Periodic solutions for third order ordinary differential equations*

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Abstract. In this paper, we introduce the concept of upper and lower solutions for third order periodic boundary value problems. We show that the monotone iterative technique is valid and obtain the extremal solutions as limits of monotone sequences. We first present a new maximum principle for ordinary differential inequalities of third order that is interesting by itself.

Keywords: periodic solution, maximum principle, upper and lower solutions, monotone method

Classification: 34B15, 34C25

1. Introduction.

The existence of solutions for third order ordinary differential equations has been widely studied in the last years and applications of third order differential equations are encountered in physics, engineering and mathematical biology. See, for instance, [1]–[6], [9] and the references therein. Recently has been considered the existence of periodic solutions of third order ordinary differential equations [2], [3], [4], [8].

In this paper, we study the existence of solutions for the following periodic boundary value problem (PBVP) for a third order ordinary differential equation

$$u''' = f(t,u),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi). \tag{1.1}$$

We first present a new maximum principle for third order ordinary differential inequalities. As it is well known, the maximum principles have applications to the question of uniqueness and continuous dependence on the boundary values for linear equations and, also, to the question of existence for nonlinear equations by means of the monotone iterative method. We apply the maximum principle to the study of the existence of periodic solutions for the periodic boundary value problem (1.1). To this purpose, we introduce a new concept of upper and lower solutions as limits of monotone iterates.

Finally, we mention some open problems and questions relative to the periodic boundary value problem and questions relative to the periodic boundary value problem (1.1) for further research.

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2. Maximum principle.

Let $m > 0, M = m^2$, and for $u \in C^1(I), v \in C^2(I)$, define the operators $L_1 u = u' - mu$, and $L_2 v = v'' - mv$, respectively. Relative to L_1 and L_2, we have the following comparison results.

Lemma 2.1. *Suppose that $L_1 u \geq 0$ on I and $u(0) \geq u(2\pi)$, then $u \leq 0$ on I.*

Lemma 2.2. *If $L_2 u \geq 0$ on I, $u(0) = u(2\pi)$, and $u'(0) \geq u'(2\pi)$, then $u \leq 0$ on I.*

For the proofs of these results see Lemma 1.2.2 and Lemma 2.1.1 in [10], respectively. They are utilized to generate monotone sequences that converge to extremal solutions of first and second order PBVP, respectively.

Now, for $u \in C^3(I)$, let $L = L_1 \circ L_2$, that is, $Lu = u''' - mu'' - mu' + Mu$. We are in a position to prove the following maximum principle for third order differential inequalities.

Theorem 2.3. *Suppose that $Lu \geq 0$ on I and $u(0) = u(2\pi), u'(0) \leq u'(2\pi)$ and $u''(0) \geq u''(2\pi)$. Then we have that $u \leq 0$ on I.*

Proof: Let $v = L_2 u$. Thus, $L_1 v \geq 0$ and $v(0) \geq v(2\pi)$. In consequence, we get that $v \leq 0$ on I, that is, $L_2(-u) \geq 0$. Now, by Lemma 2.2, we conclude that $-u \leq 0$ on I. □

3. Upper and lower solutions.

We say that $\alpha \in C^3(I)$ is a lower solution of (1.1) if there exists $m > 0$ such that

$$\alpha'''(t) - m\alpha''(t) - m\alpha'(t) \leq f(t, \alpha(t)), \; t \in I, \tag{3.1}$$

$$\alpha(0) = \alpha(2\pi), \; \alpha'(0) \geq \alpha'(2\pi), \; \alpha''(0) \leq \alpha''(2\pi). \tag{3.2}$$

Analogously, we say that $\beta \in C^3(I)$ is an upper solution of (1.1) if there exists $n > 0$ with

$$\beta'''(t) - n\beta''(t) - n\beta'(t) \geq f(t, \beta(t)), \; t \in I, \tag{3.3}$$

$$\beta(0) = \beta(2\pi), \; \beta'(0) \leq \beta'(2\pi), \; \beta''(0) \geq \beta''(2\pi). \tag{3.4}$$

Conditions (3.1) and (3.3) may seem very artificial, but in practice we have some general situations, where they are satisfied. Indeed, we have the following two cases for $\alpha \in C^3(I)$. Obviously, similar situations are valid for β.

Case I. Suppose that $\alpha \in C^3(I)$ and $\alpha'''(t) < f(t, \alpha(t))$, $t \in I$. Then we can choose $k, m > 0$, such that $\alpha'''(t) + k \leq f(t, \alpha(t))$, $t \in I$ and $m|\alpha''(t)| + m|\alpha'(t)| \leq k$, $t \in I$. Therefore, Condition (3.1) holds.

Case II. Suppose that $a \in \mathbb{R}$ is such that $0 \leq f(t, a)$, $t \in I$. Thus, defining $\alpha(t) = a$, $t \in I$, we have that (3.1) is satisfied for any $m > 0$.

Now, let us assume that there exist α, β lower and upper solutions of (1.1), respectively, such that

$$\alpha(t) \leq \beta(t), \; t \in I. \tag{3.5}$$

Suppose that $m = n$ and assume that f satisfies the following condition for $\alpha(t) \leq v \leq u \leq \beta(t)$, $t \in I$,

$$f(t,u) - f(t,v) \geq -M(u-v). \tag{3.6}$$

Let $E = \{u \in C^3(I) : u(0) = u(2\pi),\ u'(0) = u'(2\pi), u''(0) = u''(2\pi)\}$, $F = C(I)$, and consider the linear operator $L : E \to F$. The analysis of this differential operator [3] shows that $\operatorname{Ker} L = \{0\}$ since $m > 0$. Thus, $L^{-1} : F \to E$ is continuous. For $\eta \in F$, let us solve the following PBVP

$$Lv = f(t,\eta) + M\eta,\ v \in E. \tag{3.7}$$

The unique solution of this PBVP is denoted by $K\eta$, and we can define the operator $K : F \to E$. For $u \in F$, we say that $u \in [\alpha,\beta]$ if $\alpha \leq u \leq \beta$ on I.

Theorem 3.1. *The operator K has the following two properties in the sector* $[\alpha,\beta]$.

(P1) *If* $\eta \in [\alpha,\beta]$, *then* $K\eta \in [\alpha,\beta]$, *and*
(P2) *if* $\eta_1, \eta_2 \in [\alpha,\beta]$ *and* $\eta_1 \leq \eta_2$ *on* I, *then* $K\eta_1 \leq K\eta_2$ *on* I.

Proof: To show (P1), let $v = K\eta$ and $u = v - \alpha$. Thus, on using (3.1) and (3.6), we get

$$Lu = f(t,\eta) + M\eta - \alpha''' + m\alpha'' + m\alpha' - M\alpha \geq f(t,\eta) + M\eta - f(t,\alpha) - M\alpha \geq 0.$$

Now, taking into account (3.2) and Theorem 2.3, we obtain that $v \geq 0$ on I, that is, $v \geq \alpha$ on I. Similarly we obtain that $v \leq \beta$ on I holds. In order to prove (P2), note that

$$L(K\eta_2 - K\eta_1) = f(t,\eta_2) + M\eta_2 - f(t,\eta_1) - M\eta_1 \geq 0.$$

This allows us to conclude that $u \geq 0$ on I. □

Next, we present the monotone iterative technique which yields monotone sequences that converge to the extremal solutions between the lower and upper solutions.

Theorem 3.2. *Suppose that there exist* α *and* β *lower and upper solutions of* (1.1), *respectively, satisfying* (3.5) *and* (3.6). *Then there exist monotone sequences* $\{\alpha_n\} \uparrow r$ *and* $\{\beta_n\} \downarrow s$ *such that*

$$\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_m \leq \cdots \leq \beta_1 \leq \beta_0 = \beta, \quad \textit{for every} \ \ n, m \in \mathbb{N}.$$

Here, r, s *are the minimal and maximal solutions of* (1.1) *in the sector* $[\alpha,\beta]$, *respectively, that is, any solution* $u \in [\alpha,\beta]$ *of* (1.1) *is such that* $u \in [r,s]$.

