Smoothness for systems of degenerate variational inequalities with natural growth

Martin Fuchs

Abstract. We extend a regularity theorem of Hildebrandt and Widman [3] to certain degenerate systems of variational inequalities and prove Hölder-continuity of solutions which are in some sense stationary.

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0. Introduction.

We consider systems of variational inequalities of the form

$$\int_\Omega A(u)|Du|^{p-2}\, Du \cdot D(v-u)\, dx \geq \int_\Omega f(\cdot, u, Du)\cdot (v-u)\, dx \tag{0.1}$$

for all $v \in \mathbb{K} := H^{1,p}(\Omega, K)$ such that $\operatorname{spt}(u-v) \subset\subset \Omega$, where K is a convex set in $\mathbb{R}^N$ and p denotes some real number in the interval $[2, n]$, n denoting the dimension of the domain Ω. Our main purpose is to prove (partial) regularity for solutions $u \in \mathbb{K}$ of (0.1) in the case that the right-hand side is of natural growth, i.e. we require

$$|\, f(x, y, Q)\,| \leq a \cdot (\,|Q|^p + 1)$$

for some positive constant a. To my knowledge there is only a theorem of Hildebrandt and Widman [3] concerning the quadratic case $p = 2$ which can be summarized as follows:

(0.2) *If* $A \geq \lambda > 0$ *and if* $a < \lambda /\operatorname{diam} K$

is satisfied then any solution u *of* (0.1) *is of class* $C^{0,\alpha}$ *on the whole domain* Ω.

Since these authors make use of the Green's function technique it is rather clear that for general $p > 2$ one has to find completely new arguments. We start with the observation that (0.2) is sufficient to prove a Caccioppoli inequality for u giving $Du \in L^q_{\text{loc}}$ for some $q > p$ and hence partial regularity apart from a closed singular set of vanishing $\mathcal{H}^{n-q}$-measure. Of course the convexity of K is essential in two ways: it is needed to derive Caccioppoli's inequality and to show that local solutions w of $D(\,|Dw|^{p-2}\, Dw) = 0$ for boundary values u are admissible. Unfortunately we did not succeed in proving everywhere regularity by the way giving a complete extension of the above mentioned theorem of Hildebrandt and Widman. Our contribution concerns the following case: suppose that f is of the special form $f(x,y,Q) = \frac{1}{2}\, DA(y)\,|Q|^p$ and that in addition u is a stationary point of the functional $\mathcal{F}(u) := \int_\Omega A(u)\,|Du|^p\,dx$ with respect to reparametrizations of Ω. This enables us to consider blow-up sequences at possible singularities which are shown to converge strongly to a homogeneous (degree zero) tangent map u_0 in the space $H^{1,p}_{\mathrm{loc}}(\Omega)$ and from (0.2) it follows that u_0 must be trivial so that the singular set is empty. Hence our main result can be summarized as follows:

> *Suppose that $u \in \mathbb{K}$ satisfies $\frac{d}{dt/0}\,\mathcal{F}\big(u + t(v-u)\big) \ge 0$ for all $v \in \mathbb{K}$ such that* $\operatorname{spt}(u-v) \subset\subset \Omega$. *Then if* (0.2) *holds and if u is also stationary we have $u \in C^{0,\alpha}(\Omega)$.*

1. Notations and results.

We here specify our assumptions and introduce some notations which will be used throughout the paper. Let $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$, we often write B_r when x_0 is fixed and use the symbol B to denote the open unit ball with center at 0. For a compact convex set K in $\mathbb{R}^N$ and a real number $2 \le p < n$ we introduce the class $\mathbb{K} := \{u \in H^{1,p}(B,\mathbb{R}^N) : u(x) \in K \text{ a.e.}\}$ of all vector-valued Sobolev functions with values in the prescribed set K. Moreover, we are given a smooth function $A : \mathbb{R}^n \to \mathbb{R}$ with the property

$$\lambda \le A(y), \quad y \in K, \tag{1.1}$$

for some positive number λ. For the functions $u \in \mathbb{K}$ and balls $B_r(x_0) \subset B$ we then define the energy

$$\mathcal{F}\big(u, B_r(x_0)\big) := \int_{B_r(x_0)} A(u)\,|Du|^p\,dx.$$

Theorem 1.1. *Suppose $u \in \mathbb{K}$ satisfies*

$$\lim_{t\downarrow 0} t^{-1} \cdot \Big[\mathcal{F}\big(u + t(v-u), B\big) - \mathcal{F}(u,B)\Big] \ge 0 \tag{1.2}$$

for all $v \in \mathbb{K}$ with the property $\operatorname{spt}(u-v) \subset\subset B$. *Then, if the smallness condition*

$$\sup_K |DA| < 2 \cdot \lambda \cdot (\operatorname{diam} K)^{-1} \tag{1.3}$$

holds, we have $u \in C^{0,\alpha}(B')$ for some open subset B' of B such that $\mathcal{H}^{n-p}(B - B') = 0$.

