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Characteristic of convexity of Musielak-Orlicz function spaces equipped with the Luxemburg norm

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Abstract. In this paper we extend the result of [6] on the characteristic of convexity of Orlicz spaces to the more general case of Musielak-Orlicz spaces over a non-atomic measure space. Namely, the characteristic of convexity of these spaces is computed whenever the Musielak-Orlicz functions are strictly convex.

Keywords: Musielak-Orlicz space, modulus of convexity, characteristic of convexity, the Δ_2-condition

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In the sequel, (S, Σ, μ) denotes a non-atomic σ-finite measure space and Φ denotes a Musielak-Orlicz function, i.e. a function from $S \times \mathbb{R}$ into $\mathbb{R}_+$ satisfying the Carathéodory conditions which means that $\Phi(s, \cdot)$ is convex, even, continuous, and vanishing at 0, left continuous on the whole $\mathbb{R}_+$ and not identically equal to 0 for μ-a.e. $s \in S$ and $\Phi(\cdot, u)$ is a Σ-measurable function for every $u \in \mathbb{R}$. For any $A \in \Sigma$, 1_A denotes the characteristic function of A.

The Musielak-Orlicz space $L^\Phi = L^\Phi(\mu)$ is defined to be the space of all (equivalence classes of) Σ-measurable functions $x : S \to \mathbb{R}$ such that

$$I_\Phi(\lambda x) = \int_S \Phi(s, \lambda x(s))\, d\mu < \infty$$

for some $\lambda > 0$ depending on x. This space endowed with the Luxemburg norm

$$\|x\| = \|x\|_\Phi = \inf\{\lambda > 0 \mid I_\Phi(\frac{x}{\lambda}) \leq 1\}$$

is a Banach space (cf. [10], [11] and in the case of Orlicz spaces also [7], [9]).

We further denote by $G(\Phi)$ $(G(\Phi, \varepsilon))$ the set of all non-negative Σ-measurable functions g on S such that $I_\Phi(g) < \infty$ $(I_\Phi(g) \leq \varepsilon)$.

The Musielak-Orlicz function Φ is said to satisfy the Δ_2-condition if there are a null-set S_0, a positive constant K and $h \in G(\Phi)$ such that

$$\Phi(s, 2u) \leq K\Phi(s, u) \quad \text{for all} \quad s \in S \setminus S_0,\ u \geq h(s).$$

For any Banach space X, we denote by δ_X and $\varepsilon_0(X)$ the modulus of convexity and the characteristic of convexity of X, i.e.

$$\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| \mid x, y \in X, \|x\| = \|y\| = 1, \|x - y\| > \varepsilon\}$$

for any $\varepsilon \in [0,2]$, and

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] \mid \delta_X(\varepsilon) = 0\},$$

see [1], [2], [8]. To compute $\varepsilon_0(L^\Phi)$ for L^Φ generated by strictly convex Musielak-Orlicz functions we start with the following

Lemma 1. *Let Φ satisfy the Δ_2-condition and vanish only at 0 for μ-a.e. $s \in S$. Then, for every $\varepsilon > 0$ and $c > 0$, there are a null-set S_0, a constant $K = K(\varepsilon, c) > 0$ and a function $h \in G(\Phi)$ such that*

$$ch \in G(\Phi, \varepsilon),$$
$$\Phi(s, 2u) \le K\Phi(s,u) \quad \text{for all} \quad s \in S \setminus S_0,\ u \ge h(s).$$

Proof: By Lemma 1.6 in [4], there are a null-set S_0, a sequence $\{h_n\}$ with $h_n \in G(\Phi, \frac{1}{n})$ for every $n \in \mathbb{N}$, and a sequence $\{K_n\}$ of positive reals such that

$$\Phi(s, 2u) \le K_n\Phi(s,u) \quad \text{for all} \quad s \in S \setminus S_0,\ u \ge h_n(s),\ n \in \mathbb{N}.$$

