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The product of distributions on R^m

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Abstract. The fixed infinitely differentiable function $\rho(x)$ is such that $\{n\rho(nx)\}$ is a regular sequence converging to the Dirac delta function δ. The function $\delta_{\mathbf{n}}(\mathbf{x})$, with $\mathbf{x} = (x_1, \dots, x_m)$ is defined by

$$\delta_{\mathbf{n}}(\mathbf{x}) = n_1\rho(n_1x_1)\dots n_m\rho(n_mx_m).$$

The product $f \circ g$ of two distributions f and g in $\mathcal{D}'_m$ is the distribution h defined by

$$\operatorname*{N-lim}_{n_1\to\infty}\dots\operatorname*{N-lim}_{n_m\to\infty}\langle f_{\mathbf{n}}g_{\mathbf{n}}, \phi\rangle = \langle h, \phi\rangle,$$

provided this neutrix limit exists for all $\phi(\mathbf{x}) = \phi_1(x_1)\dots\phi_m(x_m)$, where $f_{\mathbf{n}} = f * \delta_{\mathbf{n}}$ and $g_{\mathbf{n}} = g * \delta_{\mathbf{n}}$.

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A commutative product of two distributions in $\mathcal{D}'_m$, the space of distributions defined on $\mathcal{D}_m$, the space of infinitely differentiable functions in m variables with compact support, was considered in [1] and a non-commutative product of two distributions in $\mathcal{D}'_m$ was considered in [6]. In the following we are going to consider a commutative product of two distributions in $\mathcal{D}'_m$ which is similar to that given in [1] but simpler to deal with.

First of all we let ρ be a fixed infinitely differentiable function with the properties

(i) $\rho(x) = 0$, $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$. It is obvious that $\{\delta_n\}$ is a sequence of functions in $\mathcal{D}$ converging to the Dirac δ function δ.

For an arbitrary distribution f in $\mathcal{D}'$ the function f_n is defined by

$$f_n(x) = (f * \delta_n)(x) = \langle f(x - t), \delta_n(t)\rangle.$$

It follows that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f.

The following definition for the product of two distributions in $\mathcal{D}'$ was given in [3]:

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. The product $f \cdot g$ is said to exist and be equal to the distribution h on the open interval (a, b), where $-\infty \le a \le b \le \infty$, if and only if*

$$\langle f \cdot g, \phi \rangle = \lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathcal{D}(a, b)$.

This definition generalizes the usual definition of a product of a distribution and an infinitely differentiable function or of a product of a distribution and a sufficiently smooth function and is clearly commutative.

The next definition for the neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'$ was given in [5].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. The neutrix product $f \circ g$ of f and g is said to exist and be equal to h on the open interval (a, b), if and only if*

$$\mathrm{N}\!-\!\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathcal{D}(a, b)$, where N is the neutrix, see van der Corput [2], having domain $\mathrm{N}' = \{1, 2, \ldots, n, \ldots\}$ and the range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \ldots$ and all functions which converge to zero in the normal sense as n tends to infinity.

Note that if

$$\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}(a, b)$, the neutrix product $f \circ g$ reduces to the product $f \cdot g$ of Definition 1 and so Definition 2 is a generalization of Definition 1. It is clear that the neutrix product $f \circ g$ is commutative.

The following theorem holds, see [1].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the open interval (a, b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on this interval.

In order to define a neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'_m$, a δ-sequence in $\mathcal{D}_m$ was defined in [1] by

$$\delta_n(\mathbf{x}) = \delta_n(x_1, \ldots, x_m) = n^m \rho(n x_1) \ldots \rho(n x_m)$$

for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to δ in the sense that

$$\lim_{n\to\infty} \langle \delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle = \langle \delta(\mathbf{x}), \phi(\mathbf{x})\rangle = \phi(\mathbf{0})$$

for all test functions ϕ in $\mathcal{D}_m$.