Definition. A function $u \in \mathbb{K}$ is a stationary point of $\mathcal{F}(\cdot, B)$ iff

$$\frac{d}{dt/0}\,\mathcal{F}(u_t, B) = 0\,, \quad u_t(x) := u\big(x + t \cdot X(x)\big), \tag{1.4}$$

holds for all vectorfields $X \in C^1_0(B, \mathbb{R}^n)$.

Theorem 1.2. *Let $u \in \mathbb{K}$ denote a stationary point of $\mathcal{F}(\cdot, B)$ which in addition satisfies* (1.2). *Then $u \in C^{0,\alpha}(B)$ provided the smallness condition* (1.3) *is satisfied.*

Remarks: 1) Theorems 1.1, 1.2 easily extend to functionals of the form

$$u \to \int_B A(u)\,(a_{\alpha\beta}\,D_\alpha u \cdot D_\beta u)^{p/2}\,dx$$

with elliptic coefficients $a_{\alpha\beta} : B \to \mathbb{R}$.

2) We conjecture that (1.2), (1.3) are sufficient to prove everywhere regularity.

3) Under suitable smallness conditions relating λ, diam$\,(K)$ and the growth constant a in

$$|f(x, y, Q)| \le a(\,|Q|^p + 1),$$

a partial regularity result in the spirit of Theorem 1.1 can be deduced for solutions $u \in K$ of the variational inequality

$$\int_B A(u)\,|Du|^{p-2}\,Du \cdot (Dv - Du)\,dx \ge$$
$$\ge \int_B f(\cdot, u, Du) \cdot (v - u)\,dx\,, \quad v \in \mathbb{K}\,,\ \mathrm{spt}(u - v) \subset\subset B\,,$$

but again we are unable to exclude singular points.

2. Proof of the partial regularity Theorem 1.1.

Clearly inequality (1.2) is equivalent to

$$\int_B A(u)\,|Du|^{p-2}\,Du \cdot D(u - v)\,dx \le \int_B \frac{1}{2}\,DA(u) \cdot (v - u)\,|Du|^p\,dx \tag{2.1}$$

for all $v \in \mathbb{K}$ such that $\mathrm{spt}(u-v) \subset\subset B$. Consider a ball $B_{2R}(x_0) \subset B$ and a cut-off function

$$\eta \in C_0^1\big(B_{2R}(x_0),\,[0,1]\big)\,, \quad \eta = 1 \text{ on } B_R(x_0)\,, \quad |D\eta| \le 2 \cdot R^{-1}\,.$$

Then

$$v := u + \eta^p(u_{2R} - u)\,, \quad u_{2R} := \fint_{B_{2R}(x_0)} u\,dx\,,$$

is admissible in (2.1) and a standard calculation using (1.3) implies Caccioppoli's inequality

$$\int_{B_R(x_0)} |Du|^p\,dx \le c_1 \cdot R^{-p} \int_{B_{2R}(x_0)} |u - u_{2R}|^p\,dx \tag{2.2}$$

for some absolute constant c_1 independent of u and the ball $B_R(x_0)$. Quoting [G] we find an exponent $q > p$ such that

$$Du \in L^q_{\mathrm{loc}}\,(B, \mathbb{R}^{nN})$$

and the following reverse Hölder inequality holds

$$\text{(2.3)}\qquad \Big(\fint_{B_R(x_0)} |Du|^q\,dx\Big)^{1/q} \le c_3 \Big(\fint_{B_{2R}(x_0)} |Du|^p\,dx\Big)^{1/p}.$$

