$\mathcal{P}$-approximable compact spaces

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Abstract. For every topological property $\mathcal{P}$, we define the class of $\mathcal{P}$-approximable spaces which consists of spaces X having a countable closed cover γ such that the "section" $X(x,\gamma) = \bigcap\{F \in \gamma : x \in F\}$ has the property $\mathcal{P}$ for each $x \in X$. It is shown that every $\mathcal{P}$-approximable compact space has $\mathcal{P}$, if $\mathcal{P}$ is one of the following properties: countable tightness, $\aleph_0$-scatteredness with respect to character, C-closedness, sequentiality (the last holds under MA or $2^{\aleph_0} < 2^{\aleph_1}$). Metrizable-approximable spaces are studied: every compact space in this class has a dense, Čech-complete, paracompact subspace; moreover, if X is linearly ordered, then X contains a dense metrizable subspace.

Keywords: $\mathcal{P}$-approximable space, Lindelöf Σ-space, compact, metrizable, C-closed, sequential, linearly ordered

Classification: 54D20, 54D30, 54E35, 54F05

1. Introduction.

It is shown by Talagrand [26] and Arhangelskii [7] that the space $C_p(X)$ of continuous real-valued functions with pointwise convergence topology is a Lindelöf Σ-space for every Eberlein compact space X. Later, Gul'ko [13] proved that, if X is compact and $C_p(X)$ is a Lindelöf Σ-space, then X is a Corson space, i.e. X is embeddable into a Σ-product of reals. Soon after the following result was established in [24]: if X is compact and $C_p(X)$ is a Lindelöf Σ-space, then there exists a countable closed cover γ of X such that the "section" $X(x,\gamma) = \bigcap\{F \in \gamma : x \in F\}$ is an Eberlein compact space for each $x \in X$. Moreover, in this case, X contains a dense metrizable subspace [17].

A.V. Arhangelskii raised a problem of investigation of those compact spaces which can be covered by a countable family of closed subsets, so that all sections have some property $\mathcal{P}$. Compact spaces satisfying the above condition are called $\mathcal{P}$-approximable. Thus, relations between the properties of sections $X(x,\gamma)$ and those of compact space X are to be found.

The paper is devoted to the consideration of some aspects of this problem. It is shown in Section 1 that $\mathcal{P}$-approximable compact space has the property $\mathcal{P}$, if $\mathcal{P}$ is one of the following properties: countable tightness, $\aleph_0$-scatteredness or C-closedness, and sequentiality (the latter result requires MA or $2^{\aleph_0} < 2^{\aleph_1}$). If all sections of a compact space X are singletons, then X is metrizable (Assertion 2.1). However, a compact space with two-point sections need not be a Fréchet–Urysohn

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space. A counterexample is the Mrówka–Franklin compact space [12], which, in addition, is not monolithic.

The behaviour of $\mathcal{P}$-approximability under countable products and passing to a continuous image is considered in Assertions 2.19–2.22. In particular, the class of metrizable-approximable compact spaces is closed under these operations (Assertion 2.21 and Corollary 2.23).

In Section 3, we consider compact spaces close to being metrizable-approximable. Assertion 3.2 claims that a compact, approximable by first-countable sections space X contains a dense, Čech-complete, paracompact, first-countable subspace. Moreover, if X has countable cellularity, then X contains a dense metrizable subspace (Corollary 3.3) (the result holds under MA+$\neg$CH).

One of our main results, Theorem 3.4, states that every linearly ordered, metrizable-approximable (even separable-approximable) compact space contains a dense metrizable subspace. However, it is unknown whether the condition of linear orderability is necessary in Theorem 3.4.

For every space X without isolated points, let $n(X)$ be the minimal number of nowhere dense sets in X covering X. By the Baire category theorem, $n(X) > \aleph_0$ for any compact space X. Theorem 3.7 claims that every metrizable-approximable compact space either contains an open, non-void, separable subset, or satisfies the equality $n(X) = \aleph_1$.

In the end, some examples of metrizable-approximable compact spaces are given. For instance, such are the two arrows space and the unit square with the lexicographic ordering.

2. When does $\mathcal{P}$-approximability of X imply that X has $\mathcal{P}$?

We begin with quite an easy result.

Assertion 2.1. *A compact space X is metrizable iff X is approximable by one-point sections.*

Proof: If X is metrizable , then X has a countable base $\mathcal{B}$. Let γ be the family consisting of the closures of all elements of $\mathcal{B}$. Then $|X(x,\gamma)| = 1$ for each $x \in X$. Conversely, suppose that γ is a countable closed cover of X such that all sections $X(x,\gamma)$ are singletons. Denote by λ the family of all finite intersections of elements of γ. Then λ is a countable network in X, and hence, by the theorem of Arhangelskii, X has countable base, i.e. X is metrizable. $\square$

A sequence $\xi = \{x_\alpha : \alpha < \omega_1\}$ of points of a space X is said to be <u>free</u> (see [4]), if $\mathrm{cl}(\xi_\beta) \cap \mathrm{cl}(\xi^\beta) = \emptyset$ for every $\beta < \omega_1$, where $\xi_\beta = \{x_\alpha : \alpha < \beta\}$ and $\xi^\beta = \{x_\alpha : \beta \le \alpha < \omega_1\}$. We call a subset A of X ξ-<u>bounded</u>, if $A \subseteq \mathrm{cl}(\xi^\beta)$ for some $\beta < \omega_1$. Denote by $\mathrm{cl}_\omega\, \xi$ the set $\bigcup\{\mathrm{cl}(\xi_\beta) : \beta < \omega_1\}$. If $\xi = \{x_\alpha : \alpha < \omega_1\}$ and $\eta = \{y_\alpha : \alpha < \omega_1\}$ are free sequences in X, then the expression $\eta \prec \xi$ means that there exists a mapping $\varphi : \omega_1 \to \omega_1$ such that $\eta_\alpha \subseteq \mathrm{cl}(\xi_{\varphi(\alpha)})$ and $\eta^\alpha \subseteq \mathrm{cl}(\xi^{\varphi(\alpha)})$ for each $\alpha < \omega_1$. One easily verifies that $\alpha \le \varphi(\alpha) < \varphi(\beta)$ whenever $\alpha < \beta < \omega_1$ (see Assertion 1.1 of [27]). These notions and notations enable us to lighten the proof of the following result.