In the following, we use an alternative definition of a δ-sequence, which is easier to work with. From now on the function $\delta_{\mathbf{n}}(\mathbf{x})$ will be defined by

$$\delta_{\mathbf{n}}(\mathbf{x}) = n_1\rho(n_1x_1)\ldots n_m\rho(n_mx_m)$$

for $n_1, \ldots, n_m = 1, 2, \ldots,$ where $\mathbf{n} = (n_1, \ldots, n_m)$. It is obvious that $\{\delta_{\mathbf{n}}\}$ is a sequence of infinitely differentiable functions converging to δ in the sense that

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \langle \delta_{\mathbf{n}}(\mathbf{x}), \phi(\mathbf{x})\rangle = \langle \delta(\mathbf{x}), \phi(\mathbf{x})\rangle = \phi(\mathbf{0})$$

for all test functions ϕ in $\mathcal{D}_m$, the result being independent of the order in which the limits are taken.

For an arbitrary distribution f in $\mathcal{D}'_m$ the function $f_{\mathbf{n}}$ is defined by

$$f_{\mathbf{n}}(\mathbf{x}) = (f * \delta_{\mathbf{n}})(\mathbf{x}) = \langle f(\mathbf{x} - \mathbf{t}), \delta_{\mathbf{n}}(\mathbf{t})\rangle,$$

where $\mathbf{t}$ is in R^m. It follows that $\{f_{\mathbf{n}}\}$ is a sequence of infinitely differentiable functions converging to f, in the sense that

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \langle f_{\mathbf{n}}(\mathbf{x}), \phi(\mathbf{x})\rangle = \langle f(\mathbf{x}), \phi(\mathbf{x})\rangle$$

for all ϕ in $\mathcal{D}_m$, the result again being independent of the order in which the limits are taken.

For our next definition and our main results we need the following lemmas, see Schwartz [7].

Lemma 1. *The vector space $\mathcal{X}_m$ generated by the functions $\phi_1(x_1)\ldots\phi_m(x_m)$, with $\phi_1, \ldots, \phi_m$ in $\mathcal{D}$, is dense in $\mathcal{D}_m$.*

Lemma 2. *The convolution product of two direct products $f_1(\mathbf{x}) \times g_1(\mathbf{y})$ and $f_2(\mathbf{x}) \times g_2(\mathbf{y})$ is equal to the direct product of the convolution products $f_1 * f_2$ and $g_1 * g_2$, if the convolution products $f_1 * f_2$ and $g_1 * g_2$ exist, where $f_1, f_2 \in \mathcal{D}'_m$ and $g_1, g_2 \in \mathcal{D}'_r$, i.e.*

$$(f_1 \times g_1) * (f_2 \times g_2) = (f_1 * f_2) \times (g_1 * g_2).$$

We also need the following lemma, see [4].

Lemma 3.

$$\int_t^{1/n} s^k \delta_n^{(q)}(s)\, ds = \sum_{i=0}^{k} \frac{(-1)^{k+i+1} k!}{i!} t^i \delta_n^{(q-k+i-1)}(t)$$

for $k = 0, 1, 2, \ldots, q-1$ *and* $q = 1, 2, \ldots$ *and*

$$\int_t^{1/n} s^q \delta_n^{(q)}(s)\, ds = \sum_{i=1}^{q} \frac{(-1)^{q+i+1} q!}{i!} t^i \delta_n^{(i-1)}(t) + (-1)^q q![1 - H_n(t)],$$

for $q = 1, 2, \ldots,$ *where*

$$H_n(t) = \int_{-1/n}^{t} \delta_n(s)\, ds.$$

The next definition is a generalization of Definition 2.

Definition 3. *Let* f *and* g *be distributions in* $\mathcal{D}'_m$ *and let* $f_{\mathbf{n}} = f * \delta_{\mathbf{n}}$ *and* $g_{\mathbf{n}} = g * \delta_{\mathbf{n}}$. *If* h *is a distribution in* $\mathcal{D}'_m$ *such that*

$$\operatorname*{N-lim}_{n_1 \to \infty} \ldots \operatorname*{N-lim}_{n_m \to \infty} \langle f_{\mathbf{n}} g_{\mathbf{n}}, \phi \rangle = \langle h, \phi \rangle,$$

or more briefly

$$\operatorname*{N-lim}_{\mathbf{n} \to \infty} \langle f_{\mathbf{n}} g_{\mathbf{n}}, \phi \rangle = \langle h, \phi \rangle$$

for all functions ϕ *in* $\mathcal{X}_m$ *with support contained in the interval* $(\mathbf{a}, \mathbf{b})$, *where* $\mathbf{a} = (a_1, \ldots, a_m)$ *and* $\mathbf{b} = (b_1, \ldots, b_m)$, *and* h *is independent of the order in which the limits are taken, we say that the neutrix product* $f \circ g$ *exists and is equal to* h *on* $(\mathbf{a}, \mathbf{b})$.

