# Deformation of Banach spaces

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*Abstract.* Using some moduli of convexity and smoothness we introduce a function which allows us to measure the deformation of Banach spaces. A few properties of this function are derived and its applicability in the geometric theory of Banach spaces is indicated.

*Keywords:* uniformly convex Banach space, uniformly smooth Banach space, modulus of convexity, modulus of smoothness

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## 1. Introduction.

The aim of this paper is to introduce and to study a function which is a kind of the modulus of deformation of Banach spaces. While the classical modulus of convexity measures the rotundity of the unit sphere in a Banach space and the modulus of smoothness classifies Banach spaces with respect to smoothness of their unit spheres, so the modulus of deformation will allow us to measure simultaneously both convexity and smoothness of a Banach space.

The mentioned modulus of deformation is introduced with help of the classical Clarkson's modulus of convexity [4] and the modulus of smoothness defined by the first author a few years ago [2] (cf. also [3]).

## 2. Modulus of convexity and its properties.

Let $(E, \|\cdot\|)$ be a given real Banach space with the zero element $\theta$. Denote by $B_E(x,r)$ the closed ball in $E$ centered at $x$ and with radius $r$. For simplicity, we shall denote by $B_E$ or $B$ the unit ball $B_E(\theta,1)$. Similarly, the symbol $S_E$ will stand for the unit sphere of the space $E$.

Let us recall that *the modulus of convexity* introduced by Clarkson [4] is a function $\delta_E : [0,2] \to [0,1]$ defined in the following way:

$$\delta_E(\varepsilon) = \inf \left[ 1 - \frac{\|x+y\|}{2} : x, y \in B_E,\ \|x-y\| \geq \varepsilon \right] .$$

One can show that this modulus can be defined equivalently as

$$\delta_E(\varepsilon) = \inf \left[ 1 - \frac{\|x+y\|}{2} : x, y \in S_E,\ \|x-y\| = \varepsilon \right]$$

(see [5], for example).

For further purposes let us recall a few properties of the function $\delta_E$ (cf. [1], [7], [11]).

The number $\varepsilon_0(E) = \sup[\varepsilon \geq 0 : \delta_E(\varepsilon) = 0]$ is called *the characteristic of convexity* of a space $E$. A space $E$ is referred to as *uniformly convex* provided $\varepsilon_0(E) = 0$. For example, the spaces $l^p$ and $L^p$ are uniformly convex whenever $1 < p < \infty$ [9].

The function $\delta_E$ is nondecreasing on the interval $[0, 2]$ and is increasing on $[\varepsilon_0(E), 2]$. Moreover, $\delta_E$ is continuous on $[0, 2)$ and may be discontinuous at the point $e = 2$ only.

For any Banach space $E$, its modulus of convexity is bounded from above by the modulus of convexity of a Hilbert space $H$ [12],

$$\delta_E(\varepsilon) \leq \delta_H(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^2\right)^{1/2} . \tag{2.1}$$

This means that Hilbert spaces are the most convex among all Banach spaces.

## 3. Modulus of smoothness.

Our goal in this section is to derive some properties of the modulus of smoothness defined in the paper [2]. This modulus seems to be defined in a more natural way than the modulus of smoothness due to Day [6].

Namely, for $\varepsilon \in [0, 2]$, let us put

$$\rho_E(\varepsilon) = \sup\left[1 - \frac{\|x + y\|}{2} : x, y \in S_E,\ \|x - y\| \leq \varepsilon\right] .$$

The function $\rho_E$ will be called *the modulus of smoothness* of a space $E$.

Recall that $E$ is a *uniformly smooth Banach space* if and only if $\lim_{\varepsilon \to 0} \frac{\rho_E(\varepsilon)}{\varepsilon} = 0$. Moreover (cf. [2]) the function $\rho_E$ is increasing on the interval $[0, 2]$ and $\rho_E(\varepsilon) \leq \frac{\varepsilon}{2}$ for $\varepsilon \in [0, 2]$. It is easily seen that $\rho_C(\varepsilon) = \frac{\varepsilon}{2}$, where $C = C[0, 1]$.

