A general upper bound in extremal theory of sequences

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Abstract. We investigate the extremal function $f(u,n)$ which, for a given finite sequence u over k symbols, is defined as the maximum length m of a sequence $v = a_1a_2...a_m$ of integers such that 1) $1 \leq a_i \leq n$, 2) $a_i = a_j, i \neq j$ implies $|i-j| \geq k$ and 3) v contains no subsequence of the type u. We prove that $f(u,n)$ is very near to be linear in n for any fixed u of length greater than 4, namely that

$$f(u,n) = O(n2^{O(\alpha(n)^{|u|-4})}).$$

Here $|u|$ is the length of u and $\alpha(n)$ is the inverse to the Ackermann function and goes to infinity very slowly. This result extends the estimates in [S] and [ASS] which treat the case $u = abababa\ldots$ and is achieved by similar methods.

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Introduction

In the Extremal theory of sequences we investigate the quantity

$$f(u,n) = \max\{|v| \mid u \not\leq v, \|v\| \leq n, v \text{ is } \|u\|\text{-regular}\}.$$

Here u and v are finite sequences of arbitrary symbols, n is a nonnegative integer, $|v|$ stands for the length of v and $\|v\|$ denotes the cardinality of $S(v)$, the set of all symbols that occur in v. If there is a subsequence s in v such that s differs from u only in the names of the symbols we write $u \leq v$ and say that v *contains* u. For instance $v_1 = 123245131$ contains both $u_1 = xxyy$ and $u_2 = ababa$. A sequence $u = a_1a_2..a_m$ is called *k-regular* if $a_i = a_j, i \neq j$ implies $|i-j| \geq k$. Example: v_1 and u_2 are 2-regular but are not 3-regular and u_1 is not 2-regular. If $u = a_1a_2..a_m$ and $a_i = a \in S(u)$ then we shall refer to a_i as to the a-letter.

The function $f(u,n)$ extends in a natural way the function $F = f(ababa,n)$ investigated at first by Davenport and Schinzel in [DS]. They proved the upper bound $F = O(n \log n / \log \log n)$ that was later improved by Szemerédi to $O(n \log^* n)$ ([Sz]). Here $\log^* n$ is the minimum number of iterations of the power function 2^m (starting with $m = 1$) which are needed to get a number greater or equal to n. The question whether $F = O(n)$ ($f(abab,n) = 2n-1$ trivially) remained open until 1986 when it was answered by Hart and Sharir in [HS] negatively. They showed

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that $F = \Theta(n\alpha(n))$ where $\alpha(n)$ goes to infinity but very slowly (a precise definition of $\alpha(n)$ will be given in the second part of this paper). M. Sharir obtained later

$$f(al_s, n) = O(n\alpha(n)^{O(\alpha(n)^{s-5})})$$

for arbitrary alternating sequence $al_s = ababab\ldots$ of the length $s \geq 5$ ([S]). Recently almost tight estimates were derived ([ASS]):

$$f(al_s, n) \leq n.2^{(\alpha(n))^{\frac{s-5}{2}} \log_2 \alpha(n) + C_s(n)} \qquad \text{for } s \geq 5 \ \text{ odd}$$

$$f(al_s, n) \leq n.2^{(\alpha(n))^{\frac{s-4}{2}} + C_s(n)} \qquad \text{for } s \geq 6 \ \text{ even}$$

$$f(al_s, n) = \Omega(n.2^{K_s.(\alpha(n))^{\frac{s-4}{2}} + Q_s(n)}) \qquad \text{for } s \geq 6 \ \text{ even}$$

where $K_s = \frac{1}{(\frac{s-4}{2})!}$ and $C_s(n)$ and $Q_s(n)$ are asymptotically smaller than the main terms. For $s = 6$ even, $f(ababab, n) = \Theta(n2^{\alpha(n)})$ ([ASS]). How complex the previous formulae may seem on the first view, one thing is clear: $f(al_s, n)$ is almost almost linear in n for all s.