In virtue of the Δ_2-condition we have $I_\Phi(ch_n) \to 0$ as $n \to \infty$ for every $c > 0$ (cf. [5, Theorem 3.3.I]). Therefore, it suffices to put $h = h_n$ and $K(\varepsilon, c) = K_n$ for sufficiently large n depending on ε and c. $\square$

We define for every $c, \sigma \in (0,1)$ and $s \in S$:

$$\begin{aligned}
q(s,u,v) &= \begin{cases} 0 & \text{if } \Phi(s, \frac{1}{2}(u+v)) = 0 \\ \frac{2\Phi(s,\frac{1}{2}(u+v))}{\Phi(s,u)+\Phi(s,v)} & \text{otherwise,} \end{cases} \\
A(c,\sigma,s) &= \{u > 0 \mid q(s,u,cu) > 1-\sigma\}, \\
h_{c,\sigma}(s) &= \sup\{u > 0 \mid u \in A(c,\sigma,s)\}, \\
p(\Phi) &= \sup\{c \in (0,1) \mid h_{c,\sigma} \in G(\Phi) \quad \text{for some} \quad \sigma \in (0,1)\}.
\end{aligned}$$

Theorem 2. *Assume that $\Phi(s,\cdot)$ is a strictly convex function on $\mathbb{R}$ for μ-a.e. $s \in S$ and let $a \in (0,2)$. Then the following statements are equivalent:*

1. $\delta_{L^\Phi(\mu)}(a) > 0$.
2. (a) $p(\Phi) > \frac{2-a}{2+a}$,
   (b) *Φ satisfies the Δ_2-condition.*

Proof: $2 \Rightarrow 1$. If 2 (a) holds, then there is a number $b \in (0,2)$, $b < a$, such that

$$p(\Phi) > c > \frac{2-a}{2+a}\,,\ c = \frac{2-b}{2+b}\,.$$

Choose $\sigma \in (0,1)$ such that $f = h_{c,\sigma} \in G(\Phi)$. We first prove the following property of Φ:

(1) There is a number $\varepsilon \in (0,1)$ such that $q(s,u,v) \le 1-\varepsilon$ whenever $\max\{|u|,|v|\} \ge f(s)$ and $2|u-v| \ge a(1-\varepsilon)|u+v|$.

First, assume that $0 \le v \le cu$. Then, in view of the definition of $p(\Phi)$, we have $q(s,u,v) \le 1-\sigma$ if $u \ge f(s)$. Here and in the sequel all inequalities in which the parameter s is used are to be understood in the sense "for μ-a.e. $s \in S$". The inequality $0 \le v \le cu$ is equivalent to: $\frac{u-v}{a} \ge \frac{b}{2a}(u+v)$ and $u,v \ge 0$. Since $b < a$ we obtain (1) for non-negative u,v. In the same way, the condition (1) can be proved for negative u,v. It remains to prove (1) in the case $u \cdot v \le 0$. So, fix u,v with $u \cdot v \le 0$. Since the function

$$f_\Phi(t) = \operatorname*{ess\,sup}_{s \in S} \sup_{u > f(s)} q(s,u,tu)$$

is increasing in (0,1], it follows that $\eta = f_\Phi(0) < 1$. Thus

$$\begin{aligned}\Phi(s,\frac{1}{2}(u+v)) &\le \Phi(s,\frac{1}{2}\max\{|u|,|v|\})\\ &\le \frac{1}{2}\Phi(s,\max\{|u|,|v|\})\\ &\le \frac{1}{2}[\Phi(s,u)+\Phi(s,v)].\end{aligned}$$

Combining this with the previous case, we obtain (1) with

$$\varepsilon = \min\{1-\frac{b}{a},\sigma,,1-\eta\}.$$

Let $\lambda \in (0,1)$ be such that $I_\Phi(\frac{2\lambda}{a}f) \le \frac{\varepsilon}{12}$. Define

$$\begin{aligned}A_k = \{s \in S \mid \quad & q(s,u,v) \le 1-\frac{1}{k}\\ & \text{if } \lambda f(s) \le \max\{|u|,|v|\} \le f(s)\\ & \text{and } 2|u-v| \ge a(1-\varepsilon)|u+v|\}.\end{aligned}$$