Using the parallelogram identity it is easy to show that

$$\rho_H(\varepsilon) = \delta_H(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^2\right)^{1/2} ,\ \varepsilon \in [0, 2],$$

where $H$ denotes an arbitrary Hilbert space.

On the other hand, repeating the argumentation from the paper [12] we can show that for any Banach space the following estimate is true:

$$\rho_H(\varepsilon) \leq \rho_E(\varepsilon)$$

for each $\varepsilon \in [0, 2]$. This yields

$$1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^2\right)^{1/2} \leq \rho_E(\varepsilon) \leq \frac{\varepsilon}{2}\, , \tag{3.1}$$

for every Banach space $E$.

The most important fact needed further on is contained in the following obvious inequality

$$\delta_E(\varepsilon) \le \rho_E(\varepsilon), \quad \varepsilon \in [0,2], \tag{3.2}$$

which is valid for an arbitrary Banach space $E$.

The remainder of this section is devoted to show that the modulus of smoothness $\rho_E$ is continuous on the interval $[0,2]$.

We start with recalling some facts concerning the geometry of two-dimensional Banach spaces [8]. Assume that $x, y$ are linearly independent elements. The set $L = L(x,y)$ defined in the way

$$L(x,y) = \{\alpha x + \beta y : \alpha \in \mathbb{R},\ \beta \ge 0\}$$

will be called *two-dimensional half-plane* (in the space $E$). In this case, $x$ is said to be diametral element of the half-plane $L$.

We have the following theorem.

**Theorem 3.1.** *Let $\mathscr{L}$ denote the family of all two-dimensional half-planes in $E$. Then*

$$\rho_E(\varepsilon) = \sup_{L \in \mathscr{L}} \rho_L(\varepsilon), \quad \varepsilon \in [0,2].$$

The proof may be done in the same fashion as the proof of the same assertion for the modulus of convexity (cf. [6], [8], [11]) and is therefore omitted.

In what follows, we shall also need the following lemma which is contained in the proof of Theorem 2 in [8].

**Lemma 3.1.** *Let $\varepsilon_1, \varepsilon_2$ be fixed positive numbers such that $\varepsilon_1 < \varepsilon_2 < 2$. Further assume that $y_1, y_2$ are linearly independent elements of the unit sphere $S_E$ such that $\|y_1 - y_2\| = \varepsilon_2$. Then in the half-plane $L(y_1, y_2)$ there exist elements $z_1, z_2 \in S_E$ such that*

$$\left(1 - \frac{\|y_1 + y_2\|}{2}\right) - \left(1 - \frac{\|z_1 + z_2\|}{2}\right) \le \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\,.$$

Now we are prepared to prove the main theorem of this section.

**Theorem 3.2.** *The modulus of smoothness $\rho_E(\varepsilon)$ is continuous on the interval $[0,2]$.*

Proof: Assume first that $\varepsilon_1, \varepsilon_2$ are fixed arbitrarily $0 < \varepsilon_1 < \varepsilon_2 < 2$. Further, let $\eta > 0$ be small enough. According to Theorem 3.1 we can find $x, y \in S_E$, $\|x - y\| = \varepsilon_2$ such that

$$\rho_E(\varepsilon_2) - \eta \le 1 - \frac{\|x+y\|}{2} \le \rho_E(\varepsilon_2)\,.$$

Next, in view of Lemma 3.1, we can find two elements $x_1, y_1$ in the half-plane $L(x,y)$ such that $x_1, y_1 \in S_E$, $\|x_1 - y_1\| = \varepsilon_1$ and

$$\left(1 - \frac{\|x+y\|}{2}\right) - \left(1 - \frac{\|x_1 + y_1\|}{2}\right) \leq \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