Proof: Let $\alpha_0 = \alpha$ and $\alpha_1 = K\alpha_0$. Thus $\alpha_1 \in [\alpha,\beta]$ by the property (P1). Now, by induction, it is easy to prove that the sequence $\{\alpha_n\}$ defined by $\alpha_{n+1} = K\alpha_n$, is such that $\alpha \leq \alpha_n \leq \alpha_{n+1} \leq \beta$, $n \in \mathbb{N}$. Moreover, $\{\alpha_n\}$ is bounded and, for every $t \in I$, we have that $\{\alpha_n(t)\} \uparrow r(t)$. On the other hand, we have that there exists a constant $C > 0$ with $|f(t, \alpha_n(t)) + M\alpha_n(t)| \le C$ for every $n \in \mathbb{N}$ and $t \in I$ since $\{\alpha_n\}$ is bounded in F and f is continuous. In consequence, $\|\alpha_{n+1}\|_E \le \|L^{-1}\| \cdot C$ and $\{\alpha_n\}$ is bounded in $C^3(I)$. This implies that $\{\alpha_n\}$ converges to r uniformly on I. Using standard arguments, we obtain that r is actually a solution of (1.1). Analogously, we obtain a sequence $\{\beta_n\} \downarrow s$, where s is a solution of (1.1).

To show that $r \le s$ are the minimal and maximal solutions of (1.1) in $[\alpha, \beta]$, let u be a solution of (1.1) in $[\alpha, \beta]$. Thus, $\alpha_1 = K\alpha \le Ku = u \subset K\beta = \beta_1$, and by induction we get that $\alpha_n \le u \le \beta_n$ for every $n \in \mathbb{N}$. Passing to the limit, we see that $r \le u \le s$ on I. $\square$

As an important and practical case we obtain the following result.

Corollary 3.3. *Suppose that there exist constants $a \le b$ and $M > 0$ such that*

$$f(t, a) \le 0 \le f(t, b), \ t \in I \quad \textit{and}$$
$$f(t, u) - f(t, v) \ge -M(u - v), \quad \textit{for} \ \ a \le v \le u \le b, \ t \in I.$$

Then there exist r and s minimal and maximal solutions of (1.1), *respectively, with $a \le r(t) \le s(t) \le b$, $t \in I$.*

4. Open problems.

It is interesting to investigate the existence of solutions of the PBVP when some or all of the inequalities that appear in the definition of lower and upper solutions are not satisfied. Some results in this direction for second order PBVP are given in [7], [10], [11].

Suppose that we have that $\alpha \ge \beta$ on I instead of (3.5). Is it possible to ensure the existence of solution for the PBVP (1.1)? For second order differential equations with periodic boundary conditions, this question is solved in [12].

If $r = s$ in Theorem 3.2, then the problem (1.1) has a unique solution in the sector $[\alpha, \beta]$. On the other hand, if $r \ne s$, then it would be interesting to study the structure of the set of solutions of (1.1) between α and β. Following the ideas of [12] and [13], we conjecture that under the conditions of Theorem 3.2, the set of solutions of (1.1) is compact and connected provided that f is monotone in u (either increasing or decreasing) for every $t \in I$ and there exists a sufficiently small constant $k > 0$ with $|f(t, u) - f(t, v)| \le k|u - v|$ for every $t \in I$, $u, v \in \mathbb{R}$.

For $\eta \in [\alpha, \beta]$, we can define $\eta_1 = K\eta$, $\eta_{n+1} = K\eta_n$, $n \ge 1$. Thus, we have a discrete dynamical system and obviously r and s are fixed points for K. For some general properties on dynamical systems see, for instance, [14]. What is the global attractor J for this discrete dynamical system? Is J stable in any sense? These questions are considered for second order PBVP in [7]. It is easy to see that any solution $u \in [\alpha, \beta]$ of (1.1) is such that $u \in J$.

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