Let $w \in H^{1,p}\big(B_R(x_0), \mathbb{R}^N\big)$ denote the unique minimizer of the functional

$$\mathcal{F}_0(v) := A(u_R)\cdot \int_{B_R(x_0)} |Dv|^p\,dx$$

for boundary values $u\,|_{\partial B_R(x_0)}$. Since $u\big(B_R(x_0)\big) \subset K$ and since K is convex, one easily checks (for example by projecting v onto the set K) that v respects the side condition and therefore is admissible in (2.1) provided we integrate over the ball $B_R(x_0)$. As in [1, Lemma 3.3] we then can prove the following comparison inequality

$$\text{(2.4)}\qquad \int_{B_R(x_0)} |Du - Dv|^p\,dx \le$$
$$\le c_4 \cdot \Big[R^{p-n} \int_{B_R(x_0)} |Du|^p\,dx\Big]^{1-p/q} \int_{B_{2R}(x_0)} |Du|^p\,dx\,.$$

Note that the proof of (2.4) combines (2.3) with standard ellipticity estimates. On the other hand we know from [5] that

$$\int_{B_\rho(x_0)} |Dv|^p\,dx \le c_r \Big(\frac{\rho}{R}\Big)^n \int_{B_R(x_0)} |Dv|^p\,dx\,, \quad 0 < \rho \le R\,,$$

which gives on account of (2.4):

Lemma 2.1. *Suppose that $u \in \mathbb{K}$ satisfies* (1.2) *and that the smallness condition* (1.3) *holds. Then there exist constants $\varepsilon, \alpha \in (0,1)$* (*independent of u*) *with the following property: If*

$$\text{(2.5)}\qquad R^{p-n} \int_{B_R(x_0)} |Du|^p\,dx < \varepsilon$$

holds for some ball $B_R(x_0) \subset B$ then $u \in C^{0,\alpha}\big(B_{R/2}(x_0)\big)$ and

$$|u(x) - u(y)| \le c \cdot |x-y|^\alpha\,, \quad x, y \in B_{R/2}(x_0)\,,$$

with $0 < c < \infty$ independent of u. □

This proves Theorem 1.1 and in view of Caccioppoli's inequality (2.2) we see that a point $x_0 \in B$ is a regular point if and only if

$$\text{(2.5)}'\qquad \fint_{B_R(x_0)} |u - u_R|^p\,dx < \varepsilon'$$

holds for some ball $B_R(x_0) \subset B$ and a suitable small constant $\varepsilon' \in (0,1)$.

3. Monotonicity and everywhere regularity.

The following lemma is essentially due to Price [4] (for $p = 2$).

Lemma 3.1. *Let* $u \in \mathbb{K}$ *satisfy* (1.4). *Then we have*

$$0 = \int_B A(u)\,|Du|^{p-2}\left[|Du|^2 \operatorname{div} X - pD_\alpha u \cdot D_\beta u D_\alpha X^\beta\right] dx \tag{3.1}$$

for all vectorfields $X \in C_0^1\,(B, \mathbb{R}^n)$. □

By applying (3.1) to fields of the form

$$X\,(x) = \gamma\,(|x|)\,x$$

for a function $\gamma \in C^1(\mathbb{R})$ such that $(0 < \rho < 1)$

$$\gamma' \leq 0, \quad \gamma = 1 \quad \text{on} \quad (-\infty, \rho/2]\,, \gamma = 0 \quad \text{on} \quad (\rho, \infty)\,,$$

we get

Lemma 3.2 (Monotonicity formula)**.** *Suppose that* $u \in \mathbb{K}$ *satisfies* (1.4). *Then*

$$\begin{aligned} R^{p-n} \int_{B_R} A(u)|Du|^p\,dx \;-\; r^{p-n} \int_{B_r} A(u)|Du|^p\,dx \\ = p \cdot \int_{B_R - B_r} A(u)|Du|^{p-2} \cdot |D_r u|^2 \cdot |x|^{p-n}\,dx \end{aligned}$$

holds for balls $B_r(0) \subset B_R(0) \subset B$.

Remarks:1) $D_r u$ denotes the radial derivative:$D_r u^i(x) := \nabla u^i(x) \cdot \frac{x}{|x|}$.

2) A similar formula is valid for balls with center $x_0 \in B$.

