On the variety Csub (D)

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Abstract. The variety of lattices generated by lattices of all convex sublattices of distributive lattices is investigated.

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0. Introduction.

Let L be a lattice. Denote by $\mathrm{Csub}(L)$ the lattice of all convex sublattices of L (including the empty set $\emptyset$). For a variety V of lattices, let $\mathrm{Csub}(V)$ denote the variety of lattices generated by $\{\mathrm{Csub}(L); L \in V\}$. In [4], it is shown that for any proper variety V of lattices, the variety $\mathrm{Csub}(V)$ is proper and that there are uncountably many varieties $\mathrm{Csub}(V)$.

The aim of this paper is to obtain some information about the least nontrivial such variety, i.e. about $\mathrm{Csub}(D)$, where D denotes the variety of all distributive lattices. We shall show that this variety is locally finite. The meet $\mathrm{Csub}(D)$ with the variety of all modular lattices will be described.

1. Preliminaries.

Any interval of a lattice L is a convex sublattice of L. Denote by $\mathrm{Int}(L)$ the lattice of all intervals of L (including $\emptyset$). Clearly, $\mathrm{Int}(L)$ is a sublattice of $\mathrm{Csub}(L)$. The one-element sublattices of a lattice L are just atoms of both $\mathrm{Int}(L)$ and $\mathrm{Csub}(L)$. If $I = [a, b]$ and $J = [c, d]$ are intervals of a lattice L, then we have in the lattice $\mathrm{Csub}(L)$

$$\begin{aligned} &I \vee J = [a \wedge c, b \vee d] \quad \text{and} \\ &I \wedge J = I \cap J = [a \vee c, b \wedge d] \quad \text{or } \emptyset \quad \text{if} \quad a \vee c \nleq b \wedge d. \end{aligned}$$

One can show (by induction) that, for any lattice term p in k variables and any $A_1, \dots, A_k \in \mathrm{Csub}(L)$ the following holds:

$$p(A_1, \dots, A_k) = \bigcup \{p(I_1, \dots, I_k); \quad I_j \subseteq A_j, I_j \in \mathrm{Int}(L)\}.$$

Thus, for any variety V of lattices, $\mathrm{Int}(L) \in V$ iff $\mathrm{Csub}(L) \in V$. Especially, the variety $\mathrm{Csub}(V)$ is generated by $\{\mathrm{Int}(L); L \in V\}$ (see [4]).

Let L be a lattice and A be a sublattice of the lattice $\mathrm{Int}(L)$. If A has the least element that is not $\emptyset$, then the meet of any pair of elements from A is a non-empty interval of L and, clearly, the mapping h of A into $L^* \times L$, where L^* denotes the dual lattice of L, defined by

$$h([a,b]) = (a,b),$$

is an embedding of A into $L^* \times L$.

Lemma 1. *Let V be a self-dual variety V of lattices and $L \in V$ be a lattice. Then any dual ideal of* $\mathrm{Int}(L)$ *generated by an atom of* $\mathrm{Int}(L)$ *belongs to V.*

Proof: Any dual ideal of $\mathrm{Int}(L)$ generated by an atom of $\mathrm{Int}(L)$ is a sublattice of $L^* \times L \in V$. □

2. Locally finite varieties.

In this section, let V denote a locally finite (any finitely generated lattice in V is finite) self-dual variety of lattices.

Theorem 1. *The variety* $\mathrm{Csub}(V)$ *is locally finite.*

Proof: Let $d(n)$ denote the cardinality of the V-free lattice with n generators. Let $A \in V$ and let C be a sublattice of $\mathrm{Int}(A)$ generated by n elements. Then there exist atoms $a_1, \dots, a_k$ of the lattice $\mathrm{Int}(A)$, $k \le n$, such that $C \subseteq \{\emptyset\} \cup [a_1) \cup \dots \cup [a_k)$. By Lemma 1, $[a_i) \in V$ and the cardinality of $C \cap [a_i)$ is at most $d(n)$. Thus the cardinality of C is at most $s(n) = 1 + n \cdot d(n)$. Since the variety $\mathrm{Csub}(V)$ is generated by $\{\mathrm{Int}(A); A \in V\}$ and for any $A \in V$ a sublattice of $\mathrm{Int}(A)$ with n generators has at most $s(n)$ elements, the variety $\mathrm{Csub}(V)$ is locally finite (see [3]). □

Lemma 2. *Let $L \in V$ be a lattice and let A be a finite sublattice of the lattice* $\mathrm{Int}(L)$. *Then A is a sublattice of* $\mathrm{Int}(K)$ *for some finite sublattice K of L.*

Proof: Denote $M_1 = \{x \in L; [x,y] \in A \text{ for some } y \in L\}$ and $M_2 = \{x \in L; [y,x] \in A \text{ for some } y \in L\}$. The sets M_1 and M_2 are finite, the sublattice K of L generated by $M_1 \cup M_2$ is finite and, clearly, A is a sublattice of $\mathrm{Int}(K)$. □

For a class K of lattices, let $H(K), S(K)$, and $P(K)$ denote the class of all homomorphic images, sublattices, and direct products of members of K, respectively. For a class K, the variety generated by K is equal to $HSP(K)$.

Theorem 2. *Let $A \in V$ be a finite lattice. Then $A \in HSP(\mathrm{Int}(B))$ for some finite lattice $B \in V$. If A is subdirectly irreducible, then $A \in HS(\mathrm{Int}(B))$ for some finite lattice $B \in V$.*

Proof: Since $A \in HSP(\{\mathrm{Int}(L); L \in V\})$, there exist lattices $L_i \in V$, $i \in I$, a sublattice C of the product of $\mathrm{Int}(L_i)$, $i \in I$, and a homomorphism f of C onto A. We can assume that C is finitely generated and so, by Theorem 1, C is finite. Thus we may suppose that I is finite. Let π_i denote the i-th projection of the product of $\mathrm{Int}(L_j)$, $j \in I$, onto $\mathrm{Int}(L_i)$. For any $i \in I$, $\pi_i(C)$ is a finite sublattice of $\mathrm{Int}(L_i)$ and, by Lemma 2, $\pi_i(C)$ is a sublattice of $\mathrm{Int}(B_i)$ for some finite sublattice B_i of L_i. We get that the lattice A belongs to $HSP(\{\mathrm{Int}(B_i); i \in I\})$. It is easy to show that for any pair of lattices A, B, $A \subseteq B$ implies $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$; thus $\mathrm{Int}(B_i)$, $i \in I$ are sublattices of $\mathrm{Int}(B)$, where B is the product of all B_i, $i \in I$; hence $A \in HSP(B)$. If A is subdirectly irreducible, then, since congruence lattices of lattices are distributive, $A \in HS(\mathrm{Int}(B))$ (see [1]). □

Corollary 1. *Let $A \in \mathrm{Csub}(V)$ be a finite subdirectly irreducible lattice. Then any dual ideal of A generated by an atom of A belongs to the variety V.*

PROOF: By Theorem 2, $A \in HS(\mathrm{Int}(B))$ for some finite lattice $B \in V$. Thus for any atom $a \in A$, the dual ideal $[a)$ of A generated by a is a homomorphic image of a sublattice of a dual ideal $[d)$ of $\mathrm{Int}(A), d \neq \emptyset$. By Lemma 1, $[d) \in V$ and so $[a) \in V$, too. □

3. The variety Csub(D).

Let D denote the class of all distributive lattices. The class D is a self-dual locally finite variety. Any finite distributive lattice is a sublattice of a finite Boolean algebra. Now we can reformulate the results of Section 2 as follows.

Theorem 3. *The following assertions hold:*

1. *The variety* Csub(D) *is locally finite.*
2. *Let $A \in \mathrm{Csub}(D)$ be a finite subdirectly irreducible lattice. Then*

(i) *$A \in HS(\mathrm{Int}(B))$ for some finite Boolean algebra B;*
(ii) *for any atom $a \in A$, the dual ideal $[a)$ is a distributive lattice.*

Since any locally finite variety is generated by its finite members, we can immediately obtain

Proposition 1. Csub$(D) = HSP(\{\mathrm{Int}(B_n); n = 2, 3, \dots\})$, *where B_n denotes the Boolean algebra with n atoms.*

Let us remark that, for any $n \geq 2$, the lattice $\mathrm{Int}(B_n)$ is simple. Indeed, if α is a nontrivial congruence relation on $\mathrm{Int}(B_n)$, then there exist intervals I, J of B_n such that $I \subseteq J, I \neq J$ and $I\alpha J$. Let c be an element from $J \setminus I$. Then $([c, c] \cap I)\alpha([c, c] \cap J)$, i.e. $\emptyset\alpha[c, c]$. Let c' be the complement of c. We can easily see that $[c', c']\alpha[0, 1]$ and that $([x, x] \cap [c', c'])\alpha([x, x] \cap [0, 1])$ for any $x \in B_n$. If $x \neq c'$, we get $\emptyset\alpha[x, x]$. If $c \notin \{0, 1\}$, then we have $\emptyset\alpha[0, 0], \emptyset\alpha[1, 1]$ and so $\emptyset\alpha[0, 1]$. Now assume that $c \in \{0, 1\}$. Let $b \in B_n \setminus \{0, 1\}$. Then $\emptyset\alpha[b, b]$ and $\emptyset\alpha[b', b']$; hence $\emptyset\alpha[0, 1]$.