Assertion 2.2. *Let X be a regular countably compact space and γ be a countable closed cover of X. If the sections $X(x,\gamma), x \in X$, contain no free sequences of length ω_1, then X has the same property, and the tightness of X is countable.*

Proof: Assume the contrary. Then there exists a free sequence ξ_0 of length ω_1 in X. Let $\gamma = \{F_n : n \in \mathbb{N}^+\}$. Assume that a free sequence $\xi(n-1)$ of length ω_1 in X is defined for some positive integer n. Consider two cases.

(a) The set $\mathrm{cl}_\omega(\xi(n-1)) \cap F_n$ is $\xi(n-1)$-bounded. Then there exists a $\beta < \omega_1$ such that $F_n \cap \mathrm{cl}_\omega(\xi(n-1)^\beta) = \emptyset$. Enumerate the set $\xi(n-1)^\beta$ in an order-preserving way, say $\{y_\alpha : \alpha < \omega_1\} = \xi(n)$. Clearly, $\xi(n)$ is a free sequence in X, and $\xi(n) \prec \xi(n-1), \mathrm{cl}_\omega(\xi(n)) \cap F_n = \emptyset$.

(b) The set $\Phi_n = F_n \cap \mathrm{cl}_\omega(\xi(n-1))$ is not $\xi(n-1)$-bounded. One easily defines a free sequence $\xi(n)$ of length ω_1 in X such that $\xi(n) \subseteq \Phi_n$ and $\xi(n) \prec \xi(n-1)$.

Let the free sequences $\xi(n), n \in \mathbb{N}$, be defined. Consider the set P of integers $n \in \mathbb{N}^+$ such that the case (a) occurs at the n-th step of our construction, and put $Q = \mathbb{N}^+ \setminus P$. By Lemma 1.4 of [27], there exists a free sequence η of length ω_1 in X such that $\eta \prec \xi(n)$ for every $n \in \mathbb{N}^+$. It follows from the construction that $\mathrm{cl}\,\eta \subseteq \bigcap\{F_n : n \in Q\}$ and $\mathrm{cl}_\omega(\eta) \cap F_n = \emptyset$ for each $n \in P$. Consequently, for any point $x \in \mathrm{cl}_\omega \eta$, we have

$$\eta \subseteq \mathrm{cl}_\omega \eta \subseteq \bigcap\{F_n : n \in Q\} = X(x,\gamma),$$

which contradicts the choice of the family γ. Thus, X contains no free sequences of length ω_1. This fact and Proposition 1.10 of [5] together imply that the tightness of X is countable. □

Since the tightness of a compact space is equal to the supremum of lengths of free sequences lying in this space (see [4]), Assertion 2.2 implies the following

Theorem 2.3. *If a compact space X is approximable by sections of countable tightness, then X has countable tightness.*

In the sequel, we use some specific notions.

Definition 2.4. A space X is said to be $\aleph_0$-scattered with respect to character, if every non-empty closed subset F of X has countable character at some point $x \in F$. In the same way, one defines the notions of τ-scattered with respect to character spaces, and τ-scattered with respect to π-character spaces.

Assertion 2.5. *Let a regular countably compact space X be approximable by sections which are $\aleph_0$-scattered with respect to character. Then X is $\aleph_0$-scattered with respect to character.*

Proof: It is sufficient to prove that X has countable character at some point. Let $\{F_n : n \in \mathbb{N}^+\}$ be an enumeration of some closed cover of X defining sections with the required property. Put $V_0 = X$. Assume that we have already defined a non-empty open subset V_k of $X, k \in \mathbb{N}$. If $V_k \setminus F_{k+1} \neq \emptyset$, then there exists a non-empty open set V_{k+1} such that $\operatorname{cl}_X V_{k+1} \subseteq V_k \setminus F_{k+1}$; otherwise $V_k \subseteq F_{k+1}$, and we choose a non-empty open set V_{k+1} so that $\operatorname{cl}_X V_{k+1} \subseteq V_k$.

Since X is countably compact, the set $\Phi = \bigcap\{V_k : k \in \mathbb{N}\} = \bigcap\{\operatorname{cl}_X V_k : k \in \mathbb{N}\}$ is not empty. Pick a point $x \in \Phi$. From the choice of sets V_n it follows that $\Phi \subseteq X(x, \gamma)$. Since the section $X(x, \gamma)$ is $\aleph_0$-scattered with respect to character, there is a point $x \in \Phi$ such that $\chi(x, \Phi) \leq \aleph_0$. Being a G_δ-set in X, Φ has countable character in X. By the result of [2], we have $\chi(x, X) \leq \chi(x, \Phi) \cdot \chi(\Phi, X) \leq \aleph_0$. □

Recall that a space X is said to be C-closed (see [15]), if every countably compact subspace of X is closed in X. It is clear that every sequential space and every space of countable pseudo-character is C-closed [15].

