# Natural sinks on $Y_\beta$

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*Abstract.* Let $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta\in\mathbf{Ord}}$ be the large source of epimorphisms in the category **Ury** of Urysohn spaces constructed in [2]. A sink $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ is called natural, if $g_\beta \circ e_\beta = g_{\beta'} \circ e_{\beta'}$ for all $\beta, \beta' \in \mathbf{Ord}$. In this paper natural sinks are characterized. As a result it is shown that **Ury** permits no $(Epi, \mathcal{M})$-factorization structure for arbitrary (large) sources.

*Keywords:* epimorphism, Urysohn space, cointersection, factorization, natural sink, periodic, cowellpowered, ordinal

*Classification:* 18A20, 18A30, 18B30, 54B30, 54C10, 54D10, 54D35, 54G20

## Introduction.

In [2] a large source $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta\in\mathbf{Ord}}$ of epimorphisms in **Ury** was constructed, showing that **Ury** is not cowellpowered. The purpose of this paper is twofolded:

(a) Every natural sink $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ is defined uniquely by $g_1 : \mathbf{Q} \to X$. $Y_\beta$ can be as large as might be required. How does $g_\beta$ look? There are rare instances in General Topology where a smallness condition has overall consequences, e.g. the arbitrary product of separable spaces fulfills the countable chain condition.

(b) Because of non-cowellpoweredness, some categorical theorems should not be applicable in **Ury**. The investigation of sinks $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ have as a result the non-existence for any $\mathcal{M}$ of a $(Epi, \mathcal{M})$-factorization structure for (large) sources.

**Notation.** $\mathbf{Q}$ ($\mathbf{Q}^+ = \mathbf{Q}^+ \cup \{0\}$) are the (positive) rationals. $\mathbf{R}$, $\mathbf{N} = \mathbf{N} \cup \{0\}$ are the real and the natural numbers, respectively. $\omega_0$ is the first infinite ordinal. $w$ is a fixed positive irrational number. $\epsilon$, $\delta$ are real numbers $> 0$. $h, l, m, n$ are elements of $\mathbf{N}$. **Ord** is the class of ordinal numbers. $\alpha, \beta, \gamma, \kappa, \lambda, \xi, \tau$ are ordinal numbers. $[0, 1)$, $[-1, +1]$ are as usual intervals of real numbers. $d(x, y)$ is the euclidean distance of $x, y \in \mathbf{R}$ or of $x, y \in \mathbf{R} \times \mathbf{R}$. If $B \subseteq \mathbf{R}$ (or $\subseteq \mathbf{R} \times \mathbf{R}$), then $d(x, B) := \inf\{d(x, b) | \, b \in B\}$. $U(x, \epsilon)$ is an $\epsilon$-neighbourhood, taken in $\mathbf{R}$, if $x \in \mathbf{R}$; taken in $\mathbf{R} \times \mathbf{R}$ if $x \in \mathbf{R} \times \mathbf{R}$. Every ordinal $\tau$ has a unique representation $\tau = \lambda + n$, where $\lambda$ is a limit ordinal and $n$ is a finite ordinal (natural number). This representation we will use often. **Top**, **Ury**, $\mathbf{T_3}$ are the categories of topological, Urysohn, and regular $T_1$ spaces, respectively. Recall that a topological space is

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called Urysohn, if distinct points have disjoint closed neighbourhoods. $clA$ is, as usual, the topological closure of $A \subseteq X$, $clA =: \overline{A}$. If $(X, \mathcal{X}) \in \mathbf{Top}$ and $A \subseteq X$, then $cl_\theta A := \{x \in X | x \in U \in \mathcal{X} \Rightarrow clU \cap A \neq \emptyset\}$, $cl_{\theta\theta} A := \bigcap \{cl_\theta clU | A \subseteq U \in \mathcal{X}\}$. $cl_\theta^{\omega_0} A := \bigcup_{\mathbf{N}} cl_\theta^n A$, where $cl_\theta^0 A := A$ and $cl_\theta^{n+1} A := cl_\theta cl_\theta^n A$. $\triangle := \{(x, x) | x \in X\}$.