An interesting problem is to describe the variety $\mathrm{Csub}(D) \cap M$, where M denotes the variety of all modular lattices. We shall show that this variety contains all finite lattices M_n having n atoms and $n + 2$ elements. Since the lattice $M_{3,3}$ pictured in Fig. 1 belongs to any variety of modular lattices that is not a subvariety of the variety $HSP(\{M_n; n = 1, 2, \dots\})$ (see [2]) and, by Theorem 3, $M_{3,3}$ does not belong to Csub(D), we can get the following result.

Theorem 4. $\mathrm{Csub}(D) \cap M = HSP(\{M_n; n = 1, 2, \dots\})$.

To prove Theorem 4, it suffices to show that any lattice M_n is a sublattice of a lattice $\mathrm{Int}(B)$ for some finite Boolean algebra B.

Lemma 3. *For any natural number $n \geq 2$, there exist subsets $A_i, B_i, i = 1, 2, \ldots, n$ of $S = \{1, 2, \ldots, \frac{n}{2}(n+1)\}$ such that the following conditions hold:*

(1) *if $i \neq j$, then $A_i \cap A_j = \emptyset$ and $B_i \cup B_j = S$;*

(2) *$A_i \not\subseteq B_j$ iff $(i,j) = (n,1)$ or $(i,j) \neq (1,n)$ and $i < j$.*

Proof: By induction on n. Let $n = 2$. Put $A_1 = \{1\}$, $A_2 = \{2\}$, $B_1 = \{1,3\}$, $B_2 = \{1,2\}$. Now suppose that $k \geq 2$ and $A'_1, A'_2, \ldots, A'_k$, $B'_1, \ldots, B'_k$ are subsets of $T = \{1, 2, \ldots, \frac{k}{2}(k+1)\}$ satisfying the conditions (1) and (2). Denote $s = \frac{k}{2}(k+1)$ and $A_i = A'_i \cup \{s+i\}$ for $i = 1, 2, \ldots, k$ and $A_{k+1} = \{s+k+1\}$. Put $B_1 = T \setminus \{s+k+1\}$ and for all i, $2 \leq i \leq k-1$, $B_i = B'_i \cup \{s+1, \ldots, s+k+1\}$, $B_k = B'_k \cup \{s+2, \ldots, s+k+1\}$, and finally $B_{k+1} = \{1, 2, \ldots, s+1\} \cup \{s+k+1\}$. One can easily verify that the sets A_i, B_i are subsets of $\{1, 2, \ldots, \frac{k+1}{2}(k+2)\}$ satisfying the required conditions (1) and (2). □

Proposition 2. *For any natural number $n \geq 2$, the lattice M_n is a sublattice of* $\mathrm{Int}(B)$ *for some finite Boolean algebra B.*

Proof: Denote by B the Boolean algebra of all subsets of the set $S = \{1, 2, \ldots, \frac{n}{2}(n+1)\}$. Let A_i, B_i $(i = 1, \ldots, n)$ be subsets of S satisfying the conditions (1) and (2) of Lemma 3. Put $I_i = [A_i, B_i], i = 1, \ldots, n$. Clearly, $I_i \in \mathrm{Int}(B)$ and for any pair $i, j, i \neq j, I_i \vee I_j = [A_i \wedge A_j, B_i \vee B_j] = [\emptyset, S]$. Since for any pair $i, j, i \neq j, A_i \not\subseteq B_j$ or $B_i \not\subseteq A_j$, we have $A_i \vee A_j \not\subseteq B_i \wedge B_j$; thus $I_i \wedge I_j = \emptyset$. We have showed that the intervals $I_1, \ldots, I_n$ together with $\emptyset$ and $[\emptyset, S]$ form a sublattice of $\mathrm{Int}(B)$ isomorphic to M_n. □

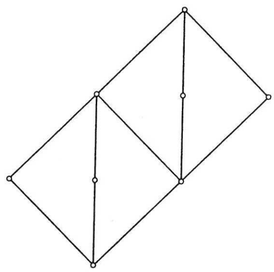

Fig. 1: $M_{3,3}$

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