Further characterizations of boundedly UC spaces

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Abstract. Following the paper [BDC1], further relations between the classical topologies on function spaces and new ones induced by hyperspace topologies on graphs of functions are introