatorname{cl}H\subset G$.*

(b) *If H and H' are the sets associated by* (a) *to the pairs of sequences $\{(F_n)_{n\in N},(G_n)_{n\in N}\}$ and $\{(F'_n)_{n\in N},(G'_n)_{n\in N}\}$ respectively, and, for each $n\in N$, $F_n\subset F'_n$ and $G_n\subset G'_n$, then $H\subset H'$.*

Proof: Let D be a pairwise monotone normality operator for (X,τ_1,τ_2) and $\{(F_n)_{n\in N},(G_n)_{n\in N}\}$ a pair of sequences satisfying the hypotheses. Since $F_1\subset\tau_j\operatorname{int}G$, there is a τ_j-open $D(F_1,X-\tau_j\operatorname{int}G)$ such that $F_1\subset D(F_1,X-\tau_j\operatorname{int}G)\subset\tau_i\operatorname{cl}D(F_1,X-\tau_j\operatorname{int}G)\subset G$. Put $C_1=D(F_1,X-\tau_j\operatorname{int}G)$. Then, $(\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_1)\subset G_1$ and then there is a τ_j-open set $A_1=D((\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_1),X-G_1)$ such that $(\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_1)\subset A_1\subset\tau_i\operatorname{cl}A_1\subset G_1$. Now let us suppose that, for $k=2,\dots,n$, we have obtained τ_i-closed sets $C_k=D(F_k\cup\tau_i\operatorname{cl}C_{k-1},X-(A_{k-1}\cap\tau_j\operatorname{int}G))$ and τ_j-open sets $A_k=D((\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_k),X-(A_{k-1}\cap G_k))$ such that $F_k\subset C_k\subset\tau_i\operatorname{cl}C_k\subset A_k\subset\tau_i\operatorname{cl}A_k\subset G_k$ and $\tau_i\operatorname{cl}C_k\subset\tau_j\operatorname{int}G,\tau_i\operatorname{cl}F\subset A_k$. Given $n+1$, since $(F_{n+1}\cup\tau_i\operatorname{cl}C_n)\subset(A_n\cap\tau_j\operatorname{int}G)$, we obtain the τ_j-open set $C_{n+1}=D(F_{n+1}\cup\tau_i\operatorname{cl}C_n,X-(A_n\cap\tau_j\operatorname{int}G))$ such that $(F_{n+1}\cup\tau_i\operatorname{cl}C_n)\subset C_{n+1}\subset\tau_i\operatorname{cl}C_{n+1}\subset(A_n\cap\tau_j\operatorname{int}G)$. As $(\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_{n+1})\subset(A_n\cap G_{n+1})$, we obtain the τ_j-open set $A_{n+1}=D((\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_{n+1}),X-(A_n\cap G_{n+1}))$ satisfying $(\tau_i\operatorname{cl}F)\cup(\tau_i\operatorname{cl}C_{n+1})\subset A_{n+1}\subset\tau_i\operatorname{cl}A_{n+1}\subset(A_n\cap G_{n+1})$. Hence, we can construct, inductively, the sequences $(C_n)_{n\in N}$ and $(A_n)_{n\in N}$ of τ_j-open sets satisfying the above relations. Put $H=\bigcup_{n=1}^{\infty}C_n$. Since H is τ_j-open and $F_n\subset C_n$ for all $n\in N$, it follows that $F\subset\tau_j\operatorname{int}H$. On the other hand, since $C_n\subset C_{n+k}\subset\tau_i\operatorname{cl}A_{n+k}\subset\tau_i\operatorname{cl}A_k\subset G_k$ for all $n,k\in N$, we have $H\subset G_k$ for all $k\in N$. This proves the part (a). In order to prove (b), note that if, for each $n\in N$, $F_n\subset F'_n$ and $G_n\subset G'_n$ we have, from the condition (ii) in Definition 1, $C_1\subset C'_1$ and $A_1\subset A'_1$. Inductively we obtain $C_n\subset C'_n$ (and $A_n\subset A'_n$) for all $n\in N$. Therefore, $H\subset H'$. □

