On a class of locally Butler groups

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Abstract. A torsionfree abelian group B is called a Butler group if $Bext(B,T)=0$ for any torsion group T. It has been shown in [DHR] that under CH any countable pure subgroup of a Butler group of cardinality not exceeding $\aleph_\omega$ is again Butler. The purpose of this note is to show that this property has any Butler group which can be expressed as a smooth union $\cup_{\alpha<\mu}B_\alpha$ of pure subgroups B_α having countable typesets.

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All groups in this paper are abelian. If p is a prime and x an element of a torsionfree group G then $h_p^G(x)$ is the p-height of x in G and $t^G(x)=t(x)$ is the type of x in G. The typeset $t(G)$ of G is the set of types of all non-zero elements of G. The corank of a pure subgroup H of G is the rank of G/H. If Π is a set of primes and T is a torsion group then we say that T is Π-primary if $T_p=0$ for all $p\notin\Pi$.

If S is a subset of a torsionfree group G , then $\langle S\rangle_*^G$ denotes the pure subgroup generated by S. A subgroup H of G is said to be a generalized regular subgroup of G if G/H is torsion and for each rank one pure subgroup J of G, $(J/J\cap H)_p=0$ for almost all primes p. A torsionfree group G is said to be locally completely decomposable if, for each prime p, the localization $G_p=Z_p\otimes G$ is completely decomposable. For the unexplained terminology and notations see [F1].

A torsionfree group B is said to be a Butler group if $Bext(B,T)=0$ for all torsion groups T, where $Bext$ is the subfunctor of Ext consisting of all balanced-exact extensions. It is known [BS] that this definition coincides with the familiar one if B has finite rank, i.e., a pure subgroup of a completely decomposable group, or, equivalently [B], a torsionfree homomorphic image of a completely decomposable group of finite rank.

Following [FV] we shall call a torsionfree group locally Butler if any its pure subgroup of finite rank is Butler. Dugas [D] proved that any Butler group, the cardinality of which does not exceed $\aleph_1$ is locally Butler. In this paper we are going to generalize this result by showing that the same property has any Butler group B expressible as a smooth union $\cup_{\alpha<\mu}B_\alpha$ of pure subgroups B_α with countable typesets. Doing this we also give for this class of groups an affirmative answer concerning the problems (1) and (2) formulated in [A].

Lemma 1. *Let X be a subgroup of a torsionfree group G with G/X torsion and $J\le G$ be of rank one. If H is a subgroup of G such that $(X+J)\cap H/X\cap H$*

is Π*-primary for some set of primes* Π*, then there is a subgroup* K *of* J *such that* J/K *is* Π*-primary and* $(X+K)\cap H = X\cap H$*.*

Proof: Decompose $J/X\cap J$ into $L/X\cap J \oplus K/X\cap J$, where $L/X\cap J$ is the Π-primary part of the torsion group $J/X\cap J$. Now consider the homomorphism $\psi : (X+J)\cap H \to J/X\cap J$ given for $h = x+j$ by the formula $\psi h = j + X\cap J$. Obviously, ψ is well-defined and it naturally induces the monomorphism $\phi : (X+J)\cap H/X\cap H \to J/X\cap J$. By hypothesis, $Im\psi = Im\phi \leq L/X\cap J$ and so the results follow easily from the inclusion $\psi((X+K)\cap H) \leq K/X\cap J$. $\square$

Lemma 2. *Let* H *be a corank one pure subgroup of a torsionfree group* G *with countable typeset. If* K *is a generalized regular subgroup of* H*, then there is a generalized regular subgroup* L *of* G *such that* $L\cap H = K$*.*

Proof: Obviously, there is an ordinal $\lambda \leq \omega$ such that $\{ t^G(g) \mid g \in G \setminus H \} = \{ t_i \mid i < \lambda \}$. For each $i < \lambda$ take a rank one pure subgroup J_i of G such that $t(J_i) = t_i$ and $J_i \cap H = 0$. Using the induction, we are going to show that for each $i < \lambda$ there is a generalized regular subgroup K_i of J_i such that $L_i = K + K_1 + ... + K_i$ meets H in K.

For $n = 1$ we have $(K\oplus J_1)\cap H = K\oplus (J_1\cap H) = K$ and so we can set $K_1 = J_1$. Assume that for some $1 < n < \lambda$ the subgroup $L_{n-1} = K + K_1 + ... + K_{n-1}$ with $L_{n-1}\cap H = K$ has been defined. Denoting $X_n = K_1 + ... + K_{n-1} + J_n$ we have $(L_{n-1} + J_n)\cap H/L_{n-1}\cap H = (K + X_n)\cap H/K = K + (X_n\cap H)/K \simeq (X_n\cap H)/X_n\cap K$.

Now $X_n/X_n\cap H \simeq (X_n+H)/H$ is torsionfree, H being pure in G, and consequently $X_n\cap H$ is a finite rank Butler group. Moreover, for $0 \neq x \in X_n\cap K$, the natural embedding induces the monomorphism $\langle x\rangle_*^{X_n\cap H}/\langle x\rangle_*^{X_n\cap K} \to \langle x\rangle_*^H/\langle x\rangle_*^K$ and so [B1] gives that the factor-group $X_n\cap H/X_n\cap K$ has a finite number of non-zero primary components, only. A simple application of Lemma 1 gives the existence of $K_n \leq J_n$ with the desired properties.