**Definition 1.** A sink $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ is called natural, if $g_\beta \circ e_\beta = g_1 \circ e_1$ for all $\beta \in \mathbf{Ord}$.
$X_0 = \mathbf{Q} \times \{0\} \times \{1\}$, $X_\alpha = \mathbf{Q} \times (\mathbf{Q} \cap [0, 1)) \times \{\alpha\}$, $\alpha > 0$,
$Y_1 = X_0$, $Y_\beta = \bigcup \{X_\alpha | \alpha < \beta\}$, $\beta > 1$,
$e_\beta : \mathbf{Q} \to Y_\beta$ is defined by $e_\beta(q) = (q, 0, 1)$ for all $q \in \mathbf{Q}$.
$K(r, s, \alpha, \epsilon) = \{(u, v, \alpha) \in X_\alpha | v > 0 \wedge d((u, v), (r - s/w, 0)) < \epsilon\}$, $0 < \epsilon < 1$.
$Y_\beta$ becomes a Urysohn space equipped with the following sets forming neighbourhood bases of $(r, s, \alpha)$ (see [2]):

$\alpha = 1$ :

$s \neq 0$ : $K(r, s, 1, \epsilon) \cup \{(r, s, 1)\} =: U(r, s, 1, \epsilon)$.

$s = 0$ : $K(r, 0, 1, \epsilon) \cup \{(u, 0, 1) | d(u, r) < \epsilon\}$.

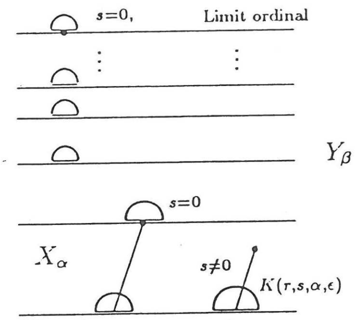

Fig. 1

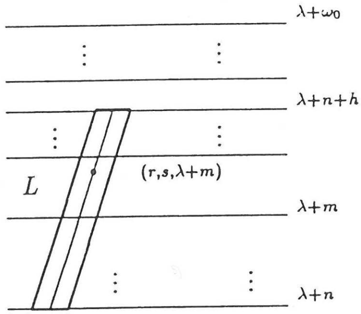

Fig. 2

$\alpha > 1$ :

$\alpha$ **limit**:

$$s \neq 0 : K(r, s, \alpha, \epsilon) \cup \{(r, s, \alpha)\} =: U(r, s, \alpha, \epsilon).$$

$$s = 0 : \{(r, 0, \alpha)\} \cup K(r, 0, \alpha, \epsilon) \cup \bigcup_{\alpha > \tau \geq \gamma} K(r, 0, \tau, \epsilon) =: U(r, 0, \alpha, \gamma, \epsilon),\ \gamma < \alpha.$$

$\alpha$ **non-limit**:

$$s \neq 0 : K(r, s, \alpha, \epsilon) \cup \{(r, s, \alpha)\} =: U(r, s, \alpha, \epsilon).$$

$$s = 0 : K(r - 1/w, 0, \alpha - 1, \epsilon) \cup K(r, 0, \alpha, \epsilon) \cup \{(r, 0, \alpha)\} =: U(r, 0, \alpha, \epsilon)$$ (see Fig. 1).

Consider now a limit ordinal $\lambda$ and $\bigcup_{\mathbf{N}} X_{\lambda+n} =: X_\lambda^\infty$. There is a bijection $\phi : X_\lambda^\infty \to \mathbf{Q} \times \mathbf{Q}^+ \times \{\lambda\}$ given by $\phi(r, s, \lambda + n) := (r, s + n, \lambda)$. We refer to this bijection in the following construction.
Define $\bigcup_{n \le l \le h+n} X_{\lambda+l} \cup \{(r, 0, \lambda + h + n + 1) | r \in \mathbf{Q}\} =: X^h_{\lambda+n}$, $L(r, s, \lambda + m, \epsilon, \lambda + n, h) := \{(u, v+l, \lambda) \in X^h_{\lambda+n} | d((u, v+l), L(r, s, \lambda+m)) \le \epsilon\}$, where $L(r, s, \lambda+m) := \{(x, w(x - r) + s + m) | x \in \mathbf{R}\} \times \{\lambda\}$, i.e. $L(r, s, \lambda + m, \epsilon, \lambda + n, h)$ is the $2\epsilon$-stripe around $L(r, s, \lambda+m)$ passing through $(r, s, \lambda+m) = \phi^{-1}(r, s+m, \lambda)$ with bottom at $\{(r, 0, \lambda+n) | r \in \mathbf{Q}\}$ and height $h$. Note that we identify via $\phi$ the point $(u, v+l, \lambda)$ with $(u, v, \lambda + l)$, where $v < 1$, i.e. $(u, v + l, \lambda) \in X_{\lambda+l}$. The meaning of $h = \omega_0$ is obvious (see Fig. 2).

