# The endocenter and its applications to quasigroup representation theory

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*Abstract.* A construction is given, in a variety of groups, of a "functorial center" called the endocenter. The endocenter facilitates the identification of universal multiplication groups of groups in the variety, addressing the problem of determining when combinatorial multiplication groups are universal.

*Keywords:* multiplication group, quasigroup, center

*Classification:* 20E10, 20F14, 20N05

The theory of quasigroup modules, or quasigroup representation theory, is equivalent to the representation theory of quotients of group algebras of certain groups associated with quasigroups; namely, the stabilizers in the so-called universal multiplication groups (cf. [Sm, p. 56] and below). Universal multiplication groups give functors from varieties of quasigroups to the variety of groups. To help identify these universal multiplication groups we offer a construction (in varieties of groups) of a subgroup we call the endocenter. This endocenter itself gives a functor from varieties of groups to the variety of abelian groups. To a certain extent, the endocenter may be regarded as a "functorial center". We also identify some universal multiplication groups, most notably in $\mathsf{HSP}\{G\}$, the variety generated by a group $G$. For a quasigroup $Q$ and for any $q \in Q$, the maps

$$\begin{aligned} & R(q) : Q \to Q; \quad x \mapsto x\, q \\ \text{and} \quad & L(q) : Q \to Q; \quad x \mapsto q\, x \end{aligned}$$

are set bijections. As such, they generate a subgroup of the symmetric group $Q!$ on $Q$. This subgroup is the <u>(combinatorial) multiplication group</u> $\operatorname{Mlt} Q$ of $Q$; i.e. $\operatorname{Mlt} Q = \langle R(q), L(q) : q \in Q \rangle_{Q!}$. Unfortunately Mlt (which assigns $\operatorname{Mlt} Q$ to $Q$) does not extend suitably to homomorphisms to give a functor [Sm, p. 28]. To overcome this failure, consider the following construction.

Suppose we have a quasigroup $Q$ and an arbitrary variety $\mathbf{V}$ of quasigroups containing $Q$. The category whose objects are quasigroups in $\mathbf{V}$ and whose morphisms are quasigroup homomorphisms will also be denoted by $\mathbf{V}$. As an algebraic category, $\mathbf{V}$ is complete and co-complete [HS, 13.12, 13.14]. In $\mathbf{V}$, form the coproduct of $Q$ with $\langle x \rangle$, the free $\mathbf{V}$-algebra on one generator. Denote this coproduct by $Q * \langle x \rangle$. Since $Q$ may be identified with its image in $Q * \langle x \rangle$ [Sm, p. 33], we can consider the subgroup of the combinatorial multiplication group of $Q * \langle x \rangle$ generated by right and left multiplications by elements of $Q$. This subgroup is the universal multiplication group $U(Q;\mathbf{V})$ of $Q$ in $\mathbf{V}$; i.e. $U(Q;\mathbf{V}) = \langle R(q), L(q) : q \in Q\rangle_{(Q*\langle x\rangle)!}$.

**Remarks.** 1. The assignment of $U(Q;\mathbf{V})$ to $Q$ gives the promised functor from the category $\mathbf{V}$ to the category $\mathbf{Gp}$ of all groups [Sm, p. 34].

2. $U(Q;\mathbf{V})$ is variety dependent in the sense that, for a given quasigroup $Q$ and varieties $\mathbf{V}_1$ and $\mathbf{V}_2$ containing $Q$, it is not necessarily the case that $U(Q;\mathbf{V}_1) = U(Q;\mathbf{V}_2)$ [Sm, p.36].

3. If $\mathbf{V}_1 \subseteq \mathbf{V}_2$ then there is a natural group epimorphism $F : U(Q;\mathbf{V}_2) \twoheadrightarrow U(Q;\mathbf{V}_1)$ [Sm, p. 55].

4. For any variety $\mathbf{V}$ of quasigroups containing $Q$, there is a natural group epimorphism $H : U(Q;\mathbf{V}) \twoheadrightarrow \mathrm{Mlt}\, Q$ [Sm, p. 55].