Theorem 2.6. *If a compact space X is approximable by C-closed sections, then X is C-closed.*

Proof: Let a subspace Y of X be countably compact. Note the following obvious fact:

(∗) If $\{F_i : i \in \mathbb{N}\}$ is a sequence of closed subsets of Y with $F_{i+1} \subseteq F_i$ for each $i \in \mathbb{N}, x \in X \setminus Y$, and $x \in \operatorname{cl} F_i$ for all i, then $x \in \operatorname{cl} \bigcap_{i=0}^{\infty} F_i$.

Choose a countable closed cover γ of X which defines C-closed sections, and put $\mu = \{\bigcap \lambda : \lambda \subseteq \gamma, |\lambda| < \aleph_0\}$. Consider the family $\mu^* = \{\operatorname{cl}_X(K \cap Y) : K \in \mu\}$. We claim that for every $x \in X \setminus Y$ and $y \in Y$, there exists $L \in \mu^*$ such that $y \in L$ and $x \notin L$. Assume the contrary, and let the assertion be wrong for some points $x \in X \setminus Y$ and $y \in Y$. Put $F = X(y, \gamma)$. Then the countably compact set $F \cap Y$ is not closed in F because (∗) implies that x is a cluster point for it. This contradicts the assumption that F is C-closed, and hence the family μ^* has the property formulated above.

Since all elements of μ^* are compact, Y is a Lindelöf Σ-space (see [20] or [8]). Being Lindelöf and countably compact, Y is compact, and hence is closed in X. □

Now we need the following result.

Assertion 2.7 (See [28, p. 162]). *Suppose that a compact space X is C-closed and $\aleph_0$-scattered with respect to character. Then X is sequential.*

Note that Assertion 2.7 remains valid for countably compact regular spaces. Assertions 2.5 and 2.7 together imply one of the main results of the paper.

Theorem 2.8. *Let a compact space X be approximable by sections which are C-closed and $\aleph_0$-scattered with respect to character. Then X is sequential.*

The following result is proved in [15].

Assertion 2.9 [$2^{\aleph_0} < 2^{\aleph_1}$ or MA]. *Every compact, C-closed space is sequential.*

Since any compact, sequential space is C-closed, Theorem 2.6 and the above assertion imply

Corollary 2.10 [$2^{\aleph_0} < 2^{\aleph_1}$ or MA]**.** *If a compact space X is approximable by sequential sections, then X is sequential.*

It is well-known that every Corson compact space is Fréchet–Urysohn and $\aleph_0$-scattered with respect to character (see [8]). Hence, Theorem 2.6 implies the following

Corollary 2.11. *If a compact space X is approximable by Corson sections, then X is sequential.*

The following result can be proved in a manner analogous to that of Assertion 2.5 (use the inequality $\pi\chi(x,X) \leq \pi\chi(x,F) \cdot \chi(F,X)$, which holds for every closed subset F of a regular space X and a point $x \in F$, see Lemma 1 of [9]).

Assertion 2.12. *If a regular countably compact space X is approximable by sections which are $\aleph_0$-scattered with respect to π-character, then X is $\aleph_0$-scattered with respect to π-character.*

By Theorem 1 of [22], a compact space X is $\aleph_0$-scattered with respect to π-character iff there is no continuous mapping of X onto the cube I^{ω_1}. Therefore, Assertion 2.12 implies the following

Corollary 2.13. *Suppose there exists a continuous mapping of a compact space X onto the cube I^{ω_1}. Then, for every countable closed cover γ of X, one can find a section $X(x,\gamma)$ with the same property.*

A space Y is said to be α-extended (see [6]) if there exists a linear ordering $<$ of Y such that the set $Y_y = \{z \in Y : z \leq y\}$ is closed in Y for each point $y \in Y$. By Theorem 7 of [6], every regular, countably compact, α-extended space Y has countable π-character at some point $x \in Y$. Since any subspace of an α-extended space is α-extended, a space under the requirements of [6, Theorem 7] is $\aleph_0$-scattered with respect to π-character. From this fact and Assertion 2.12, we deduce the following

Corollary 2.14. *If a regular, countably compact space X is approximable by α-extended sections, then X is $\aleph_0$-scattered with respect to π-character.*

Assertion 2.15. *Let τ be an infinite cardinal, and $\mathcal{K}$ be a closed cover of a space X. If $\psi(K) \leq \tau$ for each $K \in \mathcal{K}$ and $|\mathcal{K}| \leq \tau$, then $\psi(X) \leq \tau$.*

PROOF: Pick a point $x \in X$, and define the families $\mathcal{K}_1 = \{K \in \mathcal{K} : x \in K\}$, $\mathcal{K}_2 = \mathcal{K} \setminus \mathcal{K}_1$. For every $k \in \mathcal{K}_1$, choose a family λ_K of open subsets of X such that $\{x\} = K \cap (\bigcap \lambda_K)$ and $|\lambda_K| \leq \tau$. Put $\lambda = \bigcup\{\lambda_k : K \in \mathcal{K}_1\}, G_1 = \bigcap \lambda$ and $G_2 = \bigcap\{X \setminus K : K \in \mathcal{K}_2\}$. Then $\{x\} = G_1 \cap G_2$, i.e. $\psi(x,X) \leq \tau$ (note that $|\lambda| \leq \tau$ and $|\mathcal{K}_2| \leq \tau$). □

Assertion 2.16. *Let $\tau \geq 2^{\aleph_0}$ be a cardinal, and a compact space X be approximable by sections of character at most τ. Then $\chi(X) \leq \tau$.*

PROOF: Put $\mathcal{K} = \{X(x,\gamma) : x \in X\}$, where a closed countable cover γ of X gives the required approximation. Since $|\mathcal{K}| \leq 2^{\aleph_0}$, Assertion 2.15 implies that $\psi(X) \leq \tau$. It remains to note that $\chi(X) = \psi(X)$ because X is compact. □

It is easy to see that the character in Assertion 2.16 can be replaced by cellularity, density, weight, hereditary density, etc. But the following problem remains open.