The first aim of this paper is to show that the same is true for arbitrary sequence u. The second aim is to give a brief and clear idea about the techniques developed by Agarwal, Hart, Sharir and Shor for obtaining almost linear upper bounds on $f(al_s, n)$ to the reader that is not familiar with them.

In the first part we show a simple method that leads to the upper bound $f(u, n) = O(n^2)$ for all u. Then, in the second part, we use a slightly generalized method of [S] to derive the estimate

$$f(u, n) = O(n.2^{O(\alpha(n)^{|u|-4})}).$$

Part 1

We first define a modification of the function $f(u, n)$ for l-regular sequences:

$$f(u, n, l) = \max\{|v| \mid u \not\leq v, \|v\| \leq n, v \ \text{ is } l \text{ -regular}\}$$

where $l \geq \|u\|$.

Lemma 1.1. a) *$f(u, n, l)$ is defined and finite for any $n \geq 1$ and moreover $f(u, n, l) = O(|u|.\|u\|.n^{\|u\|})$.*

b) *$f(u, n, l) \leq f(u, n, k) \leq (1 + f(u, l-1, k))f(u, n, l)$ for all $l > k \geq \|u\|, n \geq 1$.*

Proof: ad a) We suppose there is at least one repetition in u, otherwise the function $f(u, n, l)$ is constant. If $n < l$ then $f(u, n, l) = n$. If $n \geq l$ then any l-regular sequence v satisfying $|v| \geq \|u\|.(\binom{n}{\|u\|}(|u|-1)+1)$ must contain u. We split $v = v_1 v_2 \ldots v_c w$ so that $|v_i| = \|v_i\| = \|u\|$ and $c = (|u|-1)\binom{n}{\|u\|} + 1$. According to the Dirichlet Principle there exist $|u|$ indices $1 \leq i_1 < i_2 < \ldots < i_{|u|} \leq c$ that $S(v_{i_1}) = S(v_{i_2}) = \ldots = S(v_{i_{|u|}})$. Thus $u \leq v_{i_1} v_{i_2} \ldots v_{i_{|u|}}$.

ad b) The first inequality is obvious. Suppose $v = a_1a_2\dots a_m$ is k-regular, does not contain u and $\|v\| \le n$. We choose a subsequence v^* of v in this way: we start with $v^* = a_1$ and $i = 1$ and search for the minimum j such that $j > i$ and v^*a_j is l-regular. If such a j exists then we put $v^* = v^*a_j$ and $i = j$ and repeat. Otherwise the algorithm terminates. Obviously $\|v^*\| \le n$ and v^* is l-regular. Moreover $|v| \le (1 + f(u, l-1, k))|v^*|$ because any interval I in v omitted by the previous algorithm satisfies $\|I\| \le l - 1$. We got the second inequality. $\square$

Definition 1.2. *Let u, v be sequences. We write $u \le\le v$ if $u \le v^*$ for all v^* obtained from v by restricting v to some $\|u\|$ symbols. Thus in this case v contains u in all possible ways.*

Lemma 1.3. *For any sequence u there exist positive integers m and s such that $u \le v$ whenever $\|v\| \ge m$ and $al_s \le\le v$.*

Before proving this lemma we derive the main result of this section.

Theorem 1.4. *$f(u, n) = O(n^2)$ for all sequences u. The constant in O depends on u.*