Remark 3 can be phrased as: "The smaller the variety, the smaller the universal multiplication group". Remark 4 can be phrased as: "A universal multiplication group can be no smaller than the combinatorial multiplication group". Since the smallest variety containing $Q$ is just $\mathsf{HSP}\{Q\}$, it would be natural to ask whether $U(Q;\mathsf{HSP}\{Q\}) \cong \mathrm{Mlt}\, Q$, i.e. whether the combinatorial multiplication group is universal. Since lack of associativity leads to complications, we will concentrate on the "easy" case of groups. Thus, from now on $G$ will denote a group and $\mathbf{V}$ an arbitrary variety of groups containing $G$. In particular, $\mathbf{V}$ could be $\mathsf{HSP}\{G\}$ but it is not required to be so. Theorem 5 below gives a sufficient condition for $U(G;\mathsf{HSP}\{G\}) \cong \mathrm{Mlt}\, G$. On the other hand, Theorems 6 and 7 furnish examples of groups with $U(G;\mathsf{HSP}\{G\}) \ncong \mathrm{Mlt}\, G$.

For a group $G$, the combinatorial multiplication group $\mathrm{Mlt}\, G$ is given by the exact sequence

$$1 \to Z(G) \xrightarrow{\Delta} G \times G \xrightarrow{F} \mathrm{Mlt}\, G \to 1,$$

where $\Delta$ is the diagonal embedding given by $\Delta : Z(G) \to G \times G;\ z \mapsto (z,z)$, and where $F$ is the group epimorphism given by $F : G \times G \twoheadrightarrow \mathrm{Mlt}\, G;\ (g_1, g_2) \mapsto L(g_1^{-1})\, R(g_2)$. Thus,

$$\mathrm{Mlt}\, G \cong G \times G / \widehat{Z}, \tag{1}$$

where $\widehat{Z} = Z(G)\Delta$. Next, we define the group epimorphism $T : G \times G \to U(G;\mathbf{V});\ (g_1, g_2) \mapsto L(g_1^{-1})\, R(g_2)$. Clearly

$$U(G;\mathbf{V}) \cong G \times G / \operatorname{Ker} T. \tag{2}$$

The map $T$ will play a prominent role throughout, as will its kernel, $\operatorname{Ker} T$. By (1) and (2) it is clear that:

$$\text{If} \quad \operatorname{Ker} T = \widehat{Z}, \quad \text{then} \quad U(G;\mathbf{V}) \cong \mathrm{Mlt}\, G. \tag{3}$$

Thus, we note that since $G$ embeds naturally in $G * \langle x \rangle$, it is always the case that

$$\operatorname{Ker} T \leq \widehat{Z}. \tag{4}$$

This discussion leads to two results:

**Proposition 1.** *If $G$ is an abelian group and $\mathbf{V}$ is any variety of abelian groups containing $G$, then $\operatorname{Ker} T = \widehat{Z}$ (and hence $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$ by (3)).*

**Proposition 2.** *If $G$ is a group such that $Z(G) = 1$ and $\mathbf{V}$ is any variety of groups containing $G$, then $\operatorname{Ker} T = \widehat{Z}$ (and hence $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$ by (3)).*

In the study of these universal multiplication groups (of groups), attention focusses on the behavior of the subgroup $\operatorname{Ker} T$. If $\operatorname{Ker} T = \widehat{Z}$ then we have seen that $U(G;\mathbf{V}) \cong \operatorname{Mlt} G$. If $\operatorname{Ker} T < \widehat{Z}$, and if $G$ satisfies suitable finiteness conditions (most trivially, if $G$ is finite), then we will see that $U(G;\mathbf{V}) \ncong \operatorname{Mlt} G$. An intrinsic description of $\operatorname{Ker} T$ would clearly be beneficial. Towards that end we offer the following