**Lemma 2.** *Let $(X, \mathcal{X})$ be a Urysohn space, $A \subseteq X$, $cl_{\theta\theta}A = A$. Define an equivalence relation by $\sim_A := A \times A \cup \triangle$. Then the quotient $X/\sim_A$ is a Urysohn space.*

Proof: Take $x \notin A$. Then there exists $U \in \mathcal{X}$ s.t. $A \subseteq U$ and $x \notin cl_\theta clU$. Hence there is $U_x$ s.t. $x \in U_x$, $clU_x \cap clU = \emptyset$. Since $A$ is $\theta$-closed, distinct points in $X - A$ can be separated by disjoint closed neighbourhoods. □

**Lemma 3.** *Let $(r, s, \alpha) \in Y_\beta$. We consider basic neighbourhoods of $(r, s, \alpha)$.*

(a) *if $s \ne 0$, then $clU(r, s, \alpha, \epsilon) = L(r, s, \alpha, \epsilon, \alpha, 1) \cup U(r, s, \alpha, \epsilon)$*
(b) *if $s = 0$, $\alpha$ non-limit, then $clU(r, 0, \alpha, \epsilon) = L(r, 0, \alpha, \epsilon, \alpha - 1, 2) \cup U(r, 0, \alpha, \epsilon)$*
(c) *if $s = 0$, $\alpha = \lambda$ limit, then $clU(r, 0, \lambda, \gamma, \epsilon) = \bigcup\{L(r, 0, \tau, \epsilon, \tau, 1) | \alpha \ge \tau \ge \gamma\} \cup U(r, 0, \lambda, \gamma, \epsilon)$*

(see Fig. 3).

Proof: A point $(u, v, \alpha)$ is in the closure of $K(r, s, \alpha)$ if $d(r - s/w, u - v/w) \le \epsilon$ and $0 \le v \le 1$. Where in case $v = 1$ the point $(u, 1, \alpha)$ is identified with $(u, 0, \alpha+1)$. □

**Lemma 4.** *If $\lambda$ is a limit ordinal and $n, m \in \mathbf{N}$, then $cl_\theta^{\omega_0} L(r, s, \lambda+m, \epsilon, \lambda+n, h) = L(r, s, \lambda + m, \epsilon, \lambda, \omega_0)$.*

Proof: By induction with the help of Lemma 3 (b). □

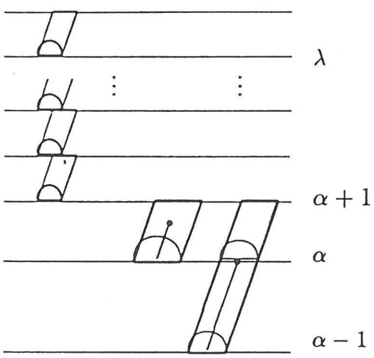

Fig.3

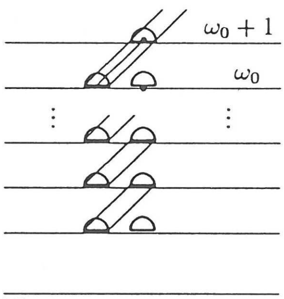

Fig.4

**Lemma 5.** *Let $\lambda$ be a limit ordinal, $\lambda < \beta$. Then $\bigcup_{\lambda \le \alpha < \beta} X_\alpha = Y_\beta - Y_\lambda$ and $cl_{\theta\theta}(Y_\beta - Y_\lambda) = Y_\beta - Y_\lambda$.*

