Harnack's properties of biharmonic functions

Emmanuel P. Smyrnelis

Abstract. Study of the equicontinuity of biharmonic functions, of the Harnack's principle and inequalities, and of their relations.

Keywords: biharmonic functions, Harnack's inequalities
Classification: 31B30, 31D05

The harmonic system of Brelot was based originally on three axioms: (1) axiom of sheaf; (2) axiom of the existence of a basis of regular open sets, and; (3) axiom of convergence. Further developments, as the integral representation, required the introduction of a new axiom, called axiom $3'$, which seemed to be stronger than axiom 3. In fact it was proved that axioms 1, 2, 3 imply axiom $3'$.

In an elliptic biharmonic space, we shall prove first the equicontinuity of biharmonic positive pairs with values less or equal to given numbers at a fixed point; next, the equivalence of Harnack's principle, Harnack's inequalities and other properties will be established.

Our framework will be an elliptic biharmonic space $(\Omega, \mathcal{H})$ with Ω connected. For the notions and notations used in this work we refer to [3].

Theorem 1.1. *The biharmonic pairs $(u_1, u_2) \geq (0,0)$ defined in a domain $U \subset \Omega$ with values at $x_0 \in U$ less or equal to given real numbers are equicontinuous at x_0.*

Proof: The only interesting case is: $u_1 > 0$, $u_2 > 0$; see [3, Proposition 2.2].

The second components u_2 being $\mathcal{H}_2$-harmonic functions, they are therefore equicontinuous at x_0; see [1]. It remains to show the equicontinuity of the first components u_1.

Let us consider our family of biharmonic pairs (u_1, u_2). In any open $\mathcal{H}$-regular set $\omega \subset \bar{\omega} \subset U$ with $x_0 \in \omega$, we know that $u_1(x) = \int u_1\, d\mu_x^\omega + \int u_2\, d\nu_x^\omega$. The functions $x \mapsto \int u_1(y)\, d\mu_x^\omega(y)$ are $\mathcal{H}_1$-harmonic in ω; as $\lambda_1 \geq u_1(x_0) \geq \int u_1\, d\mu_{x_0}^\omega$ where λ_1 is the given number of the theorem, the equicontinuity of these functions is known; see [1]. Then, for every $\varepsilon > 0$, there is a neighborhood δ of x_0 such that, independently of u_1,

$$\left| \int u_1\, d\mu_x^\omega - \int u_1\, d\mu_{x_0}^\omega \right| < \varepsilon\,.$$

On the other hand, we have

$$\int u_2\, d\nu_x^\omega - \int u_2\, d\nu_{x_0}^\omega \leq \sup u_2(\partial\omega) \int d\nu_x^\omega - \inf u_2(\partial\omega) \int d\nu_{x_0}^\omega =$$
$$= (\sup u_2(\partial\omega) - \inf u_2(\partial\omega)) \|\nu_{x_0}^\omega\| + \theta(x) \sup u_2(\partial\omega)$$

where $\theta(x) = \int d\nu_x^\omega - \int d\nu_{x_0}^\omega$; because of the continuity of the function $x \mapsto \int d\nu_x^\omega$, the first component of the biharmonic pair $(\int d\nu_x^\omega, \int d\lambda_x^\omega)$, θ is arbitrarily small in a neighborhood δ' of x_0, that is $|\theta(x)| \le \varepsilon$. Since the family of u_2 is equicontinuous at the point x_0, we choose ω such that $\sup u_2(\partial\omega) - \inf u_2(\partial\omega) < \varepsilon$; also, the Harnack's inequality in the harmonic space $(\Omega, \mathcal{H}_2)$ gives us $\sup u_2(\partial\omega) \le k u_2(x_0)$ for a suitable $k > 0$; see [1]. Consequently, if $x \in \omega \cap \delta \cap \delta'$ we have

$$u_1(x) - u_1(x_0) = \Big(\int u_1\, d\mu_x^\omega - \int u_1\, d\mu_{x_0}^\omega\Big) + \Big(\int u_2\, d\nu_x^\omega - \int u_2\, d\nu_{x_0}^\omega\Big) < \varepsilon'$$

where $\varepsilon' = \varepsilon + \varepsilon\|\nu_{x_0}^\omega\| + \varepsilon k \lambda_2$, λ_2 being the fixed number of the theorem with respect to the second components u_2; see also [3, Proposition 1.5]. Likewise,

$$\int u_2\, d\nu_x^\omega - \int u_2\, d\nu_{x_0}^\omega \ge \inf u_2(\partial\omega) \int d\nu_x^\omega - \sup u_2(\partial\omega) \int d\nu_{x_0}^\omega =$$

$$= (\inf u_2(\partial\omega) - \sup u_2(\partial\omega)) \int d\nu_{x_0}^\omega + \theta(x) \inf u_2(\partial\omega);$$

as previously, we see that, if $x \in \omega \cap \delta \cap \delta'$, $u_1(x) - u_1(x_0) \ge -\varepsilon'$. Therefore, in a neighborhood of x_0, we obtain for every pair (u_1, u_2) the inequality $|u_1(x) - u_1(x_0)| < \varepsilon'$. □

Remark 1.2. Let us consider the family of pairs $\Phi_{x_0} = \{(u_1, u_2) \in \mathcal{H}_+(U); u_1(x_0) = 1\}$. By Harnack's inequalities (see [3, Théorème 2.13] and [4, p. 109]) if K is any compact set of U, there exists a real constant $\alpha = \alpha(x_0, K)$ such that $u_1(x) \le \alpha$, $u_2(x) \le \alpha$ for every pair of Φ_{x_0} and every $x \in K$; therefore, these pairs are locally uniformly bounded.

