On the Novak number of a hyperspace

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Abstract. An estimate for the Novak number of a hyperspace with the Vietoris topology is given. As a consequence it is shown that this cardinal function can decrease passing from a space to its hyperspace.

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The motivation for this paper comes from a question posed in [3]. There it was proved (relatively to the locally finite topology, but the same reasoning applies to the Vietoris topology) that for any dense in itself T_1 space X the Novak number of $exp\,(X)$ is not greater than that of X. The point left open was whether the two cardinal numbers can actually be different.

Here we present an estimate for the Novak number of a hyperspace in terms of the netweight of the base space. Using this theorem we give a couple of examples showing that the Novak number can actually decrease passing to the hyperspace. This fact is rather surprising considering the behaviour of practically all other cardinal functions.

Given a topological space X, the hyperspace $exp\,(X)$ is the set of all non empty closed subsets of X.

If $\mathcal{S}$ is a family of subsets of X then the symbol $\langle\mathcal{S}\rangle$ denotes the set of all $A \in exp\,(X)$ satisfying $A \subset \cup\mathcal{S}$ and $A \cap S \neq \emptyset$ for every $S \in \mathcal{S}$.

The Vietoris topology on $exp\,(X)$ is defined by taking as a base all the sets of the form $\langle U_1, \dots , U_n\rangle$, where $U_1, \dots , U_n$ are open subsets of X. For more information on the Vietoris topology the reader is referred to [5].

Notice that if $C_1, \dots , C_n$ are closed subsets of X then the set $\langle C_1, \dots , C_n\rangle = exp\,(X) \setminus (\langle X \setminus C_1\rangle \cup \dots \cup \langle X \setminus C_n\rangle \cup \langle X, X \setminus (C_1 \cup \dots \cup C_n)\rangle)$ is closed in $exp\,(X)$.

A network $\mathcal{B}$ for the topological space X is a family of subsets having the property that for any open set $U \subset X$ and any $x \in U$ there exists a member $B \in \mathcal{B}$ such that $x \in B \subset U$.

The netweight of the space X, denoted by $nw(X)$, is the smallest cardinality of a network for X.

Given a dense in itself T_1 space X, the Novak number of X, denoted by $n(X)$, is the smallest cardinality of a cover of X consisting of nowhere dense sets.

More details on the Novak number can be found in [1] and [2] and the bibliography listed there.

Theorem 1. *If X is a dense in itself regular T_1 space then $n(exp\,(X)) \leq nw(X)^{\aleph_0}$.*

Proof: Let $\mathcal{B}$ be a network of X satisfying $|\mathcal{B}| = nw(X)$. As the space X is regular, we can assume that $\mathcal{B}$ consists of closed sets. Denote by $\mathcal{B}_1$ the collection of all countable infinite subsets of $\mathcal{B}$ consisting of pairwise disjoint elements. For any $\mathcal{C} = \{C_1, \dots, C_n, \dots\} \in \mathcal{B}_1$ let

$$F_{\mathcal{C}} = \cap_{n \in N^+} \langle X, C_n \rangle.$$

Since every C_n is closed, it follows that also $F_{\mathcal{C}}$ is closed. Moreover, it is clear that every point in $F_{\mathcal{C}}$ is an infinite subset of X. On the other hand, every basic open set in $exp\,(X)$ contains finite subsets of X and therefore it follows that each $F_{\mathcal{C}}$ is nowhere dense. We claim that every infinite closed subset A of X is contained in some $F_{\mathcal{C}}$. To see this, let us begin by taking two disjoint elements $C', C'' \in \mathcal{B}$ such that $C' \cap A \neq \emptyset \neq C'' \cap A$. At least one of these two sets, say C', satisfies $|A \setminus C'| \geq \aleph_0$. Let $C_1 = C'$ and suppose we have already chosen pairwise disjoint sets $C_1, \dots, C_n \in \mathcal{B}$ in such a way that $(*)$ $C_i \cap A \neq \emptyset$ for $i \leq n$ and $|A \setminus (C_1 \cup \dots \cup C_n)| \geq \aleph_0$. Then we proceed by induction selecting $C_{n+1} \in \mathcal{B}$ disjoint from $C_1, \dots, C_n$ and satisfying $(*)$. Letting $\mathcal{C} = \{C_1, \dots, C_n, \dots\}$ it is clear that $A \in F_{\mathcal{C}}$. Now let F_n be the subset of $exp\,(X)$ consisting of all subsets of X having cardinality not bigger than n. Since X is dense in itself and T_2, it is not difficult to see that F_n is closed and nowhere dense in $exp\,(X)$. To finish, observe that $\{F_n : n \in N^+\} \cup \{F_{\mathcal{C}} : \mathcal{C} \in \mathcal{B}_1\}$ is a nowhere dense cover of $exp\,(X)$ of cardinality not exceeding $nw(X)^{\aleph_0}$. □

In order to obtain our first example, we recall the construction of a certain linearly ordered topological group.