**Definition.** The <u>endocenter</u>, $Z(G;\mathbf{V})$, of a group $G$ in a variety $\mathbf{V}$ of groups is defined to be:

$$Z(G;\mathbf{V}) = \bigcap_{G \leq H \in \mathbf{V}} Z(H).$$

The relevance of this definition to representation theory, especially to the study of universal multiplication groups, is seen in

**Theorem 3.** $Z(G;\mathbf{V})\Delta = \operatorname{Ker} T$.

Proof: First note that $Z(G;\mathbf{V}) \leq Z(G * \langle x \rangle)$ since $G * \langle x \rangle \in \mathbf{V}$ and $G \leq G * \langle x \rangle$. This means that if $g \in Z(G;\mathbf{V})$, then for every $t \in G * \langle x \rangle$ we have $g^{-1}tg = t$, i.e. $(g,g) \in \operatorname{Ker} T$. Therefore, $Z(G;\mathbf{V})\Delta \leq \operatorname{Ker} T$.

Conversely, if $(g,g) \in \operatorname{Ker} T$ and $H \in \mathbf{V}$ with $G \leq H$ we need to show that $g \in Z(H)$. So given $h \in H$, we need to show $g^{-1}hg = h$. If we let $f : G \to H$ be the inclusion map, and $k : \langle x \rangle \to H$ be determined by mapping $x \mapsto h$, then since $G * \langle x \rangle$ is a $\mathbf{V}$-coproduct, there exists a unique group homomorphism $F : G * \langle x \rangle \to H$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "f"'] & G * \langle x \rangle \arrow[d, dotted, "F"] & \langle x \rangle \arrow[l] \arrow[dl, "k"] \\ & H & \end{tikzcd}$$

Since $(g,g) \in \operatorname{Ker} T$, we have $g^{-1}xg = x$. Thus,

$$\begin{aligned} F(g^{-1}xg) &= F(x), \quad \text{which implies} \\ F(g^{-1})F(x)F(g) &= F(x), \quad \text{which implies} \\ f(g^{-1})k(x)f(g) &= k(x), \quad \text{and so} \\ g^{-1}hg &= h, \end{aligned}$$

as desired. Therefore, $\operatorname{Ker} T \leq Z(G;\mathbf{V})\Delta$; and hence, $\operatorname{Ker} T = Z(G;\mathbf{V})\Delta$. □

**Remark.** In light of Theorem 3, we can recast (3) in the following form:

$$\text{If } \; Z(G;\mathbf{V}) = Z(G), \quad \text{then } \; U(G;\mathbf{V}) \cong \operatorname{Mlt} G. \tag{5}$$

The usual center of a group is not a functorial construction. By contrast, the endocenter is natural:

**Theorem 4.** *$Z(\ ;\mathbf{V})$ is a functor from $\mathbf{V}$ to $\mathbf{Gp}$.*

Proof: Given a group homomorphism $f : G \to H$, define $Z(f;\mathbf{V})$ to be the restriction of $f$ to $Z(G;\mathbf{V})$. So if $g \in Z(G;\mathbf{V})$, we must show that $f(g) \in Z(H;\mathbf{V})$, i.e. we must show that for a group $K \in \mathbf{V}$ with $H \leq K$ we have $f(g) \in Z(K)$. Hence, given $k \in K$, we must show that $f(g)^{-1}kf(g) = k$. Towards that end, define $h : \langle x \rangle \to K$ to be the unique group homomorphism determined by mapping $x \mapsto k$. Let $i : H \to K$ be the inclusion map. Since $G * \langle x \rangle$ is a $\mathbf{V}$-coproduct, there exists a unique group homomorphism $F : G * \langle x \rangle \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "f"'] & G * \langle x \rangle \arrow[dd, dotted, "F"] & \langle x \rangle \arrow[l] \arrow[ddl, "h"] \\
H \arrow[dr, "i"'] & & \\
& K &
\end{tikzcd}$$