Note that if

$$\lim_{\mathbf{n} \to \infty} \langle f_{\mathbf{n}} g_{\mathbf{n}}, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{X}_m$, we simply say that the product $f \circ g = f \cdot g$ exists and equals h.

Note further that since $\mathcal{X}_m$ is dense in $\mathcal{D}_m$, the distribution h in this definition will be uniquely defined.

The proof of Theorem 1 can be modified to give the following theorem.

Theorem 2. *Let* f *and* g *be distributions in* $\mathcal{D}'_m$ *and suppose that the neutrix products* $f \circ g$ *and* $f \circ D_i g$ (*or* $D_i f \circ g$) *exist on the open interval* $(\mathbf{a}, \mathbf{b})$. *Then the neutrix product* $D_i f \circ g$ (*or* $f \circ D_i g$) *exists and*

$$D_i(f \circ g) = D_i f \circ g + f \circ D_i g$$

on this interval, where D_i *denotes the partial derivative with respect to* x_i.

Theorem 3. *Let f and g be distributions in $\mathcal{D}'_m$ such that*

$$f(\mathbf{x}) = f_1(x_1) \times \cdots \times f_m(x_m), \quad g(\mathbf{x}) = g_1(x_1) \times \cdots \times g_m(x_m),$$

with $f_1, \dots, f_m, g_1, \dots, g_m \in \mathcal{D}'$, and suppose that the neutrix products $f_1 \circ g_1, \dots, f_m \circ g_m$ exist and equal $h_1, \dots, h_m$ respectively. Then the neutrix product $f \circ g$ exists and

$$f \circ g = h_1 \times \cdots \times h_m.$$

In particular, if the products $f_1 \cdot g_1, \dots, f_m \cdot g_m$ exist, then the product $f \cdot g$ exists and is equal to $h_1 \times \cdots \times h_m$.

Proof: Putting

$$f_{in_i}(x_i) = f_i(x_i) * \delta_{n_i}(x_i), \quad g_{in_i}(x_i) = g_i(x_i) * \delta_{n_i}(x_i),$$

for $i = 1, \dots, m$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{1n_1}(x_1) \times \cdots \times f_{mn_m}(x_m) = f(\mathbf{x}) * \delta_{\mathbf{n}}(\mathbf{x}),$$
$$g_{\mathbf{n}}(\mathbf{x}) = g_{1n_1}(x_1) \times \cdots \times g_{mn_m}(x_m) = g(\mathbf{x}) * \delta_{\mathbf{n}}(\mathbf{x}),$$

we have on applying Lemma 2

$$\langle f_{\mathbf{n}}(\mathbf{x}) g_{\mathbf{n}}(\mathbf{x}), \phi_1(x_1) \dots \phi_m(x_m) \rangle = \prod_{i=1}^{m} \langle f_{in_i}(x_i), g_{in_i}(x_i) \phi_i(x_i) \rangle$$

for all $\phi_1, \dots, \phi_m$. Now since the neutrix product $f_i \circ g_i$ exists and equals h_i, it follows that

$$\begin{aligned}
\operatorname*{N-lim}_{\mathbf{n}\to\infty} \langle f_{\mathbf{n}}(\mathbf{x}) g_{\mathbf{n}}(\mathbf{x}), \phi_i(x_i) \dots \phi_m(x_m) \rangle &= \prod_{i=1}^{m} \operatorname*{N-lim}_{\mathbf{n}\to\infty} \langle f_{in_i}(x_i), g_{in_i}(x_i) \phi_i(x_i) \rangle \\
&= \prod_{i=1}^{m} \operatorname*{N-lim}_{n_i\to\infty} \langle h_i, \phi_i \rangle \\
&= \langle h_1 \times \cdots \times h_m, \phi_1 \dots \phi_m \rangle.
\end{aligned}$$