Lemma 3. *Let (X,τ_1,τ_2) be a pairwise monotonically normal space and D a dense countable subset of $]0,1[$. Then to each pair of families $\{F(\alpha):\alpha\in D\},\{G(\alpha):\alpha\in D\}$ of subsets of X such that:* (i) *for each $\alpha\in D,F(\alpha)=\bigcup_{n=1}^{\infty}F_n(\alpha)$ and $G(\alpha)=\bigcap_{n=1}^{\infty}G_n(\alpha)$, where $F_n(\alpha)$ is τ_i-closed and $G_n(\alpha)$ is τ_j-open for all $n\in N$;* (ii) *$\tau_i\operatorname{cl}F(\alpha)\subset G(\alpha)$ and $F(\alpha)\subset\tau_j\operatorname{int}G(\alpha)$ for all $\alpha\in D$, and* (iii) *$\tau_i\operatorname{cl}F(\alpha)\subset F(\beta)$ and $G(\alpha)\subset\tau_j\operatorname{int}G(\beta)$ for $\alpha<\beta$, there is a family $\{H(\alpha):\alpha\in D\}$ of subsets of X satisfying*

(a) *$F(\alpha)\subset\tau_j\operatorname{int}H(\alpha)\subset\tau_i\operatorname{cl}H(\alpha)\subset G(\alpha)$ for all $x\in D$ and $\tau_i\operatorname{cl}H(\alpha)\subset\tau_j\operatorname{int}G(\beta)$ for $\alpha<\beta$.*

(b) *If* $\{H(\alpha) : \alpha \in D\}$ *and* $\{H'(\alpha) : \alpha \in D\}$ *are the families associated by* (a) *to the pairs of families* $\{F(\alpha) : \alpha \in D\}, \{G(\alpha) : \alpha \in D\}$, *and* $\{F'(\alpha) : \alpha \in D\}, \{G'(\alpha) : \alpha \in D\}$ *respectively, and, for each* $\alpha \in D$ *and* $n \in N$, $F_n(\alpha) \subset F'_n(\alpha)$ *and* $G_n(\alpha) \subset G'_n(\alpha)$, *then* $H(\alpha) \subset H'(\alpha)$.

PROOF: Put $D = \{d_n : n \in N\}$. Take d_1. By Lemma 2 there exists $H(d_1) \subset X$ such that $F(d_1) \subset \tau_j \operatorname{int} H(d_1) \subset \tau_i \operatorname{cl} H(d_1) \subset G(d_1)$. Now let us suppose that, for $k = 2, \dots, n$, we have obtained subsets $H(d_2), \dots, H(d_n)$, satisfying the conditions (a) and (b). Given $n+1$, define sets F and G as

$$F = F(d_{n+1}) \cup \left[\bigcup\{\tau_i \operatorname{cl} H(d_r) : d_r < d_{n+1},\ 1 \le r \le n\}\right]$$

and

$$\begin{aligned} G &= G(d_{n+1}) \ \text{ if } \ d_r < d_{n+1} \ \text{ for } \ r = 1, \dots, n, \\ G &= G(d_{n+1}) \cap \left[\bigcap\{\tau_j \operatorname{int} H(d_r) : d_r > d_{n+1},\ 1 \le r \le n\}\right] \ \text{ otherwise.} \end{aligned}$$

Following the proof of [13, Corollary 1.1] we obtain, inductively, the family $\{H(\alpha) : \alpha \in D\}$ satisfying (a). The part (b) also follows inductively from Lemma 2 (b). □

Note that the necessary conditions of the above lemmas also are sufficient conditions for the pairwise monotone normality of the space (X, τ_1, τ_2).

In order to help with reading, we include the following well-known observations which we use in the proof of the main result of this section.

Remark 3. Let X be a non-empty set and $f : X \to]0,1[$. If D is a dense countable subset of $]0,1[$ and $\{F(\alpha) : \alpha \in D\}$ a family of subsets of X such that, for each $\alpha \in D$, $f^{-1}]0,\alpha[\subset F(\alpha) \subset f^{-1}]0,\alpha]$, then $f(x) = \sup\{\alpha \in D : x \notin F(\alpha)\} = \inf\{\alpha \in D : x \in F(\alpha)\}$ for all $x \in X$. We say that f is determined by $\{F(\alpha) : \alpha \in D\}$. Conversely, given an expansive family $\{F(\alpha) : \alpha \in D\}$ of subsets of X, we may define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \sup\{\alpha \in D : x \notin F(\alpha)\} & \text{if } \{\alpha \in D : x \notin F(\alpha)\} \neq \emptyset, \\ 0 & \text{if } \{\alpha \in D : x \in F(\alpha)\} = \emptyset. \end{cases}$$

In particular, $f(x) > 0$, for all $x \in X$, if $\bigcap\{F(\alpha) : \alpha \in D\} = \emptyset$ and $f(x) < 1$, for all $x \in X$, if $\bigcup\{F(\alpha) : \alpha \in D\} = X$.

Remark 4. Let (X, τ_1, τ_2) be a bitopological space and $f : X \to]0,1[$ determined by the family $\{F(\alpha) : \alpha \in D\}$. Then f is τ_i-l.s.c. if and only if $\tau_i \operatorname{cl} F(\alpha) \subset F(\beta)$ whenever $\alpha < \beta$ and f is τ_i-u.s.c. if and only if $F(\alpha) \subset \tau_i \operatorname{int} F(\beta)$ whenever $\alpha < \beta$. Furthermore, if g is determined by $\{G(\alpha) : \alpha \in D\}$, then $g \le f$ if and only if $F(\alpha) \subset G(\beta)$ whenever $\alpha < \beta$.

Theorem 1. *A space (X,τ_1,τ_2) is pairwise monotonically normal if and only if for each pair of functions f and g defined on X such that $g \le f$, f is τ_i-l.s.c. on X, g is τ_j-u.s.c. on X and f is τ_j-u.s.c. on the τ_j-closed set $C \subset X$, one assigns a τ_i-l.s.c. and τ_j-u.s.c. function h on X such that*

(a) *$g \le h \le f$ on X and $h = f$ on C,*
(b) *if h and h' are the functions associated, by* (a), *to the pairs of functions f, g, and f', g', respectively, and $f \le f'$ and $g \le g'$ on X, then $h \le h'$.*

Proof: Necessary condition. It is enough to take functions from X into $]0,1[$. Let D be a dense countable subset of $]0,1[$. Given $f, g : X \to]0,1[$ satisfying the hypotheses, define, similarly to [13, Theorem 2], $F(\alpha) = f^{-1}]0,\alpha[$ and $G(\alpha) = g^{-1}]0,\alpha] \bigcap (f^{-1}]0,\alpha] \bigcup (X - C))$ for all $\alpha \in D$. Clearly, $F(\alpha) = \bigcup_{\beta<\alpha} f^{-1}]0,\beta]$ and $G(\alpha) = \bigcap_{\alpha<\beta} \{g^{-1}]0,\beta[\bigcap (f^{-1}]0,\beta[\bigcup (X - C))\}$. Then, $F(\alpha)$ is a countable union of τ_i-closed sets and $G(\alpha)$ is a countable intersection of τ_j-open sets. Furthermore, $\tau_i \operatorname{cl} F(\alpha) \subset G(\alpha), F(\alpha) \subset \tau_j \operatorname{int} G(\alpha)$ and, for $\alpha < \beta$, $\tau_i \operatorname{cl} F(\alpha) \subset F(\beta)$ and $G(\alpha) \subset \tau_j \operatorname{int} G(\beta)$. Consequently, the conditions (i), (ii) and (iii) of Lemma 3 are satisfied and, hence, there is a family $\{H(\alpha) : \alpha \in D\}$ of subsets of X such that $F(\alpha) \subset \tau_j \operatorname{int} H(\alpha) \subset \tau_i \operatorname{cl} H(\alpha) \subset G(\alpha)$ for all $\alpha \in D$ and $\tau_i \operatorname{cl} H(\alpha) \subset \tau_j \operatorname{int} H(\beta)$ for $\alpha < \beta$. By Remark 3, the function h determined by $\{H(\alpha) : \alpha \in D\}$, is τ_i-l.s.c. and τ_j-u.s.c. on X and $h \le f$. If G denotes the function determined by $\{G(\alpha) : \alpha \in D\}$, we deduce, by Remarks 2 and 3, that $g \le G$. Since $G \le h$, it follows $g \le h$. Furthermore, $h = f$ on C (see [13, Theorem 2]). This proves the part (a). In order to prove (b), note that for each $\alpha \in D$, we have $(f')^{-1}]0,\beta] \subset f^{-1}]0,\beta]$ for $\beta < \alpha$ and $(g')^{-1}]0,\beta] \bigcap (f^{-1}]0,\beta[\bigcup (X - C')) \subset g^{-1}]0,\beta] \bigcap (f^{-1}]0,\beta[\bigcup (X - C))$ for $\alpha < \beta$. Therefore, we can apply Lemma 3 (b). Thus, $H'(\alpha) \subset H(\alpha)$. So, $H'(\alpha) \subset H(\beta)$ for $\alpha < \beta$, and, by Remark 3, $h \le h'$. This proves the part (b).

Sufficient condition. For each pair (H,K) of disjoint subsets of X such that H is τ_i-closed and K is τ_j-closed, we define $f(x) = 1$, if $x \in X - H$ and $f(x) = 0$, if $x \in H$, and $g(x) = 1$, if $x \in K$ and $g(x) = 1$, if $x \in X - K$. Therefore, g is τ_j-u.s.c., f is τ_i-l.s.c. and $g \le f$. So, taking $C = \emptyset$, there is a τ_i-l.s.c. and τ_j-u.s.c. function $h : X \to [0,1]$ such that $g \le h \le f$. Finally, if we put $D(H,K) = h^{-1}[0,1/2[$ and $D(K,H) = h^{-1}]1/2,1]$, then it is easy to show that D is a pairwise monotone normality operator for (X,τ_1,τ_2). The proof is complete. □

Putting $C = \emptyset$ in the above theorem, we obtain an analogue of the celebrated Katětov–Tong's insertion theorem [16], [24], to pairwise monotonically normal bitopological spaces. For the sake of brevity we omit its statement.

We now give another consequence of Theorem 1.

Corollary. *Let (X,τ_1,τ_2) be a pairwise monotonically normal space. Then, for each τ_1-closed and τ_2-closed set $H \subset X$ and each τ_i-l.s.c. and τ_j-u.s.c. function $f : H \to [0,1]$, one can assign a τ_i-l.s.c. and τ_j-u.s.c. extension $\Phi(f) : X \to [0,1]$ such that if g and f are τ_i-l.s.c and τ_j-u.s.c. from H into $[0,1]$ satisfying $g \le f$, then $\Phi(g) \le \Phi(f)$.*

PROOF: Given the τ_1-closed and τ_2-closed set $H \subset X$ and the functions f and g satisfying the hypotheses, we define the functions f_1, f_2, g_1 and g_2 as follows: $f_1(x) = 0$, if $x \notin C$ and $f_1(x) = f(x)$, if $x \in C$; $f_2(x) = 1$, if $x \notin C$ and $f_2(x) = f(x)$, if $x \in C$; $g_1(x) = 0$, if $x \notin C$ and $g_1(x) = g(x)$, if $x \in C$; $g_2(x) = 1$, if $x \notin C$ and $g_2(x) = g(x)$, if $x \in C$. Clearly, f_1 is τ_j-u.s.c. on X and f_2 is τ_i-l.s.c. on X and τ_j-u.s.c. on C. Since $f_1 \leq f_2$ on X it follows from Theorem 1 (a) that there is a τ_i-l.s.c. and τ_j-u.s.c. function $\Phi(f)$ from X into $[0,1]$ such that $f_1 \leq \Phi(f) \leq f_2$ on X and $\Phi(f) = f_2 = f$ on C. Similarly, there is a τ_i-l.s.c. and τ_j-u.s.c. function $\Phi(g)$ from X into $[0,1]$ such that $g_1 \leq \Phi(g) \leq g_2$ on X and $\Phi(g) = g$ on C. Finally, if $g \leq f$ we deduce, by Theorem 1 (b), that $\Phi(g) \leq \Phi(f)$. This completes the proof. □

Note that if in the preceding corollary we put $\tau_1 = \tau_2$, then [14, Theorem 3.3] is obtained.

4. Pairwise monotone normality and quasi-metrization.

In Section 4 of [14], Heat, Lutzer and Zenor characterized metrizable spaces by assuming monotone normality of cartesian products. The key of these characterizations is Theorem 4.1 of their paper which says that if $X \times Y$ is monotonically normal, then either no countable subset of X has a limit point or Y is stratifiable. The bitopological situation is described in our next result. (It seems interesting to compare these results with those of Katětov [15] about the hereditary normality of cartesian products. See also [21, Proposition 3].)

Proposition 5. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two spaces such that the space $(X \times Y, \tau_1 \times \tau_1', \tau_2 \times \tau_2')$ is pairwise monotonically normal. Then, either no countable subset of X has τ_i-accumulation point or Y is τ_i'-stratifiable with respect to τ_j'.*

PROOF: We adopt the technique of [14, Theorem 4.1] and so we omit the details. Suppose that $M' = \{m_n : n \in N\}$ is a subset of X having a τ_i-accumulation point p. We assume that $p \in X - M'$. Let $M = M' \cup \{p\}$. By the Corollary of Proposition 4, the space $(M \times Y, (\tau_1 |_M) \times \tau_1', (\tau_2 |_M) \times \tau_2')$ is pairwise monotonically normal. We will show that Y is τ_i'-stratifiable with respect to τ_j'. Given a τ_i'-closed set $F \subset Y$, put $H_F = M' \times F$ and $K_F = \{p\} \times (Y - F)$. If we write $(\tau_i |_M) \times \tau_i' = \tau_i''$, $i = 1, 2$, then the pair (H_F, K_F) is (i,j)-separated in $(M \times Y, \tau_1'', \tau_2'')$. Therefore, if D denotes the function for $M \times Y$ described in Lemma 1, we have $H_F \subset D(H_F, K_F) \subset (M \times Y) - K_F$. Now for each $n \in N$, we define $T(F,n) = \{y \in Y : (m_n, y) \in D(H_F, K_F)\}$. Then, it is easily seen that each $T(F,n)$ is τ_j'-open. Furthermore, $F = \bigcap_{n=1}^{\infty} T(F,n) = \bigcap_{n=1}^{\infty} \tau_i' \operatorname{cl} T(F,n)$. Finally, if the τ_i'-closed set F' contains F, it follows from Lemma 1 (b) that $T(F,n) \subset T(F',n)$ for all $n \in N$. The proof is complete. □

Corollary. *A space (X, τ_1, τ_2) is pairwise stratifiable if and only if $(X \times Y, \tau_1 \times T, \tau_2 \times T)$ is pairwise monotonically normal, where $Y = \{0\} \cup \{1/n : n \in N\}$ and T is the restriction to Y of the usual topology.*

PROOF: Let (X, τ_1, τ_2) be a pairwise stratifiable space. In [12], it is proved that the countable product of pairwise stratifiable is pairwise stratifiable. Hence, $(X \times$

$Y, \tau_1 \times T, \tau_2 \times T)$ is pairwise stratifiable. Conversely, we have, by Proposition 5, that (X, τ_1, τ_2) is pairwise stratifiable. □

Corollary. *A space (X, τ_1, τ_2) is pairwise stratifiable if and only if for every quasi-metric space (Y, d), the space $(X \times Y, \tau_1 \times T(d), \tau_2 \times T(d^{-1}))$ is pairwise monotonically normal.*

In [9], Fox obtains a very nice solution to the quasi-metrization problem. Exactly, he proved that a space (X, τ_1, τ_2) is quasi-metrizable if and only if it is a pairwise stratifiable space and (X, τ_1) and (X, τ_2) are γ-spaces. Fox's theorem can be stated in a more general form as follows.

Theorem 2. *A space (X, τ_1, τ_2) is quasi-metrizable if and only if $(X \times X, \tau_1 \times \tau_2, \tau_2 \times \tau_1)$ is pairwise monotonically normal and (X, τ_1) and (X, τ_2) are γ-spaces.*

Proof: Since the necessity is almost obvious, we only prove the sufficiency. Suppose that τ_1 has a nonisolated point. Since every γ-space is first countable, it follows from Proposition 5 that (X, τ_1, τ_2) is τ_2-stratifiable with respect to τ_1. If τ_2 has also a nonisolated point,we deduce, similarly, that (X, τ_1, τ_2) is τ_1-stratifiable with respect to τ_2. By Fox's theorem, (X, τ_1, τ_2) is quasi-metrizable. Otherwise, τ_2 is the discrete topology on X and then it is clear that the space is τ_1-stratifiable with respect to τ_2. Newly, Fox's theorem proves the quasi-metrizability of (X, τ_1, τ_2). Interchanging the roles of τ_1 and τ_2, we complete the proof. □

A slight modification of the proof of the above theorem permits us to state the following variant of it.

Theorem 3. *Let (X, τ_1, τ_2) be a space such that τ_1 and τ_2 have nonisolated points. Then, (X, τ_1, τ_2) is quasi-metrizable if and only if $(X \times X, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is pairwise monotonically normal and (X, τ_1) and (X, τ_2) are γ-spaces.*

Remark 5. Consider the space (R, τ_1, τ_2), where R is the real line, τ_1 is the usual topology on R and τ_2 is the discrete topology on R. Then, $(R \times R, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is pairwise monotonically normal, (R, τ_1) and (R, τ_2) are γ-spaces, but it is well-known that (R, τ_1, τ_2) is not quasi-metrizable.

In [7] Fletcher, Hoyle III and Patty introduced the notion of a pairwise (countably) compact space. Recall that a space (X, τ_1, τ_2) is pairwise (countably) compact if and only if every proper τ_i-closed set is τ_j-(countably) compact ([22]), [5].

Lemma 4 [19]. *A space (X, τ_1, τ_2) is τ_1-semi-stratifiable with respect to τ_2, if there is $g : N \times X \to \tau_1$ such that*

(a) *$x \in g(n, x)$ for all $x \in X$ and $n \in N$,*
(b) *if, for each $n \in N, x \in g(n, x_n)$, then the sequence $(x_n)_{n \in N}$ is τ_2-convergent to x.*

A space (X, τ_1, τ_2) is said to be pairwise Hausdorff [17], if, for $x \neq y$, there is a τ_i-neighbourhood of x and a disjoint τ_j-neighbourhood of y. On the other hand, we say that (X, τ_1, τ_2) has a $\tau_1 \times \tau_2$-G_δ-diagonal, if there is a sequence $(G_n)_{n \in N}$ of $\tau_1 \times \tau_2$-open sets such that $\Delta = \bigcap_{n=1}^{\infty} G_n$, where $\Delta = \{(x, x) : x \in X\}$.

By using Lemma 4, we easily obtain the following result.

Lemma 5. *Every pairwise Hausdorff pairwise semi-stratifiable space (X, τ_1, τ_2) has a $\tau_1 \times \tau_2$-G_δ-diagonal.*

In [21], it is proved that a pairwise Hausdorff pairwise compact space (X, τ_1, τ_2) is quasi-metrizable if and only if it has a $\tau_1 \times \tau_2$-G_δ-diagonal. From this result and the preceding lemma, it follows that every pairwise Hausdorff pairwise compact pairwise semi-stratifiable space is quasi-metrizable. However, it is possible to obtain a better result as Theorem 4 shows. In order to prove this theorem, we will give some previous lemmas.

Lemma 6. *Let (X, τ_1, τ_2) be a pairwise Hausdorff space such that $\tau_1 \subset \tau_2$. If it is τ_1-semi-stratifiable with respect to τ_2 then (X, τ_1) and (X, τ_2) are semi-stratifiable spaces and have a G_δ-diagonal.*

Proof: Let $g : N \times X \to \tau_1$ be a mapping satisfying the conditions (a) and (b) in Lemma 4. If, for each $n \in N, x \in g(n, x_n)$, then the sequence $(x_n)_{n \in N}$ is τ_2-convergent to x. Since $\tau_1 \subset \tau_2$, then it also is τ_1-convergent to x. Therefore, (X, τ_1) and (X, τ_2) are semi-stratifiable spaces. Now, put, for each $n \in N$, $G_n = \bigcup_{x \in X}(g(n, x) \times g(n, x))$. Thus, $\Delta = \bigcap_{n=1}^{\infty} G_n$. This proves that (X, τ_1) and (X, τ_2) have a G_δ-diagonal. □

Lemma 7 [20]**.** *A pairwise countably compact space (X, τ_1, τ_2) such that each proper τ_1-countably compact set has a G_δ-diagonal, is pairwise compact.*

Lemma 8. *Let (X, τ_1, τ_2) be a pairwise Hausdorff pairwise countably compact τ_1-semi-stratifiable with respect to τ_2 space. Then, it is pairwise compact.*