Corollary 1.3. *The functions* $\sup\{u_1; (u_1, u_2) \in \Phi_{x_0}\}$ *and* $\inf\{u_1; (u_1, u_2) \in \Phi_{x_0}\}$ *are finite, continuous and* > 0 *in* U.

Proof: We denote the first function by $\overline{U}_1$ and the second by $\underline{U}_1$. Let $x_1 \in U$ and $\varepsilon > 0$; there exists a neighborhood δ of x_1 in U where $1 - \varepsilon \le u_1(x)/u_1(x_1) \le 1 + \varepsilon$; hence $(1-\varepsilon)\overline{U}_1(x_1) \le \overline{U}_1(x) \le (1+\varepsilon)\overline{U}_1(x_1)$ (resp. $(1-\varepsilon)\underline{U}_1(x_1) \le \underline{U}_1(x) \le (1+\varepsilon)\underline{U}_1(x_1)$). Let us consider the open sets $A = \{x \in U; \overline{U}_1(x) < +\infty\}$ and $B = \{x \in U; \overline{U}_1(x) = +\infty\}$. By the above inequality and the connectedness of U, we find that $U = A$. The continuity of $\overline{U}_1$ is proved as follows: by the previous inequality, we obtain $|\overline{U}_1(x) - \overline{U}_1(x_1)| \le \varepsilon \overline{U}_1(x_1)$; we note also that $x_1 \in A$. (We apply the same arguments for the finiteness and the continuity of $\underline{U}_1$.)

It remains to show that $\overline{U}_1 > 0$ and $\underline{U}_1 > 0$ in U. The first assertion is obvious; for the second one, we see that $(\inf\{u_1; (u_1, u_2) \in \Phi_{x_0}\})\hat{} = \underline{U}_1$. As the pair $((\inf\{u_1; (u_1, u_2) \in \Phi_{x_0}\})\hat{}, (\inf\{u_2; (u_1, u_2) \in \Phi_{x_0}\})\hat{})$ is superharmonic in U and $\underline{U}_1(x_0) = 1$, then $\underline{U}_1 > 0$ in U; see [3, Proposition 2.2]. □

Corollary 1.4. *Let* $x', x'' \in K$ *compact set* $\subset U$ *and* $(u_1, u_2) \in \mathcal{H}_+(U)$ *with* $u_1 > 0$. *Then there exist two real numbers* $\alpha > 0$, $\beta > 0$ *such that* $\alpha \le u_1(x')/u_1(x'') \le \beta$ *independently of* u_1 *and of the points* x', x''.

Proof: We apply the previous corollary on the first components of the biharmonic pairs $(u_1/u_1(x_0), u_2/u_1(x_0))$, where x_0 is a fixed point of U. Next, we use analogous arguments as in the harmonic case [1]. $\square$

Remark 1.5. By Harnack's inequalities, we see that, for a point $x_0 \in U$, $\overline{U}_1(x_0) = 0$ (resp. $\underline{U}_1(x_0) = 0$) imply $\overline{U}_2(x_0) = 0$ (resp. $\underline{U}_2(x_0) = 0$) where $\overline{U}_j = \sup u_j$, $\underline{U}_j = \inf u_j$ $(j = 1, 2)$ with $(u_1, u_2) \in \mathcal{H}_+(U)$.

Let us now recall some results:

Proposition 1.6 ([3, Théorème 2.9]). *Let $(\Omega, \mathcal{H})$ be an elliptic biharmonic space, $(h_1^n, h_2^n)_{n\in\mathbb{N}}$ an increasing sequence of biharmonic pairs in a domain $U \subset \Omega$ and $(h_1, h_2) = (\sup_n h_1^n, \sup_n h_2^n)$. Then we have three possibilities:*

1. *$(h_1, h_2) \in \mathcal{H}(U)$;*
2. *$(h_1, h_2) \equiv (+\infty, +\infty)$;*
3. *$h_1 \equiv +\infty$, $h_2 \in \mathcal{H}_2(U)$.*