The result of the theorem follows. □

If now

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m), \quad \lambda_1, \dots, \lambda_m \neq 0, \pm 1, \pm 2, \dots,$$
$$\mathbf{r} = (r_1, \dots, r_m), \quad r_1, \dots, r_m = 0, 1, 2, \dots,$$

we define

$$\operatorname{cosec}(\pi\boldsymbol{\lambda}) = \operatorname{cosec}(\pi\lambda_1) \dots \operatorname{cosec}(\pi\lambda_m),$$
$$(-1)^{\mathbf{r}} = (-1)^{r_1+\cdots+r_m}, \quad \mathbf{r}! = r_1! \dots r_m!,$$
$$\mathbf{x}_+^{\boldsymbol{\lambda}} = (x_1)_+^{\lambda_1} \times \cdots \times (x_m)_+^{\lambda_m}, \quad \mathbf{x}_-^{\boldsymbol{\lambda}} = (-\mathbf{x})_+^{\boldsymbol{\lambda}},$$
$$\mathbf{x}_+^{\mathbf{r}} = (x_1)_+^{r_1} \times \cdots \times (x_m)_+^{r_m}, \quad \mathbf{x}_-^{\mathbf{r}} = (-\mathbf{x})_+^{\mathbf{r}},$$
$$\delta^{(\mathbf{r})}(\mathbf{x}) = \delta^{(r_1)}(x_1) \times \cdots \times \delta^{(r_m)}(x_m).$$

We then have

Theorem 4. *The neutrix products* $\mathbf{x}_+^{\boldsymbol{\lambda}} \circ \mathbf{x}_-^{-\boldsymbol{\lambda}-\mathbf{r}}$ *and* $\mathbf{x}_-^{-\boldsymbol{\lambda}-\mathbf{r}} \circ \mathbf{x}_+^{\boldsymbol{\lambda}}$ *exist in* $\mathcal{D}'_m$ *and*

$$\mathbf{x}_+^{\boldsymbol{\lambda}} \circ \mathbf{x}_-^{-\boldsymbol{\lambda}-\mathbf{r}} = \mathbf{x}_-^{-\boldsymbol{\lambda}-\mathbf{r}} \circ \mathbf{x}_+^{\boldsymbol{\lambda}} = \frac{(-\pi)^m \operatorname{cosec}(\pi\boldsymbol{\lambda})}{2^m(\mathbf{r}-\mathbf{1})!}\delta^{(\mathbf{r}-\mathbf{1})}(\mathbf{x}),$$

for $\lambda_1,\ldots,\lambda_m \neq \pm1,\pm2,\ldots$ *and* $r_1,\ldots,r_m = 1,2,\ldots,$ *where*

$$\mathbf{r}-\mathbf{1} = (r_1-1,\ldots,r_m-1).$$

In particular, the products $\mathbf{x}_+^{\boldsymbol{\lambda}} \cdot \mathbf{x}_-^{-\boldsymbol{\lambda}-1}$ *and* $\mathbf{x}_-^{-\boldsymbol{\lambda}-1} \cdot \mathbf{x}_+^{\boldsymbol{\lambda}}$ *exist in* $\mathcal{D}'_m$ *for* $\lambda_1,\ldots,\lambda_m \neq 0,\pm1,\pm2,\ldots$.

Proof: In the one variable case, suppose first of all that $\lambda > -1$ and choose a non-negative integer q such that $-\lambda - r + q > -1$. Then

$$(x_+^\lambda)_n = x_+^\lambda * \delta_n = \int_{-1/n}^{x} (x-t)^\lambda \delta_n(t)\,dt,$$

$$(x_-^{-\lambda-r})_n = x_-^{-\lambda-r} * \delta_n = \frac{\Gamma(\lambda+r-q)}{\Gamma(\lambda+r)} \int_x^{1/n} (s-x)^{-\lambda-r+q}\delta_n^{(q)}(s)\,ds,$$

where Γ denotes the Gamma function. The support of $(x_+^\lambda)_n(x_-^{-\lambda-r})_n$ is clearly contained in the interval $(-1/n,1/n)$ and it follows that