Proof: Let $m = m(u)$ be as in Lemma 1.3. According to Lemma 1.1 b) we have $f(u, n) = f(u, n, \|u\|) \le (1 + f(u, m-1, \|u\|))f(u, n, m)$. We estimate $f(u, n, m)$. Suppose v is m-regular, $\|v\| \le n$, $u \not\le v$ and $|v| = f(u, n, m)$. It suffices to estimate the number c in the splitting $v = v_1v_2\dots v_cw$ where $|v_i| = \|v_i\| = m$ and $|w| \le m - 1$. Let $s = s(u)$ stand for the second number of Lemma 1.3. For any v_i there exist symbols $a, b \in S(v_i)$ such that v restricted on the symbols $\{a, b\}$ does not contain al_s. Otherwise $u \le v$ according to Lemma 1.3. But the mapping $F : \{v_1, v_2, \dots, v_c\} \to \binom{S(v)}{2}$ that maps any v_i on a pair $\{a, b\}$ mentioned above maps only at most $s - 2$ v_i's on one pair because of the property of the symbols $\{a, b\}$. Thus $c \le (s-2)\binom{n}{2}$. Finally

$$f(u, n) \le (1 + f(u, m-1, \|u\|))m(c+1) \le (1 + f(u, m-1, \|u\|))m(1 + (s-2)\binom{n}{2}).$$

Thus

$$f(u, n) = O(n^2).$$

$\square$

It remains to prove Lemma 1.3. We use the following well known:

Lemma 1.5 (Erdös P., Szekeres G. 1935 [ES]). *Any $(n-1)^2 + 1$-term sequence (of integers) contains a n-term monotone subsequence.*

Proof of Lemma 1.3: We denote by $X(k, l)$ the set of all sequences of the form $y_1y_2\dots y_l$ where $y_i = x_1x_2\dots x_k$ or $y_i = x_kx_{k-1}\dots x_1$ for k distinct symbols $x_1, x_2, \dots, x_k$. Thus $|X(k, l)| = 2^l$ and $|u| = kl$ and $\|u\| = k$ for any $u \in X(k, l)$. Since $u \le w$ for any $w \in X(\|u\|, |u|)$, it suffices to prove the following claim.

Claim. *For all positive integers k and l there exist positive integers m and s such that $w \le v$ for some $w \in X(k,l)$ whenever $\|v\| \ge m$ and $al_s \le\le v$.*

Proof of the claim: We put $s = 2l$ and $m = k_1$ where $k_l = k$ and $k_{t-1} = 4(t-1)k_t^2 + 3$ for $t = l, l-1, \dots, 2$. Suppose v meets the prescribed conditions. We prove by induction that for all $t = 1, 2 \dots, l$ there exists $w \in X(k_t, t)$ such that $w \le v$. For $t = 1$ this is obvious. Suppose it is true for $t-1 \ge 1$. We have $w \in X(k_{t-1}, t-1), w \le v$. We take a fixed w-copy U in v and split v into $t-1$ intervals $v = v_1 v_2 \dots v_{t-1}$ where v_i contains i-th part of U (i.e. y_i). U consists of $k_{t-1}(t-1)$ letters x_i^j, $j = 1 \dots t-1$, $i = 1 \dots k_{t-1}$ in v, x_i^j occur in v_j, $a < b$ implies that x_a^j precedes x_b^j and $x_1^p = x_1^q, x_2^p = x_2^q, ..$ or $x_1^p = x_{k_{t-1}}^q, x_2^p = x_{k_{t-1}-1}^q, ..$ for all p, q. It remains to give names to the symbols — say that x_i^1 is $z(i)$-letter for $i = 1, 2, \dots, k_{t-1}$. There must be other $z(i)$-letters in v besides those in U ($al_s \le\le v$). Let us consider the pairs of symbols $(z(1), z(k_{t-1})), (z(2), z(k_{t-1} - 1)), \dots, (z(L), z(k_{t-1} - L + 1))$, $L = [k_{t-1}/2] - 1$. The Dirichlet Principle implies that there are a set $M \subset \{1, 2, \dots, L\}$, $|M| \ge \frac{L}{t-1}$ and an index $r \in \{1, 2, \dots, t-1\}$ that $z(i)z(k_{t-1}-i+1)z(i)$ or $z(k_{t-1}-i+1)z(i)z(k_{t-1}-i+1)$ is a 3-term subsequence of v_r for any $i \in M$. We used that $al_s \le\le v$ and $s > 2(t-1)$. We can suppose w.l.o.g. $r = 1$. Thus we have 2-term subsequence $z(k_{t-1} - i + 1)z(i)$ of v_1 for any $i \in M$ (the opposite order than in U). The $z(L+1)$-letter x_{L+1}^1 (lies in U) splits v_1 on two intervals $v_1 = v_1' v_1''$. There are at least $|M|/2$ i's in M such that $z(i)$-letter occurs in v_1'' or there are $|M|/2$ i's in M such that $z(k_{t-1} - i + 1)$-letter occurs in v_1'. We obtained t separated areas — namely $v_1', v_1'', v_2, \dots, v_{t-1}$ — in which $z(i)$-letter occurs for at least $|M|/2$ i's. From those at least $|M|/2$ i's we choose according to Lemma 1.5 at least $\sqrt{|M|/2}$ i's in such a way that we obtain a w'-copy in v, $w' \in X([\sqrt{|M|/2}], t)$. We are finished because $[\sqrt{|M|/2}] \ge [\sqrt{L/2(t-1)}] \ge .. \ge k_t$. □

