Uniqueness of a martingale–coboundary decomposition of stationary processes

Pavel Samek, Dalibor Volný

Abstract. In the limit theory for strictly stationary processes $f \circ T^i, i \in \mathbb{Z}$, the decomposition $f = m + g - g \circ T$ proved to be very useful; here T is a bimeasurable and measure preserving transformation an $(m \circ T^i)$ is a martingale difference sequence. We shall study the uniqueness of the decomposition when the filtration of $(m \circ T^i)$ is fixed. The case when the filtration varies is solved in [13]. The necessary and sufficient condition of the existence of the decomposition were given in [12] (for earlier and weaker versions of the results see [7]).

Keywords: strictly stationary process, approximating martingale, coboundary

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1. Introduction and results.

Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space and T an automorphism on Ω, i.e. T is a bijective, bimeasurable and measure preserving mapping of Ω onto itself. Let

$$f = m + g - g \circ T \tag{1}$$

where $(m \circ T^i)$ is a martingale difference sequence, g is a measurable function. Throughout this paper, up to exactly specified cases, the equalities are to be understood to hold almost surely w.r. to P. The martingale generated by the sequence of $m \circ T^i$ is sometimes called the approximating martingale, see [7]. We have $\sum_{i=0}^{n-1}(g - g \circ T) \circ T^i = g - g \circ T^n$, hence the limit behavior of the partial sums of the process $(f \circ T^i)$ can be well approximated by those of the martingale difference sequence $(m \circ T^i)$. This fact made decomposition (1) highly useful in proving limit theorems for stationary processes (see e.g. [6], [7], [9]). For f integrable or square integrable, necessary and sufficient conditions for the existence of the decomposition are given in [12]. Here we shall be concerned with the question of the uniqueness of the decomposition (1). In this paper we shall suppose that the filtration with respect to which $(m \circ T^i)$ is a martingale difference sequence, is fixed. The other problem, i.e. the uniqueness of (1) when the filtration can be changed, is solved in [13]. Recall that a filtration of a strictly stationary martingale difference sequence $(m \circ T^i)$ is given by an invariant σ-algebra $\mathcal{M}$ where $\mathcal{M} \subset T^{-1}\mathcal{M}$ and $m = E(m|T^{-1}\mathcal{M}) - E(m|\mathcal{M})$ (see [7]).

Theorem 1. *Let f be a measurable function and $\mathcal{M} \subset \mathcal{A}$ an invariant $\sigma-$algebra, i.e. $\mathcal{M} \subset T^{-1}\mathcal{M}$. Suppose there exist functions m_1, $m_2 \in L^1$ and measurable functions g_1, g_2 such that*

$$f = m_1 + g_1 - g_1 \circ T = m_2 + g_2 - g_2 \circ T \tag{2}$$

and $(m_1 \circ T^i)$, $(m_2 \circ T^i)$ are two sequences of martingale differences, each with the filtration $T^{-i}\mathcal{M}$. Then $m_1 = m_2$ and $g_1 - g_2$ is an invariant function (i.e. $g_1 - g_2 = (g_1 - g_2) \circ T$).

As we can easily see, (2) is equivalent to

$$m_1 - m_2 = g_2 - g_1 - (g_2 - g_1) \circ T.$$

By the assumptions $((m_1 - m_2) \circ T^i)$ is a martingale difference sequence. Theorem 1 can thus be expressed in the following way:

There does not exist a nontrivial martingale difference sequence $(m \circ T^i)$ with

$$m = g - g \circ T \tag{3}$$

for some measurable function g.

When considering a martingale difference sequence we have assumed that it is integrable. Without the integrability of m the decomposition need not be unique:

Theorem 2. *There is a (nonintegrable) stationary and ergodic Markov chain $(X_i)_{i \in \mathbb{Z}}$ which satisfies*

$$E(X_{n+1} | X_k, k \leq n) = X_n, \qquad n \in \mathbb{Z},$$

i.e. for

$$Y_n = X_{n-1} - X_n$$

(3) is fulfilled (notice that (Y_n) is a non integrable martingale difference sequence).

We assume that the conditional expectation of nonintegrable random variables is defined as in [10].

2. Proofs.

For $-\infty < a < b < \infty$, $H_n(a, b; Y_1, \ldots, Y_n)$ denotes the number of upcrossings of the interval (a, b) by a finite sequence of random variables $Y_1, \ldots, Y_n$. We will need the following lemma which estimates the number of upcrossings of the sums $\sum_{i=0}^{n-1} (g - g \circ T) \circ T^i$.