$$\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+r-q)} \int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{-\lambda-r})_n x^k\,dx =$$

$$= \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} \delta_n^{(q)}(s) \int_t^s x^k(x-t)^\lambda(s-x)^{-\lambda-r+q}\,dx\,ds\,dt = \tag{1}$$

$$= n^{r-k-1}\int_{-1}^{1} \rho(u) \int_u^1 \rho^{(q)}(v) \int_u^v w^k(w-u)^\lambda(v-w)^{-\lambda-r+q}\,dw\,dv\,du,$$

where the substitutions $nt=u$, $ns=v$ and $nx=w$ have been made. Thus

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{-\lambda-r})_n x^k\,dx = 0 \tag{2}$$

for $k=0,1,2,\ldots,r-2$ and

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} \left|(x_+^\lambda)_n(x_-^{-\lambda-r})_n x^r\right|dx = 0 \tag{3}$$

In the particular case $k = r-1$, we have on making the substitution $x = t(1-y)+sy$

(4)

$$\int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} \delta_n^{(q)}(s) \int_t^s x_{r-1}(x-t)^\lambda(s-x)^{-\lambda-r+q}\,dx\,ds\,dt =$$

$$= \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} \delta_n^{(q)}(s) \int_0^1 (s-t)^{q-r+1}[t(1-y)+sy]^{r-1}y^\lambda(1-y)^{-\lambda-r+q}\,dy\,ds\,dt.$$

On expanding $(s-t)^{q-r+1}$ and $[t(1-y)+sy]^{r-1}$ in powers of s and t, it follows that this integral is a linear sum of integrals of the form

$$\int_{-1/n}^{1/n} t^{q-k}\delta_n(t)\int_t^{1/n} s^k\delta_n^{(q)}(s)\,ds\,dt$$

for $k=0,1,\dots q$.

On using Lemma 3, we see that when $k<q$ each of these integrals is a linear sum of integrals of the form

$$\int_{-1/n}^{1/n} t^{q-k+1}\delta_n(t)\delta_n^{(q-k+i-1)}(t)\,dt=0,$$

since the integrands are all odd functions.

When $k=q$ we have on using Lemma 3 again

$$\begin{aligned}\int_{-1/n}^{1/n}\delta_n(t)\int_t^{1/n} s^q\delta_n^{(q)}(s)\,ds\,dt &= \sum_{i=1}^{q}\frac{(-1)^{q+i+1}q!}{i!}\int_{-1/n}^{1/n} t^i\delta_n(t)\delta_n^{(i-1)}(t)\,dt+\\ &\quad+(-1)^q q!\int_{-1/n}^{1/n}[1-H_n(t)]\delta_n(t)\,dt\\ &=0+\frac{(-1)^q q!}{2}.\end{aligned}$$

It now follows from the equations (1) and (4) that

$$\begin{aligned}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+r-q)}\int_{-1/n}^{1/n}(x_+^\lambda)_n(x_-^{-\lambda-r})_n x^{r-1}\,dx &= \frac{(-1)^q q!}{2}\int_o^1 y^{\lambda+r-1}(1-y)^{-\lambda-r+q}\,dy\\ &=\frac{(-1)^q q!}{2}B(\lambda+r,-\lambda-r+q+1)\\ &=(-1)^q q!\Gamma(\lambda+r)\Gamma(-\lambda-r+q+1)/2\end{aligned}$$

for $\lambda\neq 0,1,2,\dots$, where B denotes the Beta function, and so

$$\begin{aligned}\int_{-1/n}^{1/n}(x_+^\lambda)_n(x_-^{-\lambda-r})_n x^{r-1}\,dx &= (-1)^q\Gamma(\lambda+r-q)\Gamma(-\lambda-r+q+1)/2\\ &=(-1)^q\pi\operatorname{cosec}\pi(\lambda+r-q)/2\\ &=(-1)^r\pi\operatorname{cosec}(\pi\lambda)/2.\end{aligned}\tag{5}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then we can write

$$\phi(x)=\sum_{k=0}^{r-1}\frac{\phi^{(k)}(0)}{k!}x^k+\frac{\phi^{(r)}(\xi x)}{r!}x^r,$$