Problem 2.17. Can one replace the character by π-character (or π-weight) in Assertion 2.16?

A space X is said to have countable o-tightness, briefly $\mathrm{ot}(X) \leq \aleph_0$, if for every family λ of open subsets of X and for each point $x \in X$ with $x \in \mathrm{cl}(\bigcup \lambda)$ there exists a countable subfamily $\mu \subseteq \lambda$ such that $x \in \mathrm{cl}(\bigcup \mu)$ (see Definition 1 of [29]).

Problem 2.18. Suppose that a compact space X is approximable by sections with countable cellularity. Does then X have countable o-tightness?

Let us consider categorical properties of classes of $\mathcal{P}$-approximable compact spaces. We begin with the following result.

Assertion 2.19. *If a topological property $\mathcal{P}$ is countably productive in the class of compact spaces, then the class of $\mathcal{P}$-approximable compact spaces is countably productive.*

Lemma 2.20. *The following properties are countably productive in the class of compact spaces:*

(a) *metrizability;*
(b) *countable tightness;*
(c) *sequentiality;*
(d) *being $\aleph_0$-scattered with respect to character;*
(e) *being $\aleph_0$-scattered with respect to π-character;*
(f) *C-closedness.*

Proof: (a) is obvious. The assertion (b) follows from the result of Malyhin [18], and (c) follows from the result of Noble [21].
(d) Consider a product $\prod = \prod_{n=0}^{\infty} X_n$, where every space X_n is compact and $\aleph_0$-scattered with respect to character. For each $n \in \mathbb{N}$, denote by π_n the projection of $\prod$ onto $\prod_n = \prod_{i=0}^{n} X_i$. It is easy to see that every compact space $\prod_n$ is $\aleph_0$-scattered with respect to character. Let F be a non-empty, closed subset of $\prod$. An easy induction enables us to define a sequence $\{x_n : n \in \mathbb{N}\}$ of points such that $x_n \in \pi_n(F)$, $\chi(x_n, \pi_n(F)) \leq \aleph_0$, and $\pi^n_m x_n = x_m$ for any integers m, n with $m < n$, where π^n_m is the projection of $\prod_n$ onto $\prod_m$. Since $\prod$ is compact and F is closed in $\prod$, there exists a point $x \in F$ such that $\pi_n(x) = x_n$ for each $n \in \mathbb{N}$. Clearly, $\chi(x, F) \leq \aleph_0$.
(e) By Theorem 5 of [22], a product $\prod = \prod_{n=0}^{\infty} X_n$ with compact factors can be mapped continuously onto the cube I^{ω_1} iff some of the factors X_n has this property. In addition, a compact space admits a continuous mapping onto I^{ω_1} iff this space is not $\aleph_0$-scattered with respect to π-character [22]. Thus, (e) is proved.
(f) follows from [14]. □

Assertion 2.19 and Lemma 2.20 immediately imply the following

Assertion 2.21. *The following classes of spaces are countably productive:*

(a) *metrizable-approximable compact spaces;*
(b) *compact spaces, approximable by sections of countable tightness;*
(c) *sequentially-approximable compact spaces;*
(d) *compact spaces approximable by sections which are $\aleph_0$-scattered with respect to character;*
(e) *compact spaces approximable by sections which are $\aleph_0$-scattered with respect to π-character;*
(f) *compact spaces, approximable by C-closed sections.*

What kind of approximation is preserved by continuous mappings? The following result gives one of possible answers.

Assertion 2.22. *Suppose that a property $\mathcal{P}$ is preserved by perfect mappings and inherited by closed subspaces. Then the class of $\mathcal{P}$-approximable compact spaces is preserved by continuous mappings.*

Proof: Let γ be a countable closed cover of a compact space X such that all sections $X(x,\gamma)$ have the property $\mathcal{P}$. Without loss of generality one can assume that γ is closed under finite intersections. Consider a continuous mapping f of X onto Y and put $\mu = \{f(P) : P \in \gamma\}$. We claim that all sections $Y(y,\mu)$ of Y have the property $\mathcal{P}$. To this end, it suffices to show that $Y(y,\mu) \subseteq f(X(x,\gamma))$ whenever $f(x) = y$. But this inclusion follows from the definition of μ and compactness of X. □

Corollary 2.23. *All classes of spaces listed in Assertion 2.21 are preserved by continuous mappings.*

3. Compact spaces close to being metrizable-approximable.

The main open problem under consideration in this section is the following one.

Problem 3.1. Does every metrizable-approximable compact space contain a dense metrizable subspace?

Here we prove several positive results concerning this problem.