Lemma. *Let the measure P be ergodic. Let g be a measurable function and F the distribution function of g, i.e. $F(x) = P(g < x)$, $x \in \mathbb{R}$. If there exists an $x \in \mathbb{R}$ such that*

$$F(x + a) - F(x) > 0 \quad \text{and} \quad F(x - a - b) > 0 \tag{4}$$

for some real numbers $0 < a < b < \infty$, *then for* $H_n = H_n(a, b; g - g \circ T, g - g \circ T^2, \dots, g - g \circ T^n)$

$$\limsup_{n\to\infty} \frac{1}{n} EH_n > 0.$$

Proof: Let us denote

$$A = \{x \le g < x + a\}, \qquad B = \{g < x - a - b\}.$$

By (4), $P(A) > 0$ and $P(B) > 0$. From Birkhoff's ergodic theorem we get

$$\frac{1}{n}\sum_{i=1}^{n} \chi\{B\} \circ T^i \xrightarrow[n\to\infty]{} P(B) \text{ a. s.},$$

where

$$\chi\{B\}(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B. \end{cases}$$

By the theorem of Jegorov, the convergence is uniform on a set the measure of which is arbitrarily close to 1. We can thus take $C \in \mathcal{A}$ and $N \in \mathbb{N}$ such that

$$P(C) > 1 - P(A)/2,$$

$$\sum_{i=1}^{N} \chi\{B\} \circ T^i \ge 1 \quad \text{on } C. \tag{5}$$

Therefore,

$$\begin{aligned} C &\subset \{\exists i, 1 \le i \le N : g \circ T^i < x - a - b\}, \\ A \cap C &\subset \{x \le g < x + a, \quad \exists i, 1 \le i \le N : g \circ T^i < x - a - b\} \\ &\subset \{x \le g < x + a, \quad \exists i, 1 \le i \le N : g - g \circ T^i > a + b\} \end{aligned}$$

and consequently for $n \ge 1$

$$\begin{aligned} A \cap & T^{-nN}(A \cap C) \subset \\ &\subset \{x \le g < x + a, \quad x \le g \circ T^{nN} < x + a, \\ &\qquad\qquad \exists i, 1 \le i \le N : g \circ T^{nN} - g \circ T^{nN+i} > a + b\} \\ &\subset \{-a < g - g \circ T^{nN} < a, \quad \exists i, 1 \le i \le N : g \circ T^{nN} - g \circ T^{nN+i} > a + b\} \\ &\subset \{g - g \circ T^{nN} < a, \quad \exists i, 1 \le i \le N : g - g \circ T^{nN+i} > b\}. \end{aligned}$$

The last event implies that the sequence $g - g \circ T^{nN}$, $g - g \circ T^{nN+1}, \dots, g - g \circ T^{nN+N}$ upcrosses the interval (a, b) at least once. Therefore

$$A \cap T^{-nN}(A \cap C) \subset \{H_{nN+N} \ge H_{nN} + 1\}, \qquad n \ge 1,$$

which together with

$$\sum_{j=1}^{n} \chi\{H_{(j+1)N} \geq H_{jN} + 1\} \leq H_{(n+1)N}$$

gives

$$\frac{1}{n} \sum_{j=1}^{n} \chi\{A \cap T^{-jN}(A \cap C)\} \leq \frac{1}{n} H_{(n+1)N}.$$

By integration we get

$$\frac{1}{n} \sum_{j=1}^{n} P(A \cap T^{-jN}(A \cap C)) \leq \frac{1}{n} EH_{(n+1)N}, \qquad n \geq 1. \tag{6}$$

It is a corollary of Birkhoff's ergodic theorem that the left-hand side of (6) converges to $E(\chi\{A\}E(\chi\{A \cap C\}|\mathcal{I}_N))$ as $n \to \infty$ where $\mathcal{I}_N = \{A' \in \mathcal{A}; T^{-N}A' = A'\}$ (see [2]). From (5) we get $P(A \cap C) > 0$, hence

$$E(\chi\{A\}E(\chi\{A \cap C\}|\mathcal{I}_N)) > 0.$$

From this and from (6) we get

$$\limsup_{n\to\infty} \frac{1}{n} EH_n \geq \frac{1}{N} \limsup_{n\to\infty} \frac{1}{n} EH_{(n+1)N} > 0.$$

□

Proof of Theorem 1: For a function h and $n \geq 1$ we denote

$$S_n(h) = \sum_{i=0}^{n-1} h \circ T^i.$$

Let

$$m = m_1 - m_2, \qquad g = g_2 - g_1.$$

Then $m \in L^1$,

$$m = g - g \circ T$$

and $(m \circ T^i)$ is a stationary sequence of martingale differences with the filtration $T^{-i}\mathcal{M}$. We are to prove $m = 0$.

$S_n(m)$ is a martingale, therefore by the Doob's upcrossing inequality (see [1])

$$EH_n(a, b; S_1(m), \ldots, S_n(m)) \leq \frac{E(S_n(m) - a)^+}{b - a} \leq \frac{E|S_n(m)|}{b - a} + \frac{|a|}{b - a}$$

for all $n \geq 1$, $-\infty < a < b < \infty$.

