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A note on universal minimal dynamical systems

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Abstract. Let $M(G)$ denote the phase space of the universal minimal dynamical system for a group G. Our aim is to show that $M(G)$ is homeomorphic to the absolute of D^{2^ω}, whenever G is a countable Abelian group.

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Let G be a group and X a compact space. We call an **action** of the group G on space X a homomorphism Φ from G into the group of homeomorphisms of X. The pair (X, G) is called a **dynamical system** and the space X a **phase space** of the system (X, G); we will write gx for $\Phi(g)(x)$, $g \in G$, $x \in X$.

A dynamical system (X, G) is called **minimal** if the set $\{gx : g \in G\}$ is dense in X for each $x \in X$. The system (X, G) is minimal iff for each non-empty open set $U \subseteq X$ there are $g_1, \dots, g_n \in G$ such that $g_1U \cup \dots \cup g_nU = X$.

Let (X, G) and (Y, G) be dynamical systems and let $\varphi : X \to Y$ be a continuous map. If $\varphi \circ g = g \circ \varphi$ for any $g \in G$ then φ is called a **homomorphism** of the system (X, G) into the system (Y, G). If in addition φ is a homeomorphism of spaces, then φ is called an **isomorphism** of dynamical systems.

The dynamical system (X, G) is called a **universal minimal dynamical system** for a group G if the following conditions hold:

(a) (X, G) is a minimal dynamical system,
(b) if (Y, G) is a minimal dynamical system then there exists a homomorphism $\varphi : (X, G) \to (Y, G)$.

The well-known results of Ellis [4; 7.13, 7.16] say that for every group G there is a universal minimal dynamical system which is unique up to an isomorphism. The phase space of this system is homeomorphic to a closed subspace of the Čech–Stone compactification of the discrete space G. Let $M(G)$ denote the phase space of the universal minimal dynamical system for a group G.

It was proved by van Douwen [3] that for every infinite Abelian group G $\pi w(M(G)) > |G|$, where $\pi w(X)$ denotes π-weight of X. Balcar and Błaszczyk [1] have shown that if (X, G) is a minimal dynamical system and X is an extremally disconnected space and G is a countable group then X is homeomorphic to the absolute of the Cantor cube $D^{\pi w(X)}$. On the other hand, it is known (cf. van Douwen [3]) that $M(G)$ has to be extremally disconnected. We will show that if G

is an Abelian group then $\pi w(M(G)) = 2^{|G|}$. Therefore $M(G)$ is homeomorphic to the absolute of D^{2^ω}, for every countable Abelian group G.

A continuous map $\varphi : X \to Y$ is **semi-open** if $\operatorname{int} \varphi(U) \neq \emptyset$ for every non-empty open set $U \subseteq X$.

The following lemma is known; see e.g. [6]. We will include its proof for completeness.

Lemma 1. *Homomorphisms of minimal dynamical systems are semi-open and "onto".*

Proof: Let φ be a homomorphism of a minimal system (X, G) into a minimal system (Y, G). If $x \in X$ then $\{g\varphi(x) : g \in G\}$ is dense in Y. Hence $\varphi(X) = \varphi(\operatorname{cl}\{gx : g \in G\}) = \operatorname{cl} \varphi(\{gx : g \in G\}) = \operatorname{cl}\{\varphi(gx) : g \in G\} = \operatorname{cl}\{g\varphi(x) : g \in G\} = Y$.

Let U be a non-empty open subset of X. Let us choose a non-empty open set V so that $\operatorname{cl} V \subseteq U$. Since (X, G) is minimal, then there exist $g_1, \dots, g_n$ from G such that $g_1 V \cup \dots \cup g_n V = X$. Thus

$$\begin{aligned} Y = \varphi(X) = \varphi(g_1 V \cup \dots \cup g_n V) = \varphi(g_1 V) \cup \dots \cup (g_n V) = \\ = g_1 \varphi(V) \cup \dots \cup g_n \varphi(V) \end{aligned}$$

and hence

$$\emptyset \neq \operatorname{int} \varphi(\operatorname{cl} V) \subseteq \operatorname{int} \varphi(U).$$

□

Lemma 2. *If there exists a semi-open map of X onto Y, then $\pi w(Y) \leq \pi w(X)$.*

The proof of the above lemma is clear. □

Let G be an Abelian group and let $\operatorname{Hom}(G, \mathbf{T})$ denote the group of all homomorphisms from G into the circle group $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. It is well known that $\operatorname{Hom}(G, \mathbf{T})$ is point-separating and the power of $\operatorname{Hom}(G, \mathbf{T})$ equals $2^{|G|}$; see [5; 22.17, 24.47].

Let the homomorphism $e : G \to \mathbf{T}^{\operatorname{Hom}(G,\mathbf{T})}$ be defined by the formula:

$$e(g)(h) = h(g), \quad \text{for } \ g \in G, \ h \in \operatorname{Hom}(G, \mathbf{T}).$$

The range $e(G)$ is a subgroup of the compact topological group $\mathbf{T}^{\operatorname{Hom}(G,\mathbf{T})}$, where $\mathbf{T}^{\operatorname{Hom}(G,\mathbf{T})}$ is regarded with the Tichonoff topology. Hence $bG = \operatorname{cl}(e(G))$ is a compact topological group. The group bG is the so-called Bohr compactification of the (discrete) Abelian group G.

It is not hard to see that G acts on bG in the following way:

$$\Phi(g)(x) = e(g) \cdot x; \quad g \in G, \quad x \in bG.$$

Then (bG, G) forms a minimal dynamical system. Indeed, if $x \in bG$ then $f_x : bG \to bG$ defined by $f_x(y) = x \cdot y$, is a homeomorphism. Hence $\{gx : g \in G\} = f_x(e(G))$ is dense in bG.

The following lemma is known; see e.g. [2; 3.6. (ii)].

Lemma 3. *If X is a topological group then $w(X) = \pi w(X)$.*

For any locally compact Abelian group K, let $\hat{K}$ denote the group of all the continuous homomorphisms of K into $\mathbf{T}$, endowed with a compact-open topology. It is known that $w(K) = w(\hat{K})$ and $bK = (\hat{K})_d\hat{}$, where X_d denotes a space X with a discrete topology; see [5; 24.14, 26.12].

Theorem. *If G is an Abelian group then $\pi w(M(G)) = 2^{|G|}$.*

Proof: Since (bG, G) is a minimal dynamical system, then there exists a homomorphism $\varphi : (M(G), G) \to (bG, G)$. Lemmas 1, 2 and 3 imply

$$w(bG) = \pi w(bG) \leq \pi w(M(G)).$$

From the above remarks, we get

$$w(bG) = w((\hat{G})_d\hat{}) = w((\hat{G})_d) = |\hat{G}| = |\operatorname{Hom}(G, \mathbf{T})| = 2^{|G|},$$

because G is a discrete space.

The inequality $\pi w(M(G)) \leq 2^{|G|}$ follows from the fact that $M(G)$ is homeomorphic to a closed subset of βG.

□

The result of [1] leads to the following:

Corollary. *If G is a countable Abelian group then $M(G)$ is homeomorphic to the absolute of D^{2^ω}.*

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