Now $g \in Z(G;\mathbf{V})$ implies that $g \in (G * \langle x \rangle)$, so that

$$\begin{aligned}
g^{-1}xg =& x, && \text{which implies} \\
F(g^{-1}xg) =& F(x), && \text{which implies} \\
F(g^{-1})F(x)F(g) =& F(x), && \text{which implies} \\
f(g^{-1})h(x)f(g) =& h(x), && \text{which implies} \\
f(g)^{-1}kf(g) =& k.
\end{aligned}$$

Thus $f(g) \in Z(K)$, and hence $f(g) \in Z(H;\mathbf{V})$. It is now easy to check that $Z(f;\mathbf{V}) : Z(G;\mathbf{V}) \to Z(H;\mathbf{V})$ is a group homomorphism and that $Z(\ ;\mathbf{V})$ is a functor. □

**Corollary.** *$Z(G;\mathbf{V})$ is fully invariant in $G$.*

Proof: Suppose $f : G \to G$ is a group endomorphism. By functorality, $Z(f;\mathbf{V})$ is a group homomorphism from $Z(G;\mathbf{V})$ to $Z(G;\mathbf{V})$. But $Z(f;\mathbf{V}) = f\,|_{Z(G;\mathbf{V})}$, so that $f$ maps $Z(G;\mathbf{V})$ to $Z(G;\mathbf{V})$. □

Anticipating the next theorem, we recall the definition of a verbal subgroup: a subgroup $H$ of a group $G$ is <u>verbal</u> if there exists a set $W$ of words such that $H = \langle w(g_1, \dots) : g_i \in G, w \in W \rangle$ [Ne, p. 5]. In the event that $\mathbf{V} = \mathsf{HSP}\{G\}$, Propositions 1 and 2 are special cases of

**Theorem 5.** *If the center $Z(G)$ of a group $G$ is verbal, then $Z(G; \mathsf{HSP}\{G\}) = Z(G)$. Thus, by (5), $U(G; \mathsf{HSP}\{G\}) \cong \operatorname{Mlt} G$.*

Proof: Since $Z(G)$ is a verbal subgroup, there exists a set $W$ of words such that $Z(G) = \langle w(g_1, \dots) : g_i \in G, w \in W \rangle$. Thus, for every $w \in W$,

$$[y, w(x_1, \dots)] = 1 \tag{6}$$

is an identity in $G$. By Birkhoff's Theorem (6) is an identity in every group $H$ in $\mathsf{HSP}\{G\}$, in particular in those $H$ for which $G \leq H$. So, given $g \in Z(G)$, since $g = w_g(g_1, \dots)$ for some $g_i \in G, w_g \in W$, and since $[y, w_g(x_1, \dots)] = 1$ is an identity in $H$, we know that $[y, g] = [y, w_g(g_1, \dots)] = 1$ for every $y \in H$. Thus, $g \in Z(H)$, i.e. $g \in Z(G; \mathsf{HSP}\{G\})$. Hence, $Z(G) \leq Z(G; \mathsf{HSP}\{G\})$ and we have $Z(G) = Z(G; \mathsf{HSP}\{G\})$, as desired. □

Many familiar groups have verbal centers. For instance abelian groups, simple groups, free groups, symmetric groups, and dihedral groups all have verbal centers. Such groups constitute a fairly large class of groups, and in light of Cayley's theorem and the fact that every group is the homomorphic image of a free group, one might be tempted to think that perhaps $U(G; \mathsf{HSP}\{G\}) \cong \operatorname{Mlt} G$ for every group $G$. Before dispelling this notion, we recall the definition of Hopfian: a group $G$ is said to be <u>Hopfian</u> if it is not isomorphic to a proper quotient of itself [Rb, p. 159].

**Theorem 6.** *If $G$ is a group such that:*

(a) $1 < Z(G) < G$;
(b) $\mathsf{HSP}\{G\} = \mathbf{Gp}$; *and*
(c) $G \times G$ *is Hopfian,*

*then* $\operatorname{Mlt} G \ncong U(G; \mathsf{HSP}\{G\})$.