Setting $L = K + \Sigma_{i<\lambda}K_i = \cup_{i<\lambda}L_i$ we have $L\cap H = (\cup_{i<\lambda}L_i)\cap H = \cup_{i<\lambda}(L_i\cap H) = K$ and it remains to show that L is generalized regular in G.

Take $0 \neq g \in L$ arbitrarily. For $g \in H$, it is $g \in L\cap H = K$ and consequently the factor-group $\langle g\rangle_*^G/\langle g\rangle_*^L = \langle g\rangle_*^H/\langle g\rangle_*^K$ has a finite number of non-zero primary components, only.

So, let $g \notin H$. There is $n < \lambda$ such that $t^G(g) = t_n = t(J_n)$. Since $r(G/H) = 1$, we have $mg = x + h$ for some $0 \neq m \in Z, x \in K_n$ and $h \in K$, H/K being torsion. The set $\Pi = \{p \mid h_p^G(mg) > h_p^G(x)\} \cup \{p \mid p|m\} \cup \{p \mid (J_n/K_n)_p \neq 0\} \cup \{p \mid (\langle h\rangle_*^H/\langle h\rangle_*^K)_p \neq 0\}$ of primes is obviously finite and for each prime $p \notin \Pi$ we have $h_p^L(x) = h_p^G(x) \geq h_p^G(mg)$, therefore $h_p^G(mg) \leq h_p^G(h) = h_p^K(h) \leq h_p^L(h)$ and consequently $h_p^L(g) = h_p^L(mg) = h_p^L(x+h) \geq h_p^L(x)\cap h_p^L(h) \geq h_p^G(mg) = h_p^G(g)$ showing that $\langle g\rangle_*^G/\langle g\rangle_*^L$ is Π-primary and finishing therefore the proof. $\square$

Lemma 3. *Let* $G = \cup_{\alpha<\mu}G_\alpha$ *be a smooth union of pure subgroups of a torsionfree group* G *where* μ *is a limit ordinal. If, for each* $\alpha < \mu$, L_α *is a generalized regular subgroup of* G_α *such that* $L_\alpha \leq L_\beta$ *and* $L_\alpha\cap G_0 = L_0$ *whenever* $\alpha \leq \beta < \mu$*, then* $L = \cup_{\alpha<\mu}L_\alpha$ *is a generalized regular subgroup of* G *satisfying* $L\cap G_0 = L_0$*.*

Proof: If $0 \neq g \in L$ is arbitrary, then $g \in L_\alpha$ for some $\alpha < \mu$ and the inclusion $\langle g\rangle_*^{L_\alpha} \leq \langle g\rangle_*^L$ induces the epimorphism $\langle g\rangle_*^{G_\alpha}/\langle g\rangle_*^{L_\alpha} \to \langle g\rangle_*^G/\langle g\rangle_*^L$, from which the assertion follows easily. □

Theorem 4. *Let $G = \cup_{\alpha<\mu} G_\alpha$ be a smooth union of pure subgroups G_α of a torsionfree group G having countable typesets. If K is a generalized regular subgroup of G_0 then there is a generalized regular subgroup L of G such that $L \cap G_0 = K$.*

Proof: By transfinite induction based on Lemmas 2 and 3. □

Corollary 5. *Let H be a pure subgroup of a torsionfree group G with countable typeset. If K is a generalized regular subgroup of H then there exists a generalized regular subgroup L of G such that $L \cap H = K$.*

Corollary 6 [D]. *Let H be a countable pure subgroup of a torsionfree group G of cardinality $\aleph_1$. If K is a generalized regular subgroup of H then there is a generalized regular subgroup L of G such that $L \cap H = K$.*

Now we are prepared to prove the main result giving a partial solution of the problems (1) and (2) stated in [A].

Theorem 7. *Let a torsionfree group G be a smooth union $G = \cup_{\alpha<\mu} G_\alpha$ of pure subgroups G_α with countable typesets. The following conditions are equivalent:*

(i) *G is locally completely decomposable and if L is a generalized regular subgroup of G and H is a pure finite rank subgroup of G, then $(H/H \cap L)_p = 0$ for almost all primes p;*
(ii) *G is locally completely decomposable and locally Butler.*

Proof: Assume (i) and let H be a rank finite pure subgroup of G. There is $\alpha < \mu$ such that $H \leq G_\alpha$ and consequently if K is a generalized regular subgroup of H, Corollary 5 gives the existence of a generalized regular subgroup M of G_α with $M \cap H = K$. A simple application of Theorem 4 leads to the existence of a generalized regular subgroup L of G satisfying $L \cap G_\alpha = M$ and hence $L \cap H = K$. By hypothesis $H/H \cap L = H/K$ has only a finite number of non-zero primary components and since H is locally completely decomposable by [F1, Th. 86.6], it is Butler by [B1]. For the converse see [A]. □

Theorem 8. *Any Butler group G expressible as a smooth union $G = \cup_{\alpha<\mu} G_\alpha$ of pure subgroups G_α with countable typesets is locally Butler.*

Proof: By [A], any Butler group satisfies the condition (i) from Theorem 7. □

Corollary 9. *Any Butler group with countable typeset is locally Butler.*

Corollary 10 [D]. *Any Butler group of cardinality $\aleph_1$ is locally Butler.*

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