Assertion 3.2. *Every first-countable-approximable compact space X contains a dense, Čech-complete, paracompact, first-countable subspace.*

Proof: By assumption, there exists a countable closed cover $\gamma = \{F_n : n \in \mathbb{N}^+\}$ of X such that every section $X(x,\gamma)$ is first-countable. Put $F_0 = X$ and $\mu_0 = \{X\}$. Assume that we have already defined a family μ_n of disjoint open subsets of X for some $n \in \mathbb{N}$, so that $\bigcup \mu_n$ is dense in X and for each $V \in \mu_n$ either $V \subseteq F_n$, or $V \cap F_n = \emptyset$. Put $V_n = \bigcup \mu_n, W_{n,0} = V_n \setminus F_{n+1}$ and $W_{n,1} = \mathrm{Int}_X(V_n \cap F_{n+1})$. Denote by $\mathcal{P}_{n+1}$ the family $\{W_{n,i} \cap U : U \in \mu_n,\ i = 0,1\}$ of disjoint open subsets of X. Clearly, $\bigcup \mathcal{P}_{n+1}$ is dense in X, because $W_{n,0} \cup W_{n,1}$ is dense in V_n. Let μ_{n+1} be a maximal disjoint family of open sets in X, the closures of which are contained in some elements of $\mathcal{P}_{n+1}$. Clearly, the set $V_{n+1} = \bigcup \mu_{n+1}$ is dense in V_n, and hence in X.

Put $Y = \bigcap_{n=0}^{\infty} V_n$. Then Y is a dense G_δ-subset of X, and hence Y is Čech-complete. Let us verify that Y is first-countable. Let y be a point of Y. For every $n \in \mathbb{N}$, there exists $O_n \in \mu_n$ with $y \in O_n$. It follows from the construction that $F = \bigcap_{n=0}^{\infty} O_n \subseteq X(y, \gamma)$; therefore, $\chi(y, F) \leq \aleph_0$. Being a closed G_δ-set in X, F has countable base in X. Consequently, $\chi(y, Y) \leq \chi(y, X) \leq \chi(y, F) \cdot \chi(F, X) \leq \aleph_0$.

A standard argument (see Assertion D of [3]) shows that there exists a perfect mapping of Y onto some metrizable space. This implies the paracompactness of Y.

□

Corollary 3.3 [MA + CH]**.** *Let a compact space X be approximable by first-countable sections. If X has countable cellularity, then X contains a dense metrizable subspace.*

PROOF: By Theorem 2.3, the tightness of X is countable. Since $c(X)t(X) \leq \aleph_0$, Corollary 3.3 of [19] implies that X contains a countable dense subset D. From Assertion 3.2, it follows that there exists a dense subset S of X such that $\chi(x, X) \leq \aleph_0$ for each $x \in S$. Since $t(X) \leq \aleph_0$, for every point $y \in D$, there is a countable set $T(y) \subseteq S$ with $y \in \operatorname{cl} T(y)$. Then the countable set $T = \bigcup\{T(y) : y \in D\}$ is dense in X and is contained in S. Clearly, T is as required. □

The following theorem is the main result of this section.

Theorem 3.4. *If a linearly ordered compact space X is approximable by separable sections, then X contains a dense metrizable subspace.*

PROOF: By the assumption of the theorem, there exists a countable closed cover $\gamma = \{F_n; n \in \mathbb{N}\}$ of X such that all sections $X(x, \gamma)$ are separable. Note that every separable subset of linearly ordered space is hereditarily separable (see [30]), and hence has countable tightness. Consequently, Theorem 2.3 implies that $t(X) \leq \aleph_0$. In turn, this implies that every increasing (or decreasing) sequence in linearly ordered compact space X is countable, i.e. X is first-countable.

Denote by $\mathcal{K}$ the family of all open sets in X which have a σ-disjoint π-base. Note that if a first-countable space Z has a σ-disjoint π-base, then Z contains a dense metrizable subspace (see [32]). Consequently, if a regular space Z contains a dense metrizable subspace, then Z has a σ-disjoint π-base.

It is easy to verify that the set $G = \bigcup \mathcal{K}$ has a σ-disjoint π-base. Indeed, let $\mathcal{Z}$ be a maximal disjoint subfamily of $\mathcal{K}$. Then $G' = \bigcup \mathcal{Z}$ is dense in G. For every $L \in \mathcal{Z}$, choose a σ-disjoint π-base $\mathcal{B}_L$ of L, and put $\mathcal{B} = \bigcup\{\mathcal{B}_L : L \in \mathcal{Z}\}$. Clearly, $\mathcal{B}$ is a σ-disjoint π-base of G, and hence G contains a dense metrizable subspace.

If G is dense in X, we are done. So let us assume that the set $O = X \setminus \operatorname{cl} G$ is not empty and deduce a contradiction. The definition of O implies that there are no open, non-empty subsets in O which have σ-disjoint π-base. In particular, all open, non-empty subsets of O are non-separable. Let $<$ be a linear ordering generating the topology of X, and suppose that a closed interval $Y = [y_1, y_2]$ is in O, $|Y| \geq \aleph_0$. Without loss of generality one can assume that the end points y_1 and y_2 of Y are not isolated in Y. An argument similar to that in the proof of Assertion 3.2 can be applied to define a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of families of open sets in X satisfying the following conditions:

(1) λ_n is a disjoint family and $\bigcup \lambda_n$ is dense in Y;
(2) a closure of each element of λ_{n+1} lies in some element of λ_n;
(3) for each $V \in \lambda_n$, either $V \subseteq F_n$, or $V \cap F_n = \emptyset$.