We now come to the proof of Theorem 1.2: Let all the assumptions of Theorem 1.2 hold; it clearly suffices to show

$$\lim_{R \downarrow 0} R^{p-n} \int_{B_R(0)} |Du|^p\,dx = 0\,, \tag{3.2}$$

i.e. $0 \in \operatorname{Reg}(u)$ ($=$ the regular set of u). To this purpose define a sequence $r_k \downarrow 0$ and consider the scaled maps $u_k(z) := u(r_k z)$, $z \in B$, which belong to the class $\mathbb{K}$ and satisfy (2.1) for all $v \in \mathbb{K}$, $\operatorname{spt}(u_k - v) \subset\subset B$. Since

$$\sup_k \|u_k\|_{H^{1,p}(B)} < \infty\,,$$

we may extract a subsequence (again denoted by u_k) such that

$$u_k \to: u_0 \quad \text{in} \quad L^p_{\text{loc}}, u_k \to u_0 \quad \text{weakly in} \quad H^{1,p}_{\text{loc}}$$

and pointwise a.e. The limit u_0 is in the class $\mathbb{K}$ and let us suppose for the moment that we already know

$$u_k \to u_0 \quad \text{strongly in} \quad H^{1,p}_{\text{loc}} . \tag{3.3}$$

We then fix an arbitrary point $\xi \in K$ and a function $\eta \in C^1_0(0,1)$, $0 \le \eta \le 1$, and apply (2.1) with u replaced by u_k and $v(x) := u_k(x) + \eta(|x|)\big(\xi - u_k(x)\big)$. ($v$ is admissible since $\operatorname{Im} v \subset K$ and $\operatorname{spt}(u_k - v) \subset\subset B$.) On account of (3.3) we may pass to the limit $k \to \infty$ in order to deduce

$$\int_B A(u_0)\, Du_0 \cdot D\left(\eta[u_0 - \xi]\right) |Du|^{p-2}\, dx \le \int_B \frac{1}{2}\, DA\, u_0 \cdot \eta(\xi - u_0)\, |Du_0|^p\, dx\, ,$$

which gives (recall (1.3))

$$\begin{aligned} \delta \cdot \int_B \eta \cdot |Du_0|^p\, dx + \\ + \int_B A(u_0)\, |Du_0|^{p-2} D_\alpha u_0 \cdot (u_0 - \xi)\eta'(|x|)x_\alpha \cdot |x|^{-1}\, dx \le 0 \end{aligned} \tag{3.4}$$

for some $\delta > 0$. By scaling (3.1) is valid also for u_k and strong convergence $u_k \to u_0$ in $H^{1,p}_{\text{loc}}$ shows that (3.1) holds for the limit u_0. Thus Lemma 3.2 extends to u_0. Applying Lemma 3.2 to u we see that

$$\Phi(t) := t^{p-n} \int_{B_t} A(u)\, |Du|^p\, dx$$

is an increasing function so that $L := \lim_{t\downarrow 0} \Phi(t)$ exists. On the other hand we have for any $0 < R < 1$

$$\begin{aligned} R^{p-n} \int_{B_R} A(u_0)\, |Du_0|^p\, dx &\underset{(3.3)}{=} \lim_{k\to\infty} R^{p-n} \int_{B_R} A(u_k)\, |Du_k|^p\, dx \\ &= \lim_{k\to\infty} (r_k \cdot R)^{p-n} \int_{B_{r_k \cdot R}} A(u)\, |Du|^p\, dx = L, \end{aligned}$$

which shows $D_r u_0 \equiv 0$. Inserting this result into (3.4) we finally arrive at

$$\int_B \eta \cdot |Du_0|^p\, dx = 0$$

so that $Du_0 = 0 \quad$ a.e. on B, and in conclusion

$$\begin{aligned} 0 = & R^{p-n} \int_{B_R(0)} |Du_0|^p\, dx = \lim_{k\to\infty} R^{p-n} \int_{B_R(0)} |Du_k|^p\, dx \\ &= \lim_{k\to\infty} (r_k \cdot R)^{p-n} \int_{B_{r_k \cdot R}(0)} |Du|^p\, dx, \end{aligned}$$

which proves (3.2).

It remains to verify (3.3): Choose a point $x \in B$ such that

$$\fint_{B_r(x)} \left|u_0 - (u_0)_r\right|^p dz < \varepsilon'$$

holds for some ball $B_r(x) \subset B$ with ε' being defined in (2.5). For k sufficiently large we then have

$$\fint_{B_r(x)} \left|u_k - (u_k)_r\right|^p dz < \varepsilon'$$

and since Lemma 2.1 applies to u_k we get the apriori estimate

$$[u_k]_{C^{0,\alpha}(B_{r/2}(x))} \leq c \leq \infty$$

for the Hölder-seminorms with c independent of k. Arzela's theorem implies $u_k \to u_0$ uniformly on $B_{r/2}(x)$, especially $u_0 \in C^{0,\alpha}\big(B_{r/2}(x)\big)$.