where $0 \le \xi \le 1$. Thus

$$\langle (x_+^\lambda)_n (x_-^{-\lambda-r})_n, \phi(x)\rangle = \sum_{k=0}^{r-1} \frac{\phi^{(k)}(0)}{k!} \int_{-1/n}^{1/n} (x_+^\lambda)_n (x_-^{-\lambda-r})_n x^k \, dx + \\ + \frac{1}{r!}\phi^{(r)}(\xi x)(x_+^\lambda)_n (x_-^{-\lambda-r})_n x^r \, dx$$

and it follows from the equations (2), (3) and (5) that

$$\operatorname*{N-lim}_{n\to\infty} \langle (x_+^\lambda)_n (x_-^{-\lambda-r})_n, \phi(x)\rangle = \frac{(-1)^r \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\phi^{(r-1)}(0),$$

proving that

$$x_+^\lambda \circ x_-^{-\lambda-r} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x) \tag{6}$$

for $\lambda > -1$, $\lambda \ne 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Note that in the case $r = 1$ the neutrix limit is not needed and so the product $x_+^\lambda \cdot x_-^{-\lambda-r}$ exists in this case.

Also note that in the case $r = 0$, the above proof shows that the product $x_+^\lambda \cdot x_-^{-\lambda}$ exists and

$$x_+^\lambda \cdot x_-^{-\lambda} = 0 \tag{7}$$

for $\lambda > -1$ and $\lambda \ne 0, 1, 2, \dots$.

A routine induction proof using the equations (6) and (7) and Theorem 2 now shows that equation (6) holds for $\lambda \ne 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$, the product existing in the case $r = 1$.

Replacing x by $-x$ and λ by $-\lambda - r$ in the equation (6) proves that

$$x_-^{-\lambda-r} \circ x_+^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x) \tag{8}$$

for $\lambda \ne 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$, the product existing in the case $r = 1$.

The results of the theorem now follows immediately on using Theorem 3 and the equations (6) and (8). □

Theorem 5. *The neutrix product* $\mathbf{x}_+^{\mathbf{r}} \circ \delta^{(\mathbf{r}+\mathbf{p})}(\mathbf{x})$ *exists in* $\mathcal{D}'_m$ *and*

$$\mathbf{x}_+^{\mathbf{r}} \circ \delta^{(\mathbf{r}+\mathbf{p})}(\mathbf{x}) = \frac{(-1)^{\mathbf{r}}(\mathbf{r}+\mathbf{p})!}{2^m \mathbf{p}!}\delta^{(\mathbf{p})}(\mathbf{x}), \tag{9}$$

for $r_1, p_1, \dots, r_m, p_m = 0, 1, 2, \dots$. *In particular, the product* $\mathbf{x}_+^{\mathbf{r}} \cdot \delta^{(\mathbf{r})}(\mathbf{x})$ *exists in* $\mathcal{D}'_m$ *for* $r_1, \dots, r_m = 0, 1, 2, \dots$.

Proof: In the one variable case we have

$$(x_+^r)_n = \int_{-1/n}^{x} (x-t)^r \delta_n(t)\, dt.$$

The support of $(x_+^r)_n\delta_n^{(r+p)}$ is clearly contained in the interval $(-1/n, 1/n)$ and it follows that

$$\begin{aligned}\int_{-1/n}^{1/n}(x_+^r)_n\delta_n^{(r+p)}(x)x^k\,dx &= \int_{-1/n}^{1/n}\delta_n(t)\int_t^{1/n}x^k(x-t)^r\delta_n^{(r+p)}(x)\,dx\,dt \\ &= n^{p-k}\int_{-1}^{1}\rho(u)\int_u^1 v^k(v-u)^r\rho^{(r+p)}(v)\,dv\,du,\end{aligned}\tag{10}$$

where the substitutions $nt = u$ and $nx = v$ have been made. Thus

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1/n}^{1/n}(x_+^r)_n\delta_n^{(r+p)}(x)x^k\,dx = 0\tag{11}$$

for $k = 0, 1, 2, \ldots, p-1$ and

$$\lim_{n\to\infty}\int_{-1/n}^{1/n}\left|(x_+^r)_n\delta_n^{(r+p)}(x)x^{p+1}\right|dx = 0\tag{12}$$