In addition, all elements of every family λ_n can be assumed convex with respect to the ordering $<$. A sequence $\xi = \{V_n : n \in \mathbb{N}\}$ is called a thread if $V_n \in \lambda_n$ and $V_{n+1} \subseteq V_n$ for each $n \in \mathbb{N}$. If ξ is a thread, then $\operatorname{cl} V_{n+1} \subseteq V_n$, and hence $\bigcap \xi \neq \emptyset$. The condition (3) implies that $T = \bigcap \xi \subseteq X(x,\gamma)$ for every point $x \in T$. Note that the set T is closed and convex in X. We claim that $|T| \leq 2$. Indeed, T is separable, being a subspace of some separable, linearly ordered section $X(x,\gamma)$, $x \in T$. Furthermore, X and T are first-countable; hence T has a σ-disjoint π-base. Since T is convex in X and $T \cap G = \emptyset$, the interior of T in X must be empty. Consequently, $|T| \leq 2$.

Now we proceed to the construction of some "new" sequence $\{\mu_n : n \in \mathbb{N}\}$ of families of disjoint open sets in Y. Let $\mathcal{P}$ be the family of all non-empty open sets W in X with $W \subseteq Y$, satisfying the property $U \setminus W \neq \emptyset$ for each $U \in \lambda$, where $\lambda = \bigcup_{n=0}^{\infty} \lambda_n$. Since the family λ is σ-disjoint and no non-empty open subset W of X with $W \subseteq Y$ has a σ-disjoint π-base, $\mathcal{P}$ is a π-base for Y.

Put $\mu_0 = \{Y\}$. Suppose that for some $n \in \mathbb{N}$, a family μ_n of disjoint, open sets in Y such that μ_n refines λ_n and $\bigcup \mu_n$ is dense in Y, is defined. Denote by Θ_n the family of all non-empty sets of the form $U \cap V$, where $U \in \mu_n$ and $V \in \lambda_{n+1}$. Clearly, the family Θ_n is disjoint, $\bigcup \Theta_n$ is dense in Y, and Θ_n refines both μ_n and λ_{n+1}. Since $\mathcal{P}$ is a π-base for Y, for every $W \in \Theta_n$ there exists a disjoint subfamily $\mathcal{P}_W \subseteq \mathcal{P}$ such that $\bigcup\{\operatorname{cl} O : O \in \mathcal{P}_W\} \subseteq W \subseteq \operatorname{cl}(\bigcup \mathcal{P}_W)$. Put $\mu_{n+1} = \bigcup\{\mathcal{P}_W : W \in \Theta_n\}$.

We claim that the family $\mu = \bigcup_{n=0}^{\infty} \mu_n$ is a π-base for Y. To prove this, we begin with the verification of the fact that an intersection of every thread from μ is a singleton. Indeed, let $W_n \in \mu_n$ and $W_{n+1} \subseteq W_n$ for each $n \in \mathbb{N}$. There exists a thread $\xi = \{V_n : n \in \mathbb{N}\}$ from λ such that $W_n \subseteq V_n \in \lambda_n$ for every $n \in \mathbb{N}$. If $|\bigcap \xi| = 1$, we are done. So assume that $|\bigcap \xi| = 2$. It follows from the construction that $V_n \setminus W_1 \neq \emptyset$ for all n. Therefore, the decreasing sequence $\{\operatorname{cl} V_n \setminus W_1 : n \in \mathbb{N}\}$ of closed subsets of Y has non-empty intersection. This implies that

$$\bigcap \xi \setminus W_1 = \bigcap \{\operatorname{cl} V_n \setminus W_1 : n \in \mathbb{N}\} \neq \emptyset.$$

Thus, $T = \bigcap_{n=0}^{\infty} W_n$ is a non-empty proper subset of $\bigcap \xi$, and hence $|T| = 1$.

Let $T = \{y\}, y \in Y$. Clearly, the thread $\{W_n : n \in \mathbb{N}\}$ is a base for Y at the point y. Put $O_n = \bigcup \mu_n, n \in \mathbb{N}$, and $S = \bigcap_{n=0}^{\infty} O_n$. Then S is dense in Y, and the restriction of the family μ to the set S constitutes a σ-discrete base for Y. Consequently, S is metrizable and μ is a σ-disjoint π-base for Y. Note that the interior of Y in X contains the interval (y_1, y_2) and hence is not empty. This contradicts the fact that $Y \cap G = \emptyset$. $\square$

It should be noted that there exists a first-countable, linearly ordered, compact space X^* with no dense metrizable subspaces (see [31]). Moreover, every first category subset of X^* is nowhere dense in X^*. Theorem 3.4 implies that X^* is not approximable by metrizable (even separable) sections.

The problem below is a weakening of Problem 3.1.

Problem 3.5. Does the equality $c(X) = d(X)$ hold for every metrizable-approximable compact space X?

For a given space X without isolated points, let $n(X)$ be the Novák number of X, i.e. the minimal cardinality of families ξ of nowhere dense sets in X with $X = \bigcup \xi$. The Baire category theorem implies that $n(X) > \aleph_0$ for every compact space X. Moreover, if a compact space X has countable cellularity, then the Martin's Axiom (MA) implies that $n(X) \geq 2^{\aleph_0}$ (see [16]). It is also known that if a metrizable space M contains no non-empty separable open sets, then $n(X) \leq \aleph_1$ (see [25]). Some delicate results on decomposition of compact spaces into sums of nowhere dense sets are obtained in [10], [23]. Here we give an estimate for $n(X)$ for a metrizable-approximable compact space X.

We need the following auxiliary result.