For any ordinal ν denote by $\Re^\nu$ the set of all functions $f : \nu \to \Re$ ordered lexicographically, that is $f < g$ if and only if $f \neq g$ and $f(\alpha) < g(\alpha)$, where $\alpha = \min\{\beta : f(\beta) \neq g(\beta)\}$. The order so defined is actually a linear order and $\Re^\nu$ can be equipped with the standard interval topology. If, moreover, we define $f + g$ by the rule $(f+g)(\alpha) = f(\alpha) + g(\alpha)$ then $\Re^\nu$ becomes a linearly ordered topological abelian group.

For any $\alpha \in \nu$ denote by ε_α the element of $\Re^\nu$ defined by $\varepsilon_\alpha(\alpha) = 1$ and $\varepsilon_\alpha(\beta) = 0$ if $\beta \neq \alpha$.

Let $\Re^{<\nu} = \cup_{\alpha \in \nu} \Re^\alpha$ and for any $\varphi \in \Re^{<\nu}$ denote by $||\varphi||$ the ordinal α such that $\varphi \in \Re^\alpha$. Let $[\varphi] = \{f \in \Re^\nu : f \mid \alpha = \varphi\}$. Observe that $[\psi] \subset [\varphi]$ if and only if $\varphi \subset \psi$.

Each $[\varphi]$ is open in $\Re^\nu$, in fact if $\varphi \in \Re^\alpha$ and $f \in [\varphi]$ then the interval $(f - \varepsilon_{\alpha+1}, f + \varepsilon_{\alpha+1})$ is contained in $[\varphi]$. Furthermore, the collection of all sets of the form $[\varphi]$ is a base for the topology of $\Re^\nu$. To see this, fix an interval (f, g) and an element $h \in (f, g)$ and let $\alpha_1 = \min\{\beta : f(\beta) \neq h(\beta)\}$ and $\alpha_2 = \min\{\beta : h(\beta) \neq g(\beta)\}$. If $\alpha = \max\{\alpha_1, \alpha_2\} + 1$ then it is easily seen that $h \in [h \mid \alpha] \subset (f, g)$.

The next proposition is somewhat related to a result of Sikorski ([6, 4.15]).

Proposition 1. *If ν is a regular cardinal then $n(\Re^\nu) > \nu$.*

Proof: It is enough to show that any family $\{A_\alpha : \alpha \in \nu\}$ of dense open subsets of $\Re^\nu$ has a non empty intersection. We construct by induction the sequence

$\{\varphi_\alpha : \alpha \in \nu\} \subset \Re^{<\nu}$ satisfying the condition

$$[\varphi_\alpha] \subset A_\alpha \cap (\cap_{\beta\in\alpha}[\varphi_\beta]).$$

Assume that the family $\{\varphi_\beta : \beta \in \alpha\}$ has already been constructed. Since ν is regular, the set $\{||\varphi_\beta|| : \beta \in \alpha\}$ is bounded in ν. Thus the function $\psi = \cup\{\varphi_\beta : \beta \in \alpha\}$ is a member of $\Re^{<\nu}$. To finish the induction, select $\varphi_\alpha \in \Re^{<\nu}$ in such a way that $[\varphi_\alpha] \subset A_\alpha \cap [\psi]$. Now let $\varphi = \cup\{\varphi_\alpha : \alpha \in \nu\}$. If φ is defined on all ν then $\varphi \in \Re^\nu$ and $\varphi \in \cap_{\alpha\in\nu}A_\alpha$. If, on the other hand, $\|\varphi\| < \nu$ then any $f \in [\varphi]$ belongs to $\cap_{\alpha\in\nu}A_\alpha$. $\square$

Recall that assuming Martin's axiom (MA) the cardinality of the continuum $\underline{c}$ is regular and, for every $\kappa < \underline{c}$, $2^\kappa \leq \underline{c}$ (see [4, Section 2.2]). Taking this into account, we see that the space $\Re^{\underline{c}}$ has a base (and a fortiori a network) of cardinality not exceeding $|\Re^{<\underline{c}}| = \sum_{\alpha\in\underline{c}} 2^{\aleph_0|\alpha|} = \underline{c}$. Thus from Theorem 1 and Proposition 1 we get:

Corollary 1 (MA)**.** *There exists a linearly ordered topological group X, namely $\Re^{\underline{c}}$, for which $n(exp\,(X)) < n(X)$.*

Under more complicate set-theoretic assumptions, it is possible to find a compact space for which the Novak number decreases passing to the hyperspace. Indeed Theorem 5.2 in [1] describes two models of ZFC in which the Novak number of N^*, the Čech-Stone remainder of N, is greater than $\underline{c}$. Taking into account that N^* is a compact space of netweight $\underline{c}$, another application of Theorem 1 gives:

Corollary 2. *It is consistent with ZFC the existence of a compact space X, namely N^*, for which $n(exp\,(X)) < n(X)$.*

Observe that, since for any compact space the locally finite topology coincides with the Vietoris topology, Corollary 2 also furnishes a direct answer to the question posed in [3].

We do not know whether the inequality $n(X) \leq 2^{n(exp\,(X))}$ holds for every dense in itself T_1 space X. Certainly, however, it cannot be improved. In fact, in one of the models described in [1], it is true that $n(N^*) = 2^{\underline{c}} = 2^{n(exp\,(N^*))}$.

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