Proof: Here we use a fact proved in [Sm, p.35]. Namely, $U(G; \mathbf{Gp}) \cong G \times G$. So suppose on the contrary that $U(G; \mathsf{HSP}\{G\}) \cong \operatorname{Mlt} G$. Then

$$\begin{aligned} G \times G &\cong U(G; \mathbf{Gp}) \\ &= U(G; \mathsf{HSP}\{G\}) && \text{[by (b)]} \\ &\cong \operatorname{Mlt} G && \text{[by assumption]} \\ &\cong G \times G / \widehat{Z} && \text{by (1).} \end{aligned}$$

This contradicts the Hopfian property of $G \times G$. Therefore, $U(G; \mathsf{HSP}\{G\}) \ncong \operatorname{Mlt} G$. □

To see that there are groups which satisfy the hypotheses of Theorem 6, consider the following

**Example.** Let $G = \langle x, y, z : [x, z] = [y, z] = 1 \rangle$; i.e. $G$ is the direct product of the free group $\langle x, y \rangle$ on two generators with the free (abelian) group $\langle z \rangle$ on one generator. We note that:

(a) $1 < Z(G) < G$ (since $Z(G) = \langle z \rangle$).
(b) $\mathsf{HSP}\{G\} = \mathbf{Gp}$ (since $\langle x, y \rangle$ is clearly a homomorphic image of $G$, and $\mathsf{HSP}\{\langle x, y \rangle\} = \mathbf{Gp}$ [MKS, p. 413]). And
(c) $G \times G$ is Hopfian (since $G$ is residually finite [MKS, pp. 116, 152] and finitely generated, so too is $G \times G$; and thus $G \times G$ is also Hopfian [MKS, p. 415]).

Applying Theorem 6 yields $U(G; \mathsf{HSP}\{G\}) \ncong \mathrm{Mlt}\, G$.

Clearly, groups satisfying the hypotheses of Theorem 6 belong to a restricted class. For instance, such groups must be infinite. The following theorem provides finite groups for which the combinatorial multiplication group is not universal.

**Theorem 7.** *If $G$ is a group such that $Z(G)$ is not fully invariant, then $Z(G; \mathbf{V}) < Z(G)$. Suppose further that for normal subgroups $N_1, N_2$ of $G$, the proper containment $N_1 < N_2$ implies that $G \times G/N_1 \ncong G \times G/N_2$. Then $U(G; \mathbf{V}) \ncong \mathrm{Mlt}\, G$.*

Proof: By the corollary to Theorem 4, $Z(G; \mathbf{V})$ is fully invariant in $G$. Since we are assuming that $Z(G)$ is not fully invariant, and since $Z(G; \mathbf{V}) \leq Z(G)$, we have that $Z(G; \mathbf{V}) < Z(G)$ as desired. The final statement follows from the first with $N_1 = Z(G; \mathbf{V})$ and $N_2 = Z(G)$. □

**Example.** The group $G = A_4 \times Z_2$ (the direct product of the alternating group of order 12 with the cyclic group of order two) has center that is not fully invariant [Rb, p. 30]. Being finite, it also satisfies the further hypothesis of the theorem. Thus, $U(G; \mathsf{HSP}\{G\}) \ncong \mathrm{Mlt}\, G$.

**Corollary.** *If $G$ is a group with center that is cyclic of prime order, but not fully invariant, and if $\mathbf{V}$ is any variety of groups containing $G$, then $Z(G; \mathbf{V}) = 1$. Thus, by (2) and Theorem 3, $U(G; \mathbf{V}) \cong G \times G$.*

**Example.** Let $G = \langle a, b, c : a^2 = b^2 = c^2 = 1, [a, c] = [b, c] = 1 \rangle$. Then $G$ is a group with simple, non-fully invariant center $Z(G) = Z_2$ (the cyclic group of order two). Hence $U(G; \mathsf{HSP}\{G\}) \cong G \times G \ncong \mathrm{Mlt}\, G$.

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(*Received* March 8, 1991)