PROOF: Take $(r, s, \alpha) \notin Y_\beta - Y_\lambda$, i.e. $(r, s, \alpha) \in Y_\lambda$. We will construct disjoint closed neighbourhoods of $(r, s, \alpha)$ and of $Y_\beta - Y_\lambda$. It is $\alpha < \lambda$ and $\lambda$ is limit ordinal. The set $\{(u, v, \gamma) \in Y_\beta | (\gamma > \alpha + 2) \vee ((\gamma = \alpha + 2) \wedge (v > 0))\}$ is an open set containing $Y_\beta - Y_\lambda$. Its closure is $\{(u, v, \gamma) \in Y_\beta | \gamma \ge \alpha + 2\}$. By Lemma 3 above, the closure of an open basic neighbourhood of $(r, s, \alpha)$ is contained in $Y_{\alpha+2}$. □

**Remark 6.** Are there non-constant natural sinks on $Y_\beta$? We must find a sink $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ coinciding on $\mathbf{Q} \subseteq Y_\beta$. As we know, each morphism $g_\beta : Y_\beta \to X$ is defined by its values on the countable set $\mathbf{Q}$. Since $Y_\gamma$ is subspace of $Y_\beta$, if $\gamma < \beta$, $g_\beta$ can be regarded as continuous extension of $g_\gamma$. Hence there are not so many morphisms into $X$. Additionally $X$ has fixed weight, character, cardinality, etc.. All these cardinality functions have no bound on the class $Y_\beta$, $\beta \in \mathbf{Ord}$. The answer is given by the following

**Example 7.**

(a) Let $\lambda$ be a limit ordinal. We apply Lemma 2 and Lemma 5. Take $A = Y_{\lambda+1} - Y_\lambda$. Then $Y_{\lambda+1}/\sim_A =: Y_\lambda \cup \{*\} =: Y_\lambda^*$ is a Urysohn space. Define $g_\beta : Y_\beta \to Y_\lambda^*$ by

$$g_\beta(r, s, \alpha) = \begin{cases} (r, s, \alpha) & \text{if } \alpha < \lambda \\ * & \text{if } \alpha \ge \lambda, \alpha < \beta. \end{cases}$$

(b) Define $\sin_\beta : Y_\beta \to [-1, +1]$ by $\sin_\beta(r, s, \alpha) = \sin(2\pi(wr - s))$ for all $(r, s, \alpha) \in Y_\beta$. Let $U(r, s, \alpha, \delta) \subseteq Y_\beta$ be a basic neighbourhood of $(r, s, \alpha)$. If $(p, q, \tau) \in U(r, s, \alpha, \delta)$, then $d(r - \overline{s}/w, p - q/w) < \delta(1 + 1/w^2)^{1/2} =: \delta c$, where

$$\overline{s} = \begin{cases} 1 & \text{if } \tau = \alpha - 1 \\ s & \text{otherwise.} \end{cases}$$

($c$ appears for geometrical reasons, as one can see in Fig. 3 or Fig. 4: some points of $U(r, s, \alpha, \epsilon)$ lie outside $L(r, s, \alpha, \epsilon, \alpha, 1)$.) Now take an $\epsilon$-neighbourhood $U(\sin(2\pi(wr_0 - s_0)), \epsilon)$. The mapping $x \mapsto \sin(2\pi wx)$ is continuous. There is a $\delta$-neighbourhood $U(r_0 - s_0/w, \delta)$ s.t. $p - q/w \in U(r_0 - s_0/w, \delta) \Rightarrow \sin(2\pi w(p - q/w)) \in U(\sin(2\pi(wr_0 - s_0)), \epsilon)$. Now assume $\tau = \alpha - 1$, then $s_0 = 0$. Take $p, q$ with $d(p - q/w, r_0 - 1/w) < \delta c$, of course $d(p - q/w + 1/w, r_0) < \delta c$,but $\sin(2\pi w(p - q/w)) = \sin(2\pi w(p - q/w + 1/w))$ and hence $\sin(2\pi(wp - q)) \in U(\sin(2\pi wr_0), \epsilon)$.