Remark 1.6. If we estimate $k_{t-1} = 4(t-1)k_t^2 + 3 \le t(2k_t)^2$ then it may be easily derived that it suffices to put in Lemma 1.3 $s = 2|u|, m = (4|u|.\|u\|)^{2^{|u|-1}}$.

Part 2

In this section we prove a result far stronger than $f(u,n) = O(n^2)$. At first we give the precise (standard) definition of $\alpha(n)$.

For any function $B : \mathbf{N} \to \mathbf{N}$ the symbol $B^{(s)}(n)$ denotes $B(B(..(B(n))..))$ (s times). We define further the functional inverse of B as $B^{-1}(n) = \min\{s \ge 1 \mid B(s) \ge n\}$. For nondecreasing and unbounded B the functional inverse B^{-1} is nondecreasing and unbounded as well. The functions $A_k(n)$ are defined by induction:

$$A_k(1) = 2, A_1(n) = 2n \text{ and } A_k(n) = A_{k-1}^{(n)}(1).$$

Thus $A_2(n) = 2^n, A_3(n) = 2^{2^{\cdot^{\cdot^{2}}}}$ n times. The Ackermann function is diagonal function of that schema: $A(n) = A_n(n)$. The function $\alpha(n)$ is defined as $\alpha(n) =$

$A^{-1}(n)$. Apart the hierarchy $A_1, A_2, \dots$ (A_{i+1} grows to infinity much faster than A_i), we have the hierarchy $\alpha_1, \alpha_2, \dots, \alpha_i = A_i^{-1}$ (α_{i+1} grows to infinity much more slowly than α_i). Thus $\alpha_1(n) = \lceil \frac{n}{2} \rceil, \alpha_2(n) = \lceil \log_2 n \rceil, \alpha_3(n) = \log^*(n), \dots$. The function α is far "lazier" than any α_i. It is easy to prove for α_i a recurrent formula $\alpha_{i+1}(n) = \min\{s \geq 1 \mid \alpha_i^{(s)}(n) = 1\}$. Thus

$$\alpha_{i+1}(\alpha_i(m)) = \alpha_{i+1}(m) - 1 \quad \text{for all } i \geq 1, m \geq 3. \tag{1}$$

Further ([ASS])

$$\alpha_{\alpha(n)+1}(n) \leq 4 \quad \text{for all } n \geq 1. \tag{2}$$

A sequence u is called a 1-*chain* if no symbol occurs repeatedly in u. $Y(k,l)$ denotes the set of all sequences of the form $y_1 y_2 \dots y_l$ where any y_i is a permutation of k fixed symbols $x_1, x_2, \dots, x_k$. $Y(k,l) \not\leq v$ means that $u \leq v$ for no $u \in Y(k,l)$. We modify a bit the function $\Psi_s(m,n)$ of [S] and introduce the function

$$\Psi_r^s(m,n) = \max\{|v| \mid v \ \text{ is } r\text{-regular}, \|v\| \leq n, v = v_1 v_2 \dots v_m$$
$$\text{where any } v_i \ \text{ is 1-chain and } Y(r,s) \not\leq v\}.$$