Then, $A_k \uparrow U$ with $\mu(S \setminus U) = 0$ by the strict convexity of Φ. Thus, in virtue of the Beppo-Levi theorem, we have

$$I_\Phi(\frac{2}{a}f1_{A_k}) \to I_\Phi(\frac{2}{a}f) \quad \text{as} \quad k \to \infty.$$

Therefore, we can pick $n \in \mathbb{N}$ with $I_\Phi(\frac{2}{a}1_{S\setminus A_n}) \le \frac{\varepsilon}{12}$. Defining

$$g_1 = \lambda f 1_{A_n} + f 1_{S\setminus A_n}$$

we estimate

$$\begin{aligned}I_\Phi(\frac{2}{a}g_1) &= I_\Phi(\frac{2}{a}\lambda f 1_{A_n}) + I_\Phi(\frac{2}{a}f1_{S\setminus A_n})\\ &\le \frac{\varepsilon}{12}+\frac{\varepsilon}{12} = \frac{\varepsilon}{6}.\end{aligned}$$

Let h be a function from Lemma 1 corresponding to $\frac{\varepsilon}{6}$ instead of ε and $\frac{2}{a}$ instead of c. Define $\tilde{g} = \max\{g_1, h\}$. Then we obtain

$$I_\Phi(\frac{2}{a}\tilde{g}) \le I_\Phi(\frac{2}{a}g_1) + I_\Phi(\frac{2}{a}h) \le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}\,.$$

Denoting $\gamma = \min\{\varepsilon, \frac{1}{n}\}$, we obtain

(2) $q(s,u,v) \le 1-\gamma$ whenever $\max\{|u|,|v|\} \ge \tilde{g}(s)$ and $2|u-v| \ge a(1-\varepsilon)|u+v|$.

Fix $x, y \in L^\Phi(\mu)$ with $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge a$. Then $I_\Phi(x) \le 1$, $I_\Phi(y) \le 1$ and $I_\Phi(\frac{x-y}{a}) \ge 1$.

Put $A = S \setminus (B \cup C)$ where the sets B, C are defined by

$$B = \{s \in S \mid 2|x(s) - y(s)| < a(1-\varepsilon)|x(s) + y(s)|\},$$
$$C = \{s \in S \mid \max\{|x(s)|, |y(s)|\} < \tilde{g}(s)\}.$$

Then

$$I_\Phi(\frac{x-y}{a}1_B) \le \frac{1-\varepsilon}{2}[I_\Phi(x1_B) + I_\Phi(y1_B)] \le 1-\varepsilon,$$
$$I_\Phi(\frac{x-y}{a}1_C) \le I_\Phi(\frac{2}{a}\tilde{g}) \le \frac{\varepsilon}{3}$$

so that

$$I_\Phi(\frac{x-y}{a}1_A) \ge 1 - I_\Phi(\frac{x-y}{a}1_B) - I_\Phi(\frac{x-y}{a}1_C) \ge 2\frac{\varepsilon}{3}\,.$$

Define further

$$D = \{s \in A \mid \frac{|x(s) - y(s)|}{2} \le \tilde{g}(s)\} \quad \text{and} \quad E = A \setminus D.$$

A repeated application of $\Phi(s, 2u) \le K\Phi(s,u)$, $u \ge h(s)$, yields

$$\Phi(s, \frac{2}{a}u) \le M\Phi(s,u),\ u \ge h(s), \quad \text{with} \quad M = K^{2-\log_2(a)}$$

so that

$$\begin{aligned}
2\frac{\varepsilon}{3} &\le I_\Phi(\frac{x-y}{a}1_A) = I_\Phi(\frac{x-y}{a}1_D) + I_\Phi(\frac{x-y}{a}1_E)\\
&\le I_\Phi(\frac{2}{a}\tilde{g}1_D) + I_\Phi(\frac{2}{a}\frac{x-y}{a}1_E)\\
&\le \frac{\varepsilon}{3} + MI_\Phi(\frac{x-y}{a}1_E)\\
&\le \frac{\varepsilon}{3} + \frac{M}{2}[I_\Phi(x1_A) + I_\Phi(y1_A)].
\end{aligned}$$