Hence we get

$$\rho_E(\varepsilon_2) - \eta - \left(1 - \frac{\|x_1 + y_1\|}{2}\right) \leq \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}$$

or equivalently

$$\rho_E(\varepsilon_2) - \left(1 - \frac{\|x_1 + y_1\|}{2}\right) \leq \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \eta.$$

Hence, taking into account Theorem 3.1 we derive

$$\rho_E(\varepsilon_2) - \rho_E(\varepsilon_1) \leq \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \eta.$$

Arbitrariness of the number $\eta$ allows us to write

$$\rho_E(\varepsilon_2) - \rho_E(\varepsilon_1) \leq \frac{2\sqrt{5}+1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

The above inequality implies that the function $\varepsilon \to \rho_E(\varepsilon)$ is continuous on the interval $(0,2)$.

In order to finish the proof, it is sufficient to notice that the continuity of the modulus $\rho_E$ at the endpoints $\varepsilon = 0$ and $\varepsilon = 2$ of the interval $[0,2]$ is a simple consequence of the inequality (3.1). This completes the proof. □

It is worthwhile to mention that another proof of the continuity (from the left side) of the modulus $\rho_E$ has been given recently by Ullan [13]. He proved also some relations between the modulus of smoothness $\rho_E$ and the modulus introduced by Day [6].

**4. Modulus of deformation.**

We start with introducing a function which is a kind of modulus of deformation. Namely, let us consider the function $d_E : [0,2] \to [0,1]$ defined in the following way:

$$d_E(\varepsilon) = \rho_E(\varepsilon) - \delta_E(\varepsilon), \quad \varepsilon \in [0,2].$$

This function will be called *the modulus of deformation* of a space $E$.

Obviously in view of (3.2) we get that $d_E$ is a nonnegative function. So we can formulate the following definition.

**Definition 4.1.** *Let $E_1, E_2$ be two Banach spaces. We say that $E_1$ is deformed less than $E_2$ whenever*

$$d_{E_1}(\varepsilon) \leq d_{E_2}(\varepsilon)$$

*for all $\varepsilon \in [0,2]$.*

Now observe that in view of the inequalities (2.1) and (3.1) we can deduce the following estimate

$$0 \leq d_E(\varepsilon) \leq \frac{\varepsilon}{2} \tag{4.1}$$

for any $\varepsilon \in [0,2]$ and for every Banach space $E$.

It is easily seen that the equality sign may be attained on both sides of (4.1). Indeed, if we take the space $C = C[0,1]$ with the standard maximum norm then it is easy to calculate that

$$\delta_C(\varepsilon) = 0 \quad \text{for} \quad \varepsilon \in [0,2]$$

and

$$\rho_C(\varepsilon) = \frac{\varepsilon}{2}, \quad \varepsilon \in [0,2].$$

Thus $d_C(\varepsilon) = \frac{\varepsilon}{2}$ for any $\varepsilon \in [0,2]$,

On the other hand, take an arbitrary Hilbert space $H$. Then, taking into account formulas for $\delta_H$ and $\rho_H$ we obtain that

$$d_H(\varepsilon) = 0, \quad \varepsilon \in [0,2].$$

This means that Hilbert spaces have the smallest deformation among all Banach spaces while the space $C$ is the worst with respect to the modulus of deformation.

In what follows we indicate some further properties of the modulus of deformation.

First of all let us notice that in view of Theorem 3.2 (cf. also Section 2) we deduce that the modulus of deformation is continuous on the interval $[0,2)$ and may be eventually discontinuous at the point $\varepsilon = 2$ only. Taking into account the properties of the moduli $\delta_E$ and $\rho_E$ we can state the following assertion:

> *The modulus $d_E$ is continuous at the point $\varepsilon = 2$ if and only if the modulus of convexity $\delta_E$ is continuous at this point.*

Particularly, if $E$ is uniformly convex then $d_E$ is continuous on the whole interval $[0,2]$.