Proof: Let $g : N \times X \to \tau_1$ be a mapping satisfying the conditions (a) and (b) in Lemma 4. Take a proper τ_1-countably compact set $F \subset X$. If, for each $n \in N, x_n \in F$ and $x \in g(n, x_n) \cap F$, then the sequence $(x_n)_{n \in N}$ is τ_2-convergent to x. But, $(x_n)_{n \in N}$ has also a τ_1-cluster point $y \in F$. Since (X, τ_1, τ_2) is pairwise Hausdorff, we deduce that $x = y$. Hence, $(F, \tau_1 |_F)$ is a T_1 countably compact semi-stratifiable space. By [6, Corollary 2.9], we have that it is compact. Therefore, $\tau_1 |_F \subset \tau_2 |_F$ [7, Theorem 10]. Thus, by Lemma 6, $(F, \tau_1 |_F)$ has G_δ-diagonal and the result now follows from Lemma 7. □

Recall that a topological space (X, τ) has a countable pseudo-character, if for each $x \in X$ there is a sequence $(V_n(x))_{n \in N}$ of open sets such that $\{x\} = \bigcap_{n=1}^{\infty} V_n(x)$.

Theorem 4. *A pairwise Hausdorff pairwise countably compact space (X, τ_1, τ_2) is quasi-metrizable if and only if it is τ_1-semi-stratifiable with respect to τ_2 and (X, τ_2) has a countable pseudo-character.*

Proof: Since the necessity is almost obvious, we only prove the sufficiency. Fix $x \in X$. Then, $\{x\} = \bigcap_{n=1}^{\infty} V_n(x)$, where each $V_n(x)$ is τ_2-open. By Lemma 8, (X, τ_1, τ_2) is pairwise compact. Therefore, by [7, Theorem 12], there is a sequence $(W_n(x))_{n \in N}$ of τ_2-open sets such that $x \in W_n(x) \subset \tau_1 \operatorname{cl} W_n(x) \subset V_n(x)$ for all $n \in N$. Put $F_n = X - W_n(x)$. Then, F_n is a proper τ_2-closed set and, hence, it is τ_1-compact. Since the subspace $(F_n, \tau_1 |_{F_n}, \tau_2 |_{F_n})$ is pairwise Hausdorff, it follows from [7, Theorem 10] that $\tau_1 |_{F_n} \subset \tau_2 |_{F_n}$. Consequently, Lemma 6 shows that $(F_n, \tau_1 |_{F_n})$ and $(F_n, \tau_2 |_{F_n})$ have a G_δ-diagonal. Thus, by [20, Corollary of Theorem 3], $(F_n, \tau_1 |_{F_n}, \tau_2 |_{F_n})$ is quasi-metrizable. In particular, $(F_n, \tau_1 |_{F_n})$ has a countable base. Similarly to the proof of [21, Proposition 5], we deduce that (X, τ_1) has a countable base. Finally, the quasi-metrizability of (X, τ_1, τ_2) follows from [20, Lemma 5]. □

Corollary. *A pairwise Hausdorff pairwise countably compact space (X, τ_1, τ_2) is quasi-metrizable if and only if $(X \times X, \tau_1 \times \tau_2, \tau_2 \times \tau_1)$ is pairwise monotonically normal.*

Proof: Assume $(X \times X, \tau_1 \times \tau_2, \tau_2 \times \tau_1)$ to be a pairwise monotonically normal space. Then it is pairwise Hausdorff and, thus, (X, τ_1, τ_2) is pairwise Hausdorff. Now suppose that there is $x \in X$ which is a τ_1-nonisolated point. Take $y \neq x$. Then there is a τ_1-open neighbourhood U of x and a disjoint τ_2-open neighbourhood V of y. Since there is a sequence $(x_n)_{n \in N}$ of distinct points of X satisfying $x_n \in U$ for all $n \in N$ and $X - V$ is countably compact, it follows that this sequence has a τ_1-accumulation point in $X - V$. By Proposition 5, X is τ_1-stratifiable with respect to τ_2. Similarly to the proof of Theorem 2, we conclude that (X, τ_1, τ_2) is pairwise stratifiable. The quasi-metrizability of (X, τ_1, τ_2) is now a consequence of Theorem 4. □

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