First, let us suppose that the measure P is ergodic. By the (L^1) ergodic theorem

$$\frac{1}{n}E|S_n(m)| \to 0, \qquad n \to \infty.$$

We have $S_n(m) = g - g \circ T^n$, hence by the Lemma

$$F(x+a) - F(x) = 0 \quad \text{or} \quad F(x-a-b) = 0$$

for all $x \in \mathbb{R}$, $a, b \in \mathbb{R}$, $0 < a < b$. This, however, is possible if and only if g is constant (and hence $m = 0$).

The nonergodic case can be easily derived using the ergodic one. Let us suppose that the family $(P^\omega; \omega \in \Omega)$ of regular conditional probabilities w.r. to P and the σ-algebra $\mathcal{I}$ of invariant sets from $\mathcal{A}$ exists (otherwise we can translate the problem to a suitable factor, see [11]). Following [11], [4], almost every (P) of the measures P^ω is ergodic and $(m \circ T^i)$ is an integrable martingale difference sequence in $(\Omega, \mathcal{A}, P^\omega)$. Therefore $m = 0$ *a.s.* (P^ω) for almost all (P) P^ω, hence $m = 0$ *a.s.* (P). □

Proof of Theorem 2: Let $\mathbb{A} = \{0, \pm 2^0, \pm 2^1, \dots\}$, $\Omega = \mathbb{A}^{\mathbb{Z}}$ and $X_n : \Omega \to \mathbb{A}$ be the n-th coordinate projection, $n \in \mathbb{Z}$. We define functions[1] μ, p on $\mathbb{A}$, $\mathbb{A} \times \mathbb{A}$:

$$\mu[0] = \frac{1}{3}, \qquad \mu[j] = \frac{1}{6|j|} \quad \text{for } j \in \mathbb{A}, \quad j \neq 0$$

$$p(0,0) = 0, \qquad p(0, \pm 1) = \frac{1}{2}$$

$$p(i,0) = p(i,2i) = \frac{1}{2} \text{ for } i \in \mathbb{A}, \quad i \neq 0,$$

$$p(i,j) = 0 \text{ for other } (i,j) \in \mathbb{A} \times \mathbb{A}.$$

Following [3], μ and p generate a stationary Markov measure P if and only if

(i) $$\sum_{i \in \mathbb{A}} \mu[i] = 1,$$

(ii) $$\sum_{j \in \mathbb{A}} p(i,j) = 1 \quad \text{for all } i \in \mathbb{A},$$

(iii) $$\sum_{i \in \mathbb{A}} \mu[i] p(i,j) = \mu[j] \quad \text{for all } j \in \mathbb{A}.$$

By [8], 9.11., Theorem 1, Lemma 2 this Markov measure P is ergodic if and only if any bounded sequence $\nu[j]$, $j \in \mathbb{A}$, satisfying

(iv) $$\sum_{j \in \mathbb{A}} \nu[j] p(i,j) = \nu[i], \quad i \in \mathbb{A},$$

[1] The construction is inspired by [5], Example 1.

is constant.
(i), (ii) and (iii) follow immediately from the definition.
Let for some bounded sequence $(\nu[i])_{i\in\mathbb{A}}$ (iv) hold, i.e.

$$\frac{1}{2}(\nu[1]+\nu[-1])=\nu[0]$$

and

$$\frac{1}{2}(\nu[0]+\nu[2i])=\nu[i], \quad i\in\mathbb{A}, \quad i\neq 0.$$

Suppose first $\nu[0]=0$. Then

$$\nu[2i]=2\nu[i], \quad i\in\mathbb{A}, \quad i\neq 0,$$

which means $\nu[i]=0$, $i\in\mathbb{A}$, since $(\nu[i])$ is bounded. In the general case we write

$$\nu[i]=(\nu[i]-\nu[0])+\nu[0]=\lambda[i]+\nu[0]. \tag{7}$$

Sequence $(\lambda[i])$ is bounded, $\lambda[0]=0$ and this solves (iv). Indeed, the constant sequences solve (iv) and the solutions of (iv) form linear space. Thus from the situation considered above we deduce $\lambda[i]=0$, $i\in\mathbb{A}$, and hence $\nu[i]=\nu[0]$, $i\in\mathbb{A}$, by (7).

The distribution of $(X_n, n\in\mathbb{Z})$ is P and therefore (X_n) is a stationary ergodic Markov chain. It follows from the Markov property that

$$E(X_{n+1}|X_k, k\leq n)=E(X_{n+1}|X_n). \tag{8}$$

By the definition of p we have

$$E(X_{n+1}|X_n=i)=\sum_{j\in\mathbb{A}} jp(i,j)=i$$

for all $i\in\mathbb{A}$. Hence $E(X_{n+1}|X_n)=X_n$, which together with (8) proves the Theorem. □

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Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

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