Lemma 3.6. *Let Y be a regular space with a σ-disjoint π-base, and suppose that $c(O) > \aleph_0$ for each non-empty open subset O of Y. Then $n(Y) \leq \aleph_1$.*

Proof: By assumption, there exists a σ-disjoint π-base $\mathcal{B} = \bigcup_{n=0}^{\infty} \gamma_n$ for Y. One easily defines a σ-disjoint π-base $\mathcal{P} = \bigcup_{n=0}^{\infty} \mu_n$ for Y such that $\mathcal{P} \subseteq \mathcal{B}$, $\bigcup \mu_n$ is dense in Y and a closure of every element of μ_{n+1} is contained in some element of $\mu_n, n \in \mathbb{N}$. Put $S = \bigcap_{n=0}^{\infty} V_n$, where $V_n = \bigcup \mu_n$ for every $n \in \mathbb{N}$. Clearly, $Y \setminus S$ is the union of countably many nowhere dense subsets of Y. If S is nowhere dense in Y, we are done. Suppose that the set $O = \operatorname{Int} \operatorname{cl} S$ is not empty. We can assume without loss of generality that $O = Y$. Let $\xi = \{U_n : n \in \mathbb{N}\}$ be a thread of $\mathcal{P}$, i.e. $U_n \in \mu_n$ and $U_{n+1} \subseteq U_n$ for each n. Then $F_\xi = \bigcap \xi$ is a closed (possibly, empty) subset of Y. Denote by f the mapping of S onto a set M which assigns to every non-empty set F_ξ a point, say, ξ. Endow M with a metrizable topology, a base of which is constituted by the sets of the form $f(U), U \in \mathcal{P}$. The mapping f is continuous and irreducible, for $\mathcal{P}$ is a π-base for Y and S is dense in Y. Therefore $f^{-1}(N)$ is nowhere dense in S whenever N is nowhere dense in M. The assumptions of the lemma and the irreducibility of f together imply that $c(W) > \aleph_0$ for every non-empty, open subset W of M. Consequently, $n(M) \leq \aleph_1$ (see [25]), and hence $n(Y) \leq \aleph_1$. □

Theorem 3.7. *If X is a metrizable-approximable compact space, then either X contains a non-empty open separable subset, or $n(X) \leq \aleph_1$.*

Proof: Suppose that X does not contain non-empty open separable subsets. Denote by $\mathcal{K}$ the family of all non-empty open subsets of X which have a σ-disjoint π-base. We claim that $c(V) > \aleph_0$ for each $V \in \mathcal{K}$. Indeed, if $v \in \mathcal{K}$ and $c(V) \leq \aleph_0$, then V has a countable π-base, and hence is separable.

Clearly, the set $O = \bigcup \mathcal{K}$ has a σ-disjoint π-base (see the first part of the proof of Theorem 3.4). Since $c(W) > \aleph_0$ for every non-empty open subset $W \subseteq O$, Lemma 3.6 implies that $n(O) \le \aleph_1$. If O is dense in X, then the proof is complete. So assume the contrary. It is sufficient to show that the set $G = X \setminus \operatorname{cl} O$ satisfies the inequality $n(G) \le \aleph_1$. Note that there are no non-empty open subsets of G with σ-disjoint π-base. Let γ be a countable closed cover of X giving a metrizable approximation for X. Apply an argument of the proof of Assertion 3.2 to define a sequence $\{\mu_n : n \in \mathbb{N}\}$ of families of open sets in X lying in G and satisfying the following conditions for every $n \in \mathbb{N}$:

(i) μ_n is a disjoint family, and $\bigcup \mu_n$ is dense in G;
(ii) for every $U \in \mu_{n+1}$, the closure $\operatorname{cl}_X U$ is contained in some element of μ_n;
(iii) if $\xi = \{V_n : n \in \mathbb{N}\}$ is a thread of $\mu = \bigcup_{n=0}^{\infty} \mu_n$ (i.e. $V_n \in \mu_n$ and $V_{n+1} \subseteq V_n$ for each n), then $\bigcap \xi \subseteq X(x, \gamma)$ for every point $x \in \bigcap \xi$.

Put $\mathcal{Z}_0 = \{\mu_n : n \in \mathbb{N}\}$. Let $\alpha < \omega_1$ and suppose that for every $\beta < \alpha$, we have defined a sequence $\mathcal{Z}_\beta = \{\mu_n^\beta : n \in \mathbb{N}\}$ of families of disjoint open sets in G so that $\mathcal{Z}_\beta$ satisfies the conditions (i)–(iii). Consider the family $\mathcal{K}_\alpha = \{\bigcup_{\beta<\alpha}\}\mathcal{Z}^\beta$. Since $|\mathcal{K}_\alpha| \le \aleph_0$, we can enumerate $\mathcal{K}_\alpha$, say, $\mathcal{K}_\alpha = \{\lambda_n : n \in \mathbb{N}\}$. Obviously, the family $\bigcup \mathcal{K}_\alpha$ of open sets in G is σ-disjoint. Denote by $\mathcal{R}_\alpha$ the family of all non-empty open subsets $V \subseteq G$ such that $U \setminus V \neq \emptyset$ for every $U \in \bigcup \mathcal{K}_\alpha$. From the definition of G, it follows that $\mathcal{R}_\alpha$ is a π-base for G. Define a family $\mathcal{Z}_\alpha$ as follows. Let μ_0^α be a maximal disjoint subfamily of $\mathcal{R}_\alpha$. Suppose that a disjoint subfamily $\mu_n^\alpha \subseteq \mathcal{R}_\alpha$ is defined so that $\bigcup \mu_n^\alpha$ is dense in G and μ_n^α refines λ_n. Denote by μ_{n+1}^α a maximal disjoint subfamily of $\mathcal{R}_\alpha$ which refines λ_{n+1} and μ_n^α. Obviously, $\bigcup \mu_{n+1}^\alpha$ is dense in G. Put $\mathcal{Z}_\alpha = \{\mu_n^\alpha : n \in \mathbb{N}\}$.