We will estimate $f(u,n)$ in four steps. We will proceed induction on s. At first we estimate $\Psi_r^3(m,n)$. Then we derive, supposing we have an upper bound on $\Psi_r^{s-1}(m,n)$, a recurrent inequality for $\Psi_r^s(m,n)$. In the third step using that inequality the upper bound considered in Step 2 is extended on $\Psi_r^s(m,n)$. Finally we estimate $f(u,n)$ by appropriate $\Psi_r^s(m,n)$.

Step 1.

Lemma 2.1. *$\Psi_r^3(m,n) \leq 2rn$.*

PROOF: Suppose v is r-regular, $\|v\| \leq n$ and $Y(r,3) \not\leq v$ (we ignore here the first variable in Ψ). We split $v = v_1 v_2 \dots v_c w$ where $|v_i| = \|v_i\| = r$ and $|w| < r$. Any v_i must contain the first letter or the last letter of some symbol (otherwise $u \leq v$ for some $u \in Y(r,3)$). Thus

$$|v| = cr + |w| \leq (2\|v\| - |w|)r + |w| \leq 2rn.$$

□

Step 2.

Lemma 2.2. *Suppose $\Psi_r^{s-1}(m,n) \leq F_{s-1}(m)m + G_{s-1}(m)n$ for $m,n \geq 1$ for some nondecreasing functions $F_{s-1}, G_{s-1} : \mathbf{N} \to \mathbf{N}$. Then for any partition $m = m_1 + \dots + m_b, m_i \geq 1, 1 < b < m$ there exists a partition $n = n_0 + n_1 + \dots + n_b, n_i \geq 0$ such that*

$$\Psi_r^s(m,n) \leq \textstyle\sum_{i=1}^b \Psi_r^s(m_i,n_i) + 2\Psi_r^s(b,n_0)G_{s-1}(m) + mH_{s-1}(m) \tag{3}$$

where $H_{s-1}(m) = 3(r-1) + 2F_{s-1}(m) + 2(r-1)G_{s-1}(m)$.

Proof: We start with a preliminary consideration. Suppose an r-regular sequence u is splitted into o 1-chains $u = u_1 u_2 \dots u_o$. Then a subsequence v of u need not be r-regular but it suffices to delete at most $(r-1)(o-1)$ letters from v and what remains is r-regular. This consideration will be used in this proof and then again in the fourth step.

Let v be r-regular, $\|v\| \leq n$, $Y(r,s) \not\leq v$, v consists of m 1-chains and $|v| = \Psi_r^s(m,n)$. We group 1-chains of v in b layers (the partition $m = m_1 + \ldots + m_b$ is given) $L_1, L_2, \ldots, L_b$ where L_i consists of m_i 1-chains. Thus $v = L_1 L_2 \ldots L_b$. We split any L_i in three subsequences v_i^1, v_i^2 and v_i^3, v_i^1 consists of those letters that occur only in L_i (i.e. $S(v_i^1) \cap S(L_j) = \emptyset$ for $i \neq j$), v_i^2 consists of those that occur also before L_i and v_i^3 consists of the remaining ones (i.e. do not occur before L_i but occur after L_i). Obviously

$$\Psi_r^s(m,n) = |v| = \textstyle\sum_{i=1}^b |v_i^1| + \sum_{i=1}^b |v_i^2| + \sum_{i=1}^b |v_i^3|. \tag{4}$$

The upper bound on the first term in (4) is clearly

$$\textstyle\sum_{i=1}^b (\Psi_r^s(m_i,n_i) + (m_i - 1)(r-1)) = \sum_{i=1}^b \Psi_r^s(m_i,n_i) + (m-b)(r-1)$$

where $n_i = \|v_i^1\|$. We come naturally to the partition $n = n_0 + n_1 + \ldots + n_b$, n_0 is the number of all symbols figurating in all v_i^2, v_i^3. Observe that $Y(r, s-1) \not\leq v_i^2, v_i^3$ for all i. This fact enables us to estimate the remaining two terms in (4). We do it only for the second one, the third one is treated similarly. According to the hypothesis

$$\begin{aligned}\textstyle\sum_{i=1}^b |v_i^2| &\leq \textstyle\sum_{i=1}^b (F_{s-1}(m_i)m_i + G_{s-1}(m_i)\|v_i^2\| + (m_i-1)(r-1)) \leq \\ &\leq F_{s-1}(m)m + G_{s-1}(m)\textstyle\sum_{i=1}^b \|v_i^2\| + (m-b)(r-1).\end{aligned}$$

We transform any v_i^2 to w_i by taking any $a \in S(v_i^2)$ just once (the 1-chain w_i is a subsequence of v_i^2). The sequence $w = w_1 w_2 \ldots w_b$ meets (after deleting at most $(b-1)(r-1)$ letters) all conditions to be estimated by $\Psi_r^s(b, n_0)$. Thus

$$\textstyle\sum_{i=1}^b \|v_i^2\| = |w| \leq \Psi_r^s(b,n_0) + (b-1)(r-1).$$

We substitute all derived bounds in (4):

$$\begin{aligned}\Psi_r^s(m,n) &\leq \textstyle\sum_{i=1}^b \Psi_r^s(m_i,n_i) + (m-b)(r-1) + \\ &\quad + 2[F_{s-1}(m)m + G_{s-1}(m)(\Psi_r^s(b,n_0) + (b-1)(r-1)) + (m-b)(r-1)].\end{aligned}$$

We got (3). $\square$

Step 3.

Lemma 2.3. *Let F_{s-1}, G_{s-1} and H_{s-1} be as in Lemma 2.2. Then for any $m, n \geq 1$, $k \geq 2$*

$$\Psi_r^s(m,n) \leq \alpha_k(m)m.H_{s-1}(m).(5G_{s-1}(m))^{k-2} + 2n.(2G_{s-1}(m))^{k-1}. \tag{5}$$

Proof: For $m \leq 4$ (5) holds because of the trivial inequality $\Psi_r^s(m,n) \leq mn$. We prove (5) induction on k, for k fixed induction on m. We start with $k = 2$. It suffices to verify induction on m the estimate

$$\Psi_r^s(m,n) \leq H_{s-1}(m)\lceil \log_2 m \rceil m + 4G_{s-1}(m)n$$

((5) for $k = 2$) using the inequality

$$\Psi_r^s(m,n) \leq \Psi_r^s(\lfloor \frac{m}{2} \rfloor, n_1) + \Psi_r^s(\lceil \frac{m}{2} \rceil, n_2) + 4G_{s-1}(m)n_0 + mH_{s-1}(m)$$

((3) for $b = 2$). It is left to the reader.

In case $k > 2$, $m \geq 3$ we put in (3) $b = \lceil \frac{m}{\alpha_{k-1}(m)} \rceil$, $m_i \leq \lceil \frac{m}{b} \rceil \leq \alpha_{k-1}(m)$. Thus $\alpha_k(m_i) \leq \alpha_k(m) - 1$ (according to (1)) and $b\alpha_{k-1}(b) \leq b\alpha_{k-1}(m) \leq 2m$. We estimate the term $\Psi_r^s(m_i, n_i)$ in (3) by (5) for k, m_i, and the term $\Psi_r^s(b, n_0)$ by (5) for $k-1, b$. Then

$$\begin{aligned}
&\Psi_r^s(m,n) \leq \textstyle\sum_{i=1}^{b} (H_{s-1}(m_i)(5G_{s-1}(m_i))^{k-2}\alpha_k(m_i)m_i + 2(2G_{s-1}(m_i))^{k-1}n_i) + \\
&+ (H_{s-1}(b)(5G_{s-1}(b))^{k-3}\alpha_{k-1}(b)b + 2(2G_{s-1}(b))^{k-2}n_0)2G_{s-1}(m) + mH_{s-1}(m) \leq \\
&\leq H_{s-1}(m)(5G_{s-1}(m))^{k-2}(\alpha_k(m) - 1)m + 2(2G_{s-1}(m))^{k-1}(n - n_0) + \\
&+ H_{s-1}(m)((5G_{s-1}(m))^{k-2} - 1)m + 2(2G_{s-1}(m))^{k-1}n_0 + mH_{s-1}(m) \leq \\
&\leq H_{s-1}(m)(5G_{s-1}(m))^{k-2}\alpha_k(m)m + 2(2G_{s-1}(m))^{k-1}n.
\end{aligned}$$

□

Lemma 2.4. *For any $s \geq 4$ the inequality*

$$\Psi_r^s(m,n) \leq m(10r)^{\alpha^{s-3}(m) + 4\alpha^{s-4}(m)} + n(4r)^{\alpha^{s-3}(m) + 2\alpha^{s-4}(m)} \quad m, n \geq 1 \tag{6}$$

holds.

Proof: We consider the functions $\overline{F}_s, \overline{G}_s$, $s \geq 3$ that are defined by the following recurrent relations (we write $\overline{F}_s$ instead $\overline{F}_s(m)$, $\overline{G}_s$ instead $\overline{G}_s(m)$ and α instead of $\alpha(m)$ for the sake of brevity):

$\overline{F}_3 = 0$, $\overline{G}_3 = 2r$
$\overline{F}_s = 4(3(r-1) + 2\overline{F}_{s-1} + 2(r-1)\overline{G}_{s-1})(5\overline{G}_{s-1})^{\alpha-1}$, $\overline{G}_s = 2(2\overline{G}_{s-1})^{\alpha}$.

Induction on s shows that

$$\Psi_r^s(m,n) \le \overline{F}_s(m)m + \overline{G}_s(m)n$$

for any $m, n \ge 1, s \ge 3$. Indeed, for $s = 3$ it follows from Step 1 and for general s we obtain this inequality from (5) where we put $k = \alpha(m) + 1$ and use (2). We count explicit upper bounds on both functions. Clearly $\overline{G}_s = 2.4^{\alpha^{s-4}+\alpha^{s-5}+..+\alpha}.(4r)^{\alpha^{s-3}}$ for $s \ge 5$ and $\overline{G}_4 = 2(4r)^\alpha$. Hence $\overline{G}_s \le (4r)^{\alpha^{s-3}+2\alpha^{s-4}}$ for $s \ge 4$.

Further $\overline{F}_4 = \frac{2}{5}(4r-1-\frac{3}{r})(10r)^\alpha \ge \overline{G}_4$ and therefore $\overline{F}_s \ge \overline{G}_s$ for all $s \ge 4$. Thus $\overline{F}_s \le 4(3(r-1) + 2r\overline{F}_{s-1})(5\overline{F}_{s-1})^{\alpha-1} \le 4r(5\overline{F}_{s-1})^\alpha$. If we solve this recurrent relation as an equation then an upper bound on $\overline{F}_s$ is obtained. We start with $\overline{F}_4 \le 2r(10r)^\alpha$ and derive

$$\overline{F}_s \le (2r)^{\alpha^{s-4}}.(4r)^{\alpha^{s-5}+..+1}.5^{\alpha^{s-4}+..+\alpha}.(10r)^{\alpha^{s-3}} \le (10r)^{\alpha^{s-3}+4\alpha^{s-4}}. \qquad \square$$

Step 4.

Lemma 2.5.

$$f(u,n) \le 2\|u\|.2^{|u|-4}.n.(10\|u\|)^{2\alpha^{|u|-4}(n)+8\alpha^{|u|-5}(n)} \tag{7}$$

for any sequence u, $|u| \ge 5$.