(c) Define $P_\beta : Y_\beta \to [0, 1]$,$\beta \ge 1$, by

$$P_\beta(r, s, \alpha) = \begin{cases} \frac{1}{1 + (r - \frac{s+n}{w})^2} & \text{if } \alpha = n < \omega_0 \\ 0 & \text{if } \alpha \ge \omega_0, \alpha < \beta. \end{cases}$$

The proof of continuity is similar to (b): The mapping $x \mapsto \frac{1}{1+x^2}$ is continuous. Fix $U(P_\beta(r_0, s_0, n), \epsilon) \subseteq [0, 1]$. There is $\delta > 0$ s.t. $p - \frac{q+n}{w} \in$

$U((r_0 - \frac{s_0+n}{w}), \delta c)$ implies $\frac{1}{1+(p-\frac{q+n}{w})^2} \in U(P_\beta(r_0, s_0, n), \epsilon)$, showing that even $P_\beta[L(r_0, s_0, n, \delta c, 0, \omega_0)] \subseteq U(P_\beta(r_0, s_0, n), \epsilon)$. Finally $P_\beta$ is arbitrary small on $U(r_0, 0, \omega_0, n, \delta)$, if $n$ increases.

(d) Take $Y^*_\lambda$,$\lambda > \omega_0$, from (a) and a limit ordinal $\xi < \lambda$. If $(r, s, \alpha) \in Y^*_\lambda$, $\alpha = \xi + n$, then $cl_{\theta\theta} L(r, s, \alpha, \epsilon, \xi, \omega_0) = L(r, s, \alpha, \epsilon, \xi, \omega_0) =: F$ and $Y^*_\lambda / \sim_F$ is a Urysohn space. Combination with (a) gives many different sinks $(g_\beta : Y_\beta \to Y^*_\lambda / \sim_F)_{\beta \in \mathbf{Ord}}$.

**Definition 8.** Let $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ be a natural sink.

(a) $(g_\beta)$ is called periodic, if for all $\beta > \omega_0$ and for all $\alpha, \tilde{\alpha} \geq \omega_0$; $\alpha, \tilde{\alpha} < \beta$: $g_\beta(r, s, \alpha) = g_\beta(r, s, \tilde{\alpha})$.

(b) $(g_\beta)$ is called eventually periodic, if there exists an ordinal $\tau$, s.t. for all $\beta > \tau$ and for all $\alpha, \tilde{\alpha} \geq \tau$; $\alpha, \tilde{\alpha} < \beta$: $g_\beta(r, s, \alpha) = g_\beta(r, s, \tilde{\alpha})$.

**Theorem 9.** *Let $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ be a natural sink and let $X$ be a $T_3$-space. Then $(g_\beta)$ is periodic.*