Further let us observe that a space $E$ is strictly convex if and only if $d_E(2) = 0$.

Moreover, keeping in mind the inequality (2.1) we deduce easily that

$$\lim_{\varepsilon \to 0} \frac{d_E(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\rho_E(\varepsilon)}{\varepsilon}.$$

Thus we derive the following assertion.

**Theorem 4.1.** *A Banach space $E$ is uniformly smooth if and only if*

$$\lim_{\varepsilon\to 0}\frac{d_E(\varepsilon)}{\varepsilon}=0.$$

Finally let us observe that the function $d_E(\varepsilon)$ is increasing on the interval $[0,\varepsilon_0(E)]$.

In order to illustrate our considerations let us take the following example.

**Example 4.1.** Fix a number $\lambda$, $\lambda>1$, and consider the plane $\mathbb{R}^2$ with the norm $\|\cdot\|_\lambda$ defined in the following way

$$\|(x_1,x_2)\|_\lambda=\max\left\{\lambda|x_1|,\ \sqrt{x_1^2+x_2^2}\right\}.$$

The space $\mathbb{R}^2$ with the norm $\|\cdot\|_\lambda$ will be denoted by $\mathbb{R}^2_\lambda$.

It can be calculated that the modulus of convexity of the space $\mathbb{R}^2_\lambda$ has the form

$$\delta_{\mathbb{R}^2_\lambda}(\varepsilon)=\begin{cases}0 & \text{for } 0\le\varepsilon\le 2\sqrt{1-\frac{1}{\lambda^2}}\,.\\ 1-\lambda\sqrt{1-\frac{\varepsilon^2}{4}} & \text{for } 2\sqrt{1-\frac{1}{\lambda^2}}\le\varepsilon\le\frac{2\lambda}{\sqrt{1+\lambda^2}}\,.\\ 1-\sqrt{1-\frac{\varepsilon^2}{4\lambda^2}} & \text{for } \frac{2\lambda}{\sqrt{1+\lambda^2}}\le\varepsilon\le 2.\end{cases}$$

Hence we obtain that $\varepsilon_0(\mathbb{R}^2_\lambda)=2\sqrt{1-\frac{1}{\lambda^2}}$, $\delta_{\mathbb{R}^2_\lambda}(2)=1-\sqrt{1-\frac{1}{\lambda^2}}$.

Similarly, we can derive

$$\rho_{\mathbb{R}^2_\lambda}(\varepsilon)=\begin{cases}1-\sqrt{1-\frac{\varepsilon^2}{4\lambda^2}} & \text{for } 0\le\varepsilon\le\frac{2\lambda}{\sqrt{1+\lambda^2}}\,,\\ 1-\lambda\sqrt{1-\frac{\varepsilon^2}{4}} & \text{for } \frac{2\lambda}{\sqrt{1+\lambda^2}}\le\varepsilon\le 2.\end{cases}$$

It is easy to check (for example, in the case $\lambda=2$) that the function $d_{\mathbb{R}^2_\lambda}(\varepsilon)$ is increasing on the interval $[0,\varepsilon_0(\mathbb{R}^2_\lambda)]$, where $\varepsilon_0(\mathbb{R}^2_\lambda)=\sqrt{3}$.

Moreover, the function $d_{\mathbb{R}^2_\lambda}(\varepsilon)$ is decreasing on the interval $[\sqrt{3},\frac{4}{\sqrt{5}}]$, which implies that this function attains its local maximum at the point $\varepsilon=\varepsilon_0(\mathbb{R}^2_\lambda)$.

By the way, it is easy to show that the function $\varepsilon\to\delta_{\mathbb{R}^2_\lambda}(\varepsilon)$ is not convex on the interval $[0,2]$ (cf. [10]).

In the light of the above example we can raise the following question.

> *Does the function $\varepsilon\to d_E(\varepsilon)$ attain its local maximum at the point $\varepsilon=\varepsilon_0(E)$?*

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