Proof: We will find the upper bound $nE_s(n)$ ($E_s(n)$ is a nondecreasing function) on the quantity

$$\max\{|v| \mid v \text{ is } r\text{-regular}, \|v\| \le n, Y(r,s) \not\le v\}.$$

It suffices because $u \le v$ for any $v \in Y(\|u\|, |u|-1)$ except $u = aa\ldots a$ (i times) but $f(aa\ldots a, n) = n(i-1)$. We derive for E_s a recurrent relation. Let v be r-regular, $\|v\| \le n$ and $Y(r,s) \not\le v$. We split $v = v_1 l_1 v_2 l_2 \ldots v_n l_n$ where $l_1, \ldots, l_n$ are the last letters of all $x \in S(v)$. Observe that $Y(r, s-1) \not\le v_i$ and hence $|v| = \sum_{i=1}^n |v_i| + n \le (\sum_{i=1}^n \|v_i\|)E_{s-1}(n) + n$. The sum $\sum_{i=1}^n \|v_i\|$ may be estimated by $\Psi_r^s(n,n) + (n-1)(r-1)$ (we use the same trick as in Lemma 2.2 — replace v_i by 1-chain of the length $\|v_i\|$). Thus

$$|v| \le 2nE_{s-1}(n).(10r)^{\alpha^{s-3}(n)+4\alpha^{s-4}(n)}$$

by (6). Hence we may choose

$E_3(n) = 2r$ (see Step 1)
$E_s(n) = 2E_{s-1}(n).(10r)^{\alpha^{s-3}(n)+4\alpha^{s-4}(n)}$.

The solution of this relation is:

$$E_s(n) = 2r.2^{s-3}.(10r)^{\alpha^{s-3}(n)+\alpha^{s-4}(n)+..+\alpha(n)+4\alpha^{s-4}(n)+..+4}.$$

If replaced r by $\|u\|$ and s by $|u|-1$ then (7) is obtained. $\square$

Concluding remarks

We achieved the exponent $\alpha^{|u|-4}(n)$ in (7) by induction starting with $s = 3$. It is possible that this bound might be improved to (roughly) $\alpha^{\frac{1}{2}|u|}(n)$ but it would require computations far more complex as in [ASS].

More interesting than the best value in (7) is perhaps the fact that $f(u,n)$ is almost linear for any sequence u. Hence a double induction must be used in some form whenever we want to obtain a superlinear lower bound on $f(u,n)$ (cf. [HS], [ASS], [K], [FH] and [WS]). Methods giving such "huge" functions as $n^{\frac{7}{6}}$ or $n \log\log n$ or $n \log^* n$ cannot be successful. It is a remarkable difference in comparison with extremal problems concerning graphs or hypergraphs (Turán theory). Here most common functions are $n^\beta, \beta > 1$. A certain hybrid occurs in Davenport-Schinzel theory of matrices in [FH] where the maximum number of 1's in a 0-1 matrix (of the size $n \times n$) which does not contain a forbidden subconfiguration is investigated. Here $n\alpha(n)$ figurates as an upper bound as well as $n^{\frac{3}{2}}$ and $n \log n$.

For obtaining a good general upper bound on $f(u,n)$ only basic features of u — such as the length and the number of symbols — were important. It is demonstrated by the fact that we worked instead of u itself with the sets $X(k,l)$ resp. $Y(k,l)$ that are determined by $|u|$ and $\|u\|$. It is probable that this changes if we start to investigate finer properties of the asymptotic growth of $f(u,n)$. But except for the case $u = al_s$ where we know the magnitude of $f(u,n)$ with high precision due the deep result of [ASS] only little about that function is known. One of the basic questions is to determine the set

$$Lin = \{u \mid f(u,n) = O(n) \}$$

— see [AKV] and [Kl] for a partial solution.

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