Let S_0 denote the interior singular set of u_0. The preceding arguments show

$$S_0 \subset \Sigma_0 := \{x \in B : \liminf_{r \downarrow 0} \fint_{B_r(x)} |u_0 - (u_0)_r|^p \, dz > 0\} \,,$$

so that $\mathcal{H}^{n-p}(S_0) \leq \mathcal{H}^{n-p}(\Sigma_0) = 0$. Fix a number $t \in (0,1)$ and some small $\delta > 0$ and choose a covering

$$\Sigma_0 \cap B_t \subset \bigcup_{i=1}^{\infty} B_i \,, \quad B_i := B_{r_i}(x_i) \subset\subset B \,,$$

with the property $\sum_{i=1}^{\infty} r_i^{n-p} < \delta$. Then we have the following estimate for the energies on the set $0 =: \bigcup\limits_{i=1}^{\infty} B_i$:

$$\begin{aligned}
\int_O |Du_k|^p \, dx \leq \sum_{i=1}^{\infty} \int_{B_i} |Du_k|^p \, dx \\
&\leq (\text{monotonicity formula for } u_k) \leq c \cdot \sum_{i=1}^{\infty} r_i^{n-p} \int_B |Du_k|^p \, dx \\
&= c \cdot \sum_{i=1}^{\infty} r_i^{n-p} \left(r_k^{p-n} \int_{B_{r_k}} |Du|^p \, dx\right) \\
&\leq (\text{monotonicity formula}) \leq c' \cdot \delta \cdot \int_B |Du|^p \, dx \,.
\end{aligned}$$

In order to control the energies on the remaining part we choose $\eta \in C_0^1\big(B,\,[0,1]\big)$ such that $\eta \equiv 1$ on $\bar{B}_t - O$ and $\operatorname{spt}\eta \cap S_o = \emptyset$. For $k \in \mathbb{N}$ we have

$$
(3.5)_k \qquad
\begin{aligned}
&\int_B A(u_k)\,|Du_k|^{p-2} Du_k \cdot D\,(u_k - v)\,dx\\
&\qquad \le \int_B \frac{1}{2}\,DA\,(u_k)\cdot (v-u_k)\,|Du_k|^p\,dx\,,\\
&\qquad\quad v \in \mathbb{K}\,,\ \operatorname{spt}(u_k - v) \subset\subset B\,;
\end{aligned}
$$

choosing $v := u_k + \eta^p \cdot (u_\ell - u_k)$ in $(3.5)_k$ and $v := u_\ell + \eta^p (u_k - u_\ell)$ in $(3.5)_\ell$ we arrive at

$$
\begin{aligned}
&\int_B \Big(A(u_k)\,Du_k \cdot D(u_k - u_\ell)\,|Du_k|^{p-2}\\
&\qquad - A(u_\ell)\,Du_\ell \cdot D(u_k - u_\ell)\,|Du_\ell|^{p-2}\Big)\cdot \eta^p\,dx\\
&\le c_1 \cdot \int_B |D\eta^p| \cdot |u_k - u_\ell| \cdot \{|Du_\ell|^{p-1} + |Du_k|^{p-1}\}\,dx\\
&\qquad + c_2 \cdot \int_B \eta^p \cdot |u_k - u_\ell| \cdot \{|Du_\ell|^p + |Du_k|^p\}\,dx,
\end{aligned}
$$

which turns into an estimate of the form ($\tau > 0$ a positive constant)

$$
\begin{aligned}
&\tau\cdot \int_B \eta^p \cdot |Du_k - Du_\ell|^p\,dx\\
&\quad \le c_3 \cdot \int_B |u_k - u_\ell| \cdot \Big(\,|D\eta^p| \cdot \{|Du_\ell|^{p-1} + |Du_k|^{p-1}\}\\
&\qquad + \eta^p \cdot \big\{|Du_k|^p + |Du_\ell|^p\big\}\Big)\,dx.
\end{aligned}
$$

Recalling $\sup\big\{|u_\ell(x) - u_k(x)| : x \in \operatorname{spt}\eta\big\} \xrightarrow[\ell,k\to\infty]{} 0$ we see $\int_B \eta^p |Du_\ell - Du_k|^p\,dx \xrightarrow[\ell,k\to\infty]{} 0$ so that $\{Du_k\}$ is a Cauchy-sequence in $L^p_{\mathrm{loc}}(B)$ which completes the proof of (3.3). $\square$

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Fachbereich Mathematik, Arbeitsgruppe 6, Technische Hochschule, Schlossgartenstrasse 7, D–6100 Darmstadt, FRG

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