From this inequality, we conclude that

$$I_\Phi(x1_A) + I_\Phi(y1_A) \geq r = \frac{2\varepsilon}{3M}$$

which implies

$$\begin{aligned}
1 - I_\Phi(\frac{1}{2}(x+y)) &\geq \frac{1}{2}[I_\Phi(x) + I_\Phi(y)] - I_\Phi(\frac{1}{2}(x+y)) \\
&\geq \frac{1}{2}[I_\Phi(x1_A) + I_\Phi(y1_A)] - I_\Phi(\frac{1}{2}(x+y)1_A) \\
&\geq \frac{1}{2}[I_\Phi(x1_A) + I_\Phi(y1_A)] - \frac{1}{2}(1-\gamma)[I_\Phi(x1_A) + I_\Phi(y1_A)] \\
&= \frac{\gamma}{2}[I_\Phi(x1_A) + I_\Phi(y1_A)] \geq \frac{1}{2}\gamma r = \vartheta,
\end{aligned}$$

what is equivalent to

(3) $I_\Phi(\frac{1}{2}(x+y)) \leq 1 - \vartheta$.

Let w be a function from $(0,1)$ into itself such that $\|x\| \leq 1 - w(\delta)$ whenever $I_\Phi(x) \leq 1 - \delta$ (such a function exists by the Δ_2-condition, cf. [4, Lemma 1.5]). Then inequality (3) yields

$$\|\frac{1}{2}(x+y)\| \leq 1 - w(\vartheta), \text{ i.e., } \delta_{L^\Phi(\mu)}(a) \geq w(\vartheta) > 0$$

which finishes the proof of the implication $2 \Rightarrow 1$.

$1 \Rightarrow 2$. If Φ does not satisfy the Δ_2-condition, then $L^\Phi(\mu)$ contains an isometric copy of ℓ_∞ (cf. [3]). Therefore $\delta_{L^\Phi(\mu)}(a) \leq \delta_{\ell_\infty}(a) = 0$ for any $a \in (0,2]$.

Assume now that Φ satisfies the Δ_2-condition but not 2 (a). Fixing an arbitrary $b \in (0,a)$ we then get $p(\Phi) < c = \frac{2-b}{2+b}$ and therefore

$$I_\Phi(h_{c,\sigma}) = \infty \text{ for all } \sigma \in (0,1).$$

Take an arbitrary such σ and denote $g = h_{c,\sigma}$. From the definition of g and the continuity of Φ we can conclude that $q(s, g(s), cg(s)) = 1 - \sigma$ whenever $g(s) < \infty$.

Put $H = \{s \mid g(s) = \infty\}$. If H is a null-set, then we put $f = g$, otherwise we choose $u_0 > 0$ and $C \subset H$ with $I_\Phi(u_0 1_C) = 2$ and define $f(s)$ by $\inf\{u > u_0 \mid q(s,u,cu) > 1-\sigma\}$ on C and by 0 on $S \backslash C$. In any case, f is real valued, measurable and satisfies $I_\Phi(f) \geq 2$ and

(4) $\Phi(s, \frac{1+c}{2} f(s)) \geq \frac{1-\sigma}{2}[\Phi(s, f(s)) + \Phi(s, cf(s))]$.