In this particular case $k = p$ we have from the equation (10)

$$\begin{aligned}\int_{-1/n}^{1/n}(x_+^r)_n\delta_n^{(r+p)}(x)x^p\,dx &= \int_{-1/n}^{1/n}\delta_n(t)\int_t^{1/n}x^p(x-t)^r\delta_n^{(r+p)}(x)\,dx\,dt \\ &= \int_{-1/n}^{1/n}\delta_n(t)\int_t^{1/n}x^{r+p}\delta_n^{(r+p)}(x)\,dx\,dt,\end{aligned}$$

all other integrals in the sum, obtained by expanding $(x-t)^r$ by the binomial theorem, being zero by Lemma 3. On using Lemma 3 again, it now follows that

$$\begin{aligned}\int_{-1/n}^{1/n}(x_+^r)_n\delta_n^{(r+p)}(x)x^p\,dx &= (-1)^{r+p}(r+p)!\int_{-1/n}^{1/n}\delta_n(t)[1-H_n(t)]\,dt \\ &= (-1)^{r+p}(r+p)!/2,\end{aligned}\tag{13}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then on using the equations (11), (12) and (13), it follows as in the proof of Theorem 4 that

$$\operatorname*{N-lim}_{n\to\infty}\langle(x_+^r)_n\delta_n^{(r+p)}(x)\phi(x)\,dx\rangle = \frac{(-1)^{r+p}(r+p)!}{2p!}\phi^{(p)}(0),$$

proving that

$$x_+^r\delta^{(r+p)}(x) = \frac{(-1)^r(r+p)!}{2p!}\delta^{(p)}(x)\tag{14}$$

for $r, p = 0, 1, 2, \ldots$. Note that in the case $p = 0$ the neutrix limit is not needed and so the product $x_+^r \cdot \delta^{(r)}(x)$ exists in this case.

The result of the theorem now follows on using Theorem 3 and the equation (14). □

Corollary. *The neutrix product* $\mathbf{x}_-^{\mathbf{r}} \circ \delta^{(\mathbf{r}+\mathbf{p})}(\mathbf{x})$ *exists in* $\mathcal{D}'_m$ *and*

$$\mathbf{x}_-^{\mathbf{r}} \circ \delta^{(\mathbf{r}+\mathbf{p})}(\mathbf{x}) = \frac{(-1)^{\mathbf{r}}(\mathbf{r}+\mathbf{p})!}{2^m \mathbf{p}!}\delta^{(\mathbf{p})}(\mathbf{x}),$$

for $r_1, p_1, \dots, r_m, p_m = 0, 1, 2, \dots$ *. In particular, the product* $\mathbf{x}_-^{\mathbf{r}} \circ \delta^{(\mathbf{r})}(\mathbf{x})$ *exists in* $\mathcal{D}'_m$ *for* $r_1, \dots, r_m = 0, 1, 2, \dots$ *.*

Proof: The result follows immediately on replacing x by $-x$ in the equation (9). □

Theorem 6. *The neutrix product* $\delta^{(\mathbf{r})}(\mathbf{x}) \circ \delta^{(\mathbf{p})}(\mathbf{x})$ *exists and*

$$\delta^{(\mathbf{r})}(\mathbf{x}) \circ \delta^{(\mathbf{p})}(\mathbf{x}) = \mathbf{0}$$

for $r_1, p_1, \dots, r_m, p_m = 0, 1, 2, \dots$ *.*

Proof: It follows from the equation (14) with $r = 0$ that

$$x_+^0 \circ \delta^{(p)}(x) = \frac{1}{2}\delta^{(p)}(x)$$

for $p = 0, 1, 2, \dots$. Using Theorem 1, it follows that

$$\delta(x) \circ \delta^{(p)}(x) = \frac{1}{2}\delta^{(p+1)}(x) - x_+^0 \delta^{(p+1)}(x) = 0$$

for $p = 0, 1, 2, \dots$. It can now be proved easily by induction that

$$\delta^{(r)}(x) \circ \delta^{(p)}(x) = 0$$

for $p = 0, 1, 2, \dots$. The result of the theorem follows on using Theorem 3. □

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