Proof: Take $x \in X$. For every neighbourhood $W_x$ of $x$ there is a neighbourhood $V_x$ of $x$ with $clV_x \subseteq W_x$, and of course $cl_\theta clV_x = clV_x$, $cl_\theta^{\omega_0} clV_x = clV_x$. If $A \subseteq Y_\beta$, then $g_\beta[cl_\theta^{\omega_0} A] \subseteq cl_\theta^{\omega_0} g_\beta[A] = \overline{g_\beta[A]}$. Take a basic neighbourhood $U(r, s, n, \delta) =: U$ of $(r, s, n)$. Then $cl_\theta^{\omega_0}[U] = L(r, s, n, \delta, 0, \omega_0) \cup U$. For every neighbourhood $W$ of $g_\beta(r, s, n)$ there is $\delta > 0$ s.t. $g_\beta[L(r, s, n, \delta, 0, \omega_0)] \subseteq W$. Consider $(p, 0, \omega_0)$ and $(p, 0, \omega_0+1)$. For every neighbourhood $U(g_\beta(p, 0, \omega_0))$ there is a neighbourhood $U(p, 0, \omega_0, k, \delta)$ s.t. $g_\beta[L(p, 0, \omega_0, \delta, \omega_0, \omega_0) \cup \bigcup_{n \geq k} L(p, 0, n, \delta, 0, \omega_0)] \subseteq U(g_\beta(p, 0, \omega_0))$. Assume $g_\beta(p, 0, \omega_0) \neq g_\beta(p, 0, \omega_0 + 1)$. Then we have open sets $U_1, V_1, U_0$ s.t. $g_\beta(p, 0, \omega_0 + 1) \in U_1 \subseteq \overline{U_1} \subseteq V_1$, $g_\beta(p, 0, \omega_0) \in U_0 = U(g_\beta(p, 0, \omega_0))$, $V_1 \cap U_0 = \emptyset$. Now take an open basic neighbourhood $U(p, 0, \omega_0+1, \delta) =: U^*$ fulfilling $g_\beta[U^*] \subseteq U_1$. Then $clU^* = L(p, 0, \omega_0+1, \delta, \omega_0, 2) \cup U^* = L(p-1/w, 0, \omega_0, \delta, \omega_0, 2) \cup U^*$. Let $O^* \subseteq g_\beta^{-1}[V_1]$ be an open set containing $\overline{U^*}$. Then for all $(u, 0, \omega_0) \in \overline{U^*}$ there is $\delta_u < \delta, k_u \in \mathbf{N}$ s.t. $K(u, 0, l, \delta_u) \subseteq O^*$ for all $l \geq k_u$ (see Fig. 4). Take a $(x, 0, l) \in K(u, 0, l, \delta_u)$, $l \geq \max\{k_u, k\}$. Then $(x, 0, l) \in L(p, 0, l+1, \delta, 0, \omega_0)$. But $g_\beta(x, 0, l) \in V_1$ and $g_\beta(x, 0, l) \in g_\beta[L(p, 0, l+1, \delta, 0, \omega_0)] \subseteq U_0$, contradicting $V_1 \cap U_0 = \emptyset$. Hence $g_\beta$ assumes the same values on $\mathbf{Q}_{\omega_0} := \{(r, 0, \omega_0) | r \in \mathbf{Q}\}$ and $\mathbf{Q}_{\omega_0+1} := \{(r, 0, \omega_0+1) | r \in \mathbf{Q}\}$. Since $cl_\theta \mathbf{Q}_{\omega_0} = \overline{X_{\omega_0}}$ and $cl_\theta \mathbf{Q}_{\omega_0+1} = \overline{X_{\omega_0+1}}$, this is also true for $X_{\omega_0}$ and $X_{\omega_0+1}$ (there is another argument: the two mappings $h_1 := g_\beta / X_{\omega_0+1}$ and $h_0$ defined by $h_0(r, s, \omega_0+1) := g_\beta(r, s, \omega_0)$ are continuous and coincide on $\mathbf{Q}_{\omega_0+1}$). The function $\tilde{g_\beta} : Y_\beta \to X$ defined by

$$\tilde{g_\beta}(r, s, \alpha) = \begin{cases} g_\beta(r, s, \alpha) & \text{if } \alpha < \omega_0 \\ g_\beta(r, s, \omega_0) & \text{if } \alpha \geq \omega_0, \alpha < \beta \end{cases}$$

is continuous and coincides with $g_\beta$ on $\mathbf{Q}$, hence $g_\beta = \tilde{g_\beta}$. To show continuity, take $K(r - 1/w, 0, \alpha - 1, \epsilon) \cup K(r, 0, \alpha, \epsilon) \cup \{(r, 0, \alpha)\} =: U$, an $\epsilon$-neighbourhood of $(r,0,\alpha)$, $\alpha > \omega_0$ non-limit. Then $\tilde{g_\beta}[U] = \tilde{g_\beta}[K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0+1,\epsilon) \cup \{(r,s,\omega_0)\}] = \tilde{g_\beta}[K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0,\epsilon) \cup \{(r,s,\omega_0)\}] = g_\beta\ [K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0+1,\epsilon) \cup \{(r,s,\omega_0)\}]$. $\alpha$ limit or $s \neq 0$ is not a problem. □

**Theorem 10.** *Let* $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ *be a natural sink in* **Ury**, *then* $(g_\beta)$ *is eventually periodic.*