We choose $B \in \Sigma$ with $I_\Phi(f1_B) + I_\Phi(cf1_B) = 2$ and put

$$r(s) = \Phi(s, f(s)) - \Phi(s, cf(s)).$$

There is a set $A \subset B$ such that

$$\int_A r(s)\,d\mu = \int_{B\setminus A} r(s)\,d\mu$$

which is equivalent to

$$I_\Phi(f1_A) + I_\Phi(cf1_{B\setminus A}) = I_\Phi(cf1_A) + I_\Phi(f1_{B\setminus A}) = 1.$$

Define $x = f1_A + cf1_{B\setminus A}$ and $y = cf1_A + f1_{B\setminus A}$. We then have

$$\begin{aligned} I_\Phi(x) = I_\Phi(y) = \|x\| = \|y\| = 1, \\ |x-y| = (1-c)f1_B = \frac{2b}{2+b}f1_B, \\ x+y = (1+c)f1_B = \frac{4}{2+b}f1_B \end{aligned}$$

and hence

$$\frac{|x-y|}{b} = \frac{x+y}{2}.$$

So, in view of the inequality (4), we get

$$\begin{aligned} I_\Phi\left(\frac{x-y}{b(1-\sigma)}\right) &= I_\Phi\left(\frac{x+y}{2(1-\sigma)}\right) \\ &\geq \frac{1}{1-\sigma} I_\Phi\left(\frac{x+y}{2}\right) \\ &\geq \frac{1}{2}[I_\Phi(x) + I_\Phi(y)] = 1, \end{aligned}$$

whence $\|x-y\| \geq b(1-\sigma)$ and $\|\frac{1}{2}(x+y)\| \geq 1-\sigma$. This means that

$$\delta_{L^\Phi(\mu)}(b(1-\sigma)) \leq \sigma.$$

Letting $\sigma \to 0$ and $b \to a$ we obtain the desired conclusion $\delta_{L^\Phi(\mu)}(a) = 0$ and the proof is finished. □

As an immediate consequence of Theorem 2 we obtain

Theorem 3. *If Φ is strictly convex then*

$$\varepsilon_0(L^\Phi(\mu)) = \begin{cases} \frac{2(1-p(\Phi))}{1+p(\Phi)} & \textit{if } \Phi \textit{ satisfies the } \Delta_2\textit{-condition} \\ 2 & \textit{otherwise.} \end{cases}$$

Remark 1. Theorem 3 is not true when the strict convexity condition for Φ is dropped as the following example shows:

Take $S = [0, 2)$ with the Lebesgue measure μ and

$$\Phi(s,u) = \begin{cases} |u| & |u| \le 1 \\ u^2 & |u| > 1. \end{cases}$$

Straightforward calculations show that Φ satisfies the Δ_2-condition and $p(\Phi) = 1$ so that $\frac{2(1-p(\Phi))}{1+p(\Phi)} = 0$. But, for $x = 1_{[0,1)}$ and $y = 1_{[1,2)}$, we have $\|x\| = \|y\| = 1$ and $\|x + y\| = \|x - y\| = 2$ whence $\varepsilon_0(L^\Phi(\mu)) = 2$.

Remark 2. The parameter $p(\Phi)$ can also be computed in the following way:

$$p(\Phi) = \sup\{p(\Phi, g) \mid g \in G(\Phi)\}$$

where

$$p(\Phi, g) = \sup\{c \in (0,1) \mid f_{\Phi,g}(c) < 1\},$$
$$f_{\Phi,g}(c) = \operatorname{ess\,sup}_s \sup\{q(s,u,cu) \mid u > g(s)\}.$$

Indeed, if $p(\Phi) > c$, then $g = h_{c,\sigma} \in G(\Phi)$ for some $\sigma \in (0, 1)$ so that $f_{\Phi,g}(c) \le 1-\sigma$ and $p(\Phi, g) \ge c$.

Vice versa, if $p(\Phi, g) > c$ for $g \in G(\Phi)$ then $f_{\Phi,g}(c) = 1 - \sigma < 1$ whence $h_{c,\sigma} \le q$ μ-a.e. so that $h_{c,\sigma} \in G(\Phi)$ and $p(\Phi) \ge c$.

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