Proposition 1.7 ([3, Proposition 2.11]). *Let $(\Omega, \mathcal{H})$ be an elliptic biharmonic space, ω an $\mathcal{H}$-regular domain and (f_1, f_2) be a couple of extended real-valued functions on $\partial\omega$ such that $\int^* f_1\, d\mu_x^\omega + \int^* f_2\, d\nu_x^\omega$, $\int_* f_1\, d\mu_x^\omega + \int_* f_2\, d\nu_x^\omega$ are well defined for $x \in \omega$. If f_1 is $\mu_{x_0}^\omega$-summable and f_2 is $\nu_{x_0}^\omega$-summable (x_0 is a fixed point of ω), then f_1 is μ_x^ω-summable and f_2 is ν_x^ω and λ_x^ω-summable for every $x \in \omega$; moreover, in this case, the pair $(\int f_1\, d\mu_x^\omega + \int f_2\, d\nu_x^\omega, \int f_2\, d\lambda_x^\omega)$ is biharmonic in ω.*

Proposition 1.8. *Let $(\Omega, \mathcal{H})$ be an elliptic biharmonic space, U a domain of Ω, K a compact set $\subset U$, $x_0 \in U$. Then, for every pair $(u_1, u_2) \in \mathcal{H}_+(U)$ we have the (Harnack's) inequalities:*

1. *$\sup u_1(K) \le \alpha u_1(x_0)$,*
2. *$\sup u_2(K) \le \alpha u_j(x_0)$ $(j = 1, 2)$*

where $\alpha = \alpha(K, x_0)$ is a positive constant.

This result improves Théorème 2.13 of [3] (see also [4, p. 109]); its proof follows from the same arguments.

Theorem 1.9. *The following results are equivalent.*

(i) *Proposition* 2.2 *from* [3] *and Theorem* 1.1.
(ii) *Proposition* 1.6.
(iii) *Proposition* 1.7.
(iv) *Proposition* 1.8.

Proof: (i) $\Rightarrow$ (ii): Let $(h_1^n, h_2^n)_{n\in\mathbb{N}}$ be an increasing sequence of biharmonic pairs with the upper envelope (h_1, h_2) in a domain $U \subset \Omega$; we can suppose that $(h_1^n, h_2^n) \ge (0, 0)$. By Corollary 1.4, we have $h_1^n(x') \le \beta h_1^n(x'')$ with $n \in \mathbb{N}$ and $h_1(x') \le \beta h_1(x'')$; therefore, if $h_1(x') = +\infty$ then $h_1 \equiv +\infty$ and if $h_1(x'') < +\infty$ then $h_1 < +\infty$ in U and the continuity of h_1 follows from the local uniform convergence. The corresponding harmonic result applied to the second components of pairs gives us either $h_2 \equiv +\infty$ or $h_2 < +\infty$ and h_2 continuous in U. The possibility $h_1 < +\infty$, $h_2 \equiv +\infty$ in U does not occur. Indeed, in any $\mathcal{H}$-regular open set $\omega \subset \bar{\omega} \subset U$, we have, for $x \in \omega$, $h_1(x) = \int h_1\, d\mu_x^\omega + \int h_2\, d\nu_x^\omega$; the fact that $T\nu_x^\omega \neq \emptyset$ [3, Lemma 2.4, Proposition 2.5], leads to a contradiction.

(ii) $\Rightarrow$ (iv): This implication follows from the proof of Proposition 1.8.

(iv) $\Rightarrow$ (ii): We may assume that $(h_1^n, h_2^n)_{n \in \mathbb{N}}$ is an increasing sequence of positive biharmonic pairs. Then, if K is any compact subset of the domain U and if x_0 is a fixed point of U, we have:

$$\sup h_1^n(K) \leq \alpha h_1^n(x_0),$$
$$\sup h_2^n(K) \leq \alpha h_j(x_0)$$

where $n \in \mathbb{N}$ and $j = 1, 2$. Now we proceed as in the implication (i) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (iii): The elliptic version of Théorème 1.33 of [3] (see also [3, Proposition 2.11]) gives us the proof.

(iv) $\Rightarrow$ (i): Having first shown the equicontinuity of pairs of type $(h_1, 0)$ and of the second components of the pairs (u_1, u_2) (see [1, p. 14–24], we use the same arguments as in the proof of Theorem 1.1 to prove the part "(iv) $\Rightarrow$ Theorem 1.1"; the part "(iv) $\Rightarrow$ Proposition 2.2 of [3]" follows from Harnack's inequalities (1) and (2) of Proposition 1.8.

$\square$

References

[1] Brelot M., *Axiomatique des fonctions harmoniques*, Les Presses de l'Université de Montréal, 1966.

[2] Smyrnelis E.P., *Axiomatique d'un problème de Dirichlet dans les espaces biharmoniques*, C.R. Acad. Sc. Paris, t. 274, Série A, 1972, 1897–1900.

[3] ———, *Axiomatique des fonctions biharmoniques*, Ann. Inst. Fourier, 25 (1), 35–97, 1975; 26 (3), 1–47, 1976.

[4] ———, *Une propriété de moyenne des fonctions biharmoniques*, Bull. Sc. math., 2ème série, 109, 1985, 103–111, 446.

Equipe d'Analyse (tour 46), Université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France

(*Received* December 6, 1991)