Proof: Let $\chi(X)$ be the character of $X$. Let $\lambda$ be a regular limit ordinal with $(cof(\lambda) =)\ \lambda > \max\{\chi(X), \omega_0\}$. Take a space $Y_\beta$ with $\beta > \lambda$. Look at the bottom edge of $X_\lambda \subseteq Y_\beta$: $\{(p,0,\lambda) | p \in \mathbf{Q}\}$. Let $\mathcal{U}(g_\beta(p,0,\lambda))$ be a neighbourhood base of cardinality $\leq \chi(X)$ of the point $g_\beta(p,0,\lambda)$ in $X$. For every $V \in \mathcal{U}(g_\beta(p,0,\lambda))$ there exists a neighbourhood $U(p,0,\lambda,\kappa_V^p,\epsilon_V^p) =: U$ of $(p,0,\lambda)$ in $Y_\beta$ fulfilling $g_\beta[U] \subseteq V$. We have $|\{(\kappa_V^p,\epsilon_V^p) | V \in \mathcal{U}(g_\beta(p,0,\lambda))\}| < \chi(X)$, then also $\sup\{\kappa_V^p | V \in \mathcal{U}(g_\beta(p,0,\lambda))\} =: \kappa^p < \lambda$ and $\sup\{\kappa^p | p \in \mathbf{Q}\} =: \kappa < \lambda$. This shows that for every $V \in \mathcal{U}(g_\beta(p,0,\lambda))$, $p \in \mathbf{Q}$, there is $U := U(p,0,\lambda,\kappa,\epsilon_V^p)$ s.t. $g_\beta[U] \subseteq V$. Now take $\tau < \lambda$, $\tau \geq \kappa$. We show $g_\beta(p,0,\lambda) = g_\beta(p,0,\tau)$. If we assume the contrary, then there are neighbourhoods $U(g_\beta(p,0,\lambda))$ and $U(g_\beta(p,0,\tau))$ of $g_\beta(p,0,\lambda)$ and $g_\beta(p,0,\tau)$ respectively with disjoint intersection. Of course $g_\beta[clU(p,0,\lambda,\kappa,\epsilon^p_{U(g_\beta(p,0,\lambda))})] \subseteq clU(g_\beta(p,0,\lambda))$, but $(p,0,\tau) \in clU(p,0,\lambda,\kappa,\epsilon^p_{U(g_\beta(p,0,\lambda))})$ is a contradiction. We are ready to show $g_\beta(p,0,\tau) = g_\beta(p,0,\tau+1)$ for a limit ordinal $\tau < \lambda, \tau \geq \kappa$. (There is such a limit ordinal.) But this is easy, since $g_\beta(p,0,\tau) = g_\beta(p,0,\lambda) = g_\beta(p,0,\tau+1)$. By the same argument such as in the proof of the previous Theorem 9 we get $g_\beta = \tilde{g_\beta}$ for all $\beta \in \mathbf{Ord}$, where

$$\tilde{g_\beta}(r,s,\alpha) = \begin{cases} g_\beta(r,s,\alpha) & \text{if } \alpha < \tau \\ g_\beta(r,s,\lambda) & \text{if } \alpha \geq \tau, \alpha < \beta. \end{cases}$$

□

**Remark 11.** The proof illustrates that for showing $g_\beta(p,0,\tau) = g_\beta(p,0,\tau+1)$ for all $p \in \mathbf{Q}$ we only need the Hausdorff property of $X$. To show $g_\beta = \tilde{g_\beta}$ we need the Urysohn separation axiom, of course.

**Example 12.** Let $\tau > 1$. Define a new Urysohn space $Y_\tau^{\tau+1}$ on the set $Y_{\tau+1} = Y_\tau \cup X_\tau$, where

(a) the neighbourhoods of the points $(r,s,\alpha)$, $\alpha < \tau$, and of $(r,s,\tau)$, $s \neq 0$, are not changed.

(b) neighbourhood bases of $(r,0,\tau) \in X_\tau$ are $U(r,0,\tau,\gamma,\epsilon) \cup K(r-1/w,0,\tau,\epsilon)$. Note that $(r,1,\tau) \notin X_\tau$ by definition. Define $g_\beta : Y_\beta \to Y_\tau^{\tau+1}$ by

$$\tilde{g_\beta}(r,s,\alpha) = \begin{cases} (r,s,\alpha) & \text{if } \alpha < \tau \\ (r,s,\tau) & \text{if } \alpha \geq \tau, \alpha < \beta. \end{cases}$$

$(g_\beta)$ is an eventually periodic sink, if $\tau > \omega_0$. The maps $g_\beta$ in case $\beta > \tau$ are pressing down all $X_\alpha$ above $\tau$ onto the modified $X_\tau$. The topology on $Y_\tau^{\tau+1}$ is coarser then on $Y_{\tau+1}$.

**Remark 13.**

(a) The proof of the previous Theorem will apply for arbitrary $X$ which has a coarser $T_3$-topology. This shows that in all eventually periodic, and non periodic examples the codomain has no coarser $T_3$-topology ($Y_{\omega_0}$ has!).

(b) All sinks are uniquely defined by $g_1 : \mathbf{Q} \to X$. In all the examples, except 7 (b) and 7 (c), $g_1$ is the identity on $\mathbf{Q}$. This illustrates that in determining $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ , $g_1$ is as important as the codomain $X$ of the sink.

(c) Theorem 9 is a strong restriction, however the arithmetic sum of the functions in 7 (b) and 7 (c) provides a new sink.

(d) After these preparations have been completed it becomes an easy task to show that **Ury** allows no $(Epi, \mathcal{M})$-factorization structure for (large) sources and any $\mathcal{M}$. Since **Ury** has coequalizers this means that **Ury** is not (Epi, extremal Mono Source)-category (see [1, Proposition 15.8 (3)]).

**Lemma 14.** *Let a sink $(g_i : X_i \to Y)_I$ in* **Top** *be given s.t. for all $j \in I$ and all $x_j, y_j \in X_j$, $x_j \neq y_j$, there is a sink $(h_i : X_i \to Z_{x_j y_j})_I$ with $h_j(x_j) \neq h_j(y_j)$ and a continuous map $h : Y \to Z_{x_j y_j}$ s.t. $h \circ g_j = h_j$, then for all $i \in I$, $g_i : X_i \to Y$ is injective.*

Proof: Assume we can find $j \in I$, $x_j, y_j \in X_j$,$x_j \neq y_j$, s.t. $g_j(x_j) = g_j(y_j)$. Take $h_i, h$ as in the Lemma. Then $h_j(x_j) = h \circ g_j(x_j) = h \circ g_j(y_j) = h_j(y_j)$, contradicting $h_j(x_j) \neq h_j(y_j)$. □

**Theorem 15.** *The large source $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta \in \mathbf{Ord}}$ consisting of epimorphisms has no cointersection in* **Ury**.

Proof: Assume there is a cointersection $(f_\beta : Y_\beta \to Z)_{\beta \in \mathbf{Ord}}$. Take $x_\beta, y_\beta \in Y_\beta$, $x_\beta \neq y_\beta$. There is a limit ordinal $\lambda > \beta$. Take the natural sink $(g_\beta : Y_\beta \to Y_\lambda^{\lambda+1})$ from Example 12 or the natural sink $(g_\beta : Y_\beta \to Y_\lambda^*)$ from Example 7. By Lemma 14, all $f_\beta$ are injective, which is impossible. □

**Corollary 16.** **Ury** *is no $(Epi, \mathcal{M})$-category for any $\mathcal{M}$.*

Proof: If we had some $\mathcal{M}$ giving a factorization structure for large sources, then by [1, Corollary 15.16 (1)] cointersections exist, which contradicts Theorem 15. □

**Remark 17.** If we only allow $T_3$-spaces as codomains of a natural sink, then by Theorem 9 each such sink factorizes through $Y_{\omega_0}^{\omega_0+1}$, which gives some kind of cointersection.

## References

[1] Adámek J., Herrlich H., Strecker G.E., *Abstract and Concrete Categories*, Wiley & Sons 1990.

[2] Schröder J., *The category of Urysohn spaces is not cowellpowered*, Top. Appl. **16** (1983), 237–241.

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