For every $\alpha < \omega_1$ and $n \in \mathbb{N}$, the set $V_n^\alpha = \bigcup \mu_n^\alpha$ is open and dense in G. Therefore, the subset of G complementary to $S_\alpha = \bigcap_{n=0}^{\infty} V_n^\alpha$ is meager in G and is so in X. To complete the proof, it suffices to show that the set $S = \bigcap_{\alpha<\omega_1} S_\alpha$ is empty. Assume the contrary: let $S \neq \emptyset$ and $x \in S$. Then for every $\alpha < \omega_1$, there exists a thread $\xi_\alpha = \{V_n^\alpha : n \in \mathbb{N}\}$ such that $x \in V_{n+1}^\alpha \subseteq V_n^\alpha \in \mu_n^\alpha, n \in \mathbb{N}$. Put $F_\alpha = \bigcap \xi_\alpha$ for every $\alpha < \omega_1$. From the construction, it follows that $F_0 \subseteq X(x, \gamma)$, F_α is closed in X and $F_\alpha \subseteq F_\beta$ whenever $\beta < \alpha < \omega_1$. Thus, $\nu = \{F_\alpha : \alpha < \omega_1\}$ is a decreasing sequence of closed sets in the compact metrizable space $X(x, \gamma)$; hence this sequence stabilizes at some step $\alpha < \omega_1$. However, the definition of $\mathcal{R}_{\alpha+1}$ implies that $V_n^\alpha \setminus V_0^{\alpha+1} \neq \emptyset$ for each $n \in \mathbb{N}$, because $V_0^{\alpha+1} \in \mu_0^{\alpha+1} \subseteq \mathcal{R}_{\alpha+1}$ and $V_n^\alpha \in \mu_n^\alpha \subseteq \bigcup \mathcal{K}_\alpha$. Consequently, the set $F_\alpha \setminus V_0^{\alpha+1} = \bigcap\{\operatorname{cl}_X V_n \setminus V_0^{\alpha+1} : n \in \mathbb{N}\}$ is not empty. This means that $F_{\alpha+1} \subseteq V_0^{\alpha+1} \cap F_\alpha$ is a proper subset of F_α, which contradicts the stabilization of ν. □

The examples below show the difference between metrizable and metrizable-approximable compact spaces. We begin with a (far from complete) list of sections' properties which cannot be extended over all the space.

Example 3.8. Every metrizable compact space is approximable by one-point sections (Assertion 2.1). Hence, compact spaces approximable by scattered, left-separated, connected, or zero-dimensional sections need not be scattered, left-separated, etc.

Example 3.9. Let X be the Mrówka–Franklin compact space (see [12]), that is, a compactification of a countable infinite discrete set N obtained by adding some uncountable discrete (in itself) set $\mathcal{A}$ which is identified with a maximal almost disjoint family of infinite subsets of N, and of a point x^* "at infinity" which compactifies the locally compact space $N \cup \mathcal{A}$. Neighborhoods of a point $A \in \mathcal{A}$ in X have the form $A \setminus T$, where T is a finite set in N. Let us verify that X is approximable by sections of cardinality at most 2. Since $|\mathcal{A}| \leq 2^{\aleph_0}$, there exists a countable family Θ of subsets of $\mathcal{A}$ separating the points of $\mathcal{A}$. In different words, for every distinct elements A, B of $\mathcal{A}$, one can find $U \in \Theta$ with $A \in U \not\ni B$. Put $\gamma_1 = \{\{n\} : n \in \mathbb{N}\}$, $\gamma_2 = \{U \cup \{x^*\} : U \in \Theta\}$ and $\gamma = \gamma_1 \cup \gamma_2 \cup \{x^*\}$. Obviously, γ is a countable closed cover of X. If either $x \in N$ or $x = x^*$, then $X(x, \gamma) = \{x\}$. If $x \in \mathcal{A}$, then the definition of the family Θ implies that $X(x, \gamma) = \{x, x^*\}$. Thus, $|X(x, \gamma)| \leq 2$ for each $x \in X$.

One can show that every open subset of $N \cup \mathcal{A}$ is pseudocompact. This readily implies that no sequence in N converges to x^*. Since N is dense in X, the space X is not Fréchet (see Exercise 3.6.1 (a) of [11]). Moreover, a locally compact space $N \cup \mathcal{A}$ is not metalindelöf. Finally, X is not monolithic, for it is separable and contains an uncountable discrete subset. Thus, metrizable-approximable compact spaces need not be Fréchet, or hereditarily metalindelöf, or monolithic.

Example 3.10. Let X be the double arrow space (see [1], or Exercise 3.10. C of [11]). Then X is approximable by sections of cardinality at most 2. Indeed, denote by I the closed unit interval and identify X with a subspace of the space $I \times \{0, 1\}$ endowed with the lexicographic ordering $<$. Let S_0 and S_1 be rational points of $I \times \{0\}$ and of $I \times \{1\}$ respectively, and γ be the family of all closed intervals $[s_0, s_1]$ with $s_0 \in S_0$ and $s_1 \in S_1$. An easy verification shows that $|X(x, \gamma)| \leq 2$ for each $x \in X$. Consequently, perfectly normal, hereditarily separable compact spaces approximable by two-point sections need not be metrizable.

An analogous argument shows that the lexicographically ordered unit square I^2 is metrizable-approximable, whereas $\chi(I^2) = \aleph_0$ and $c(I^2) = 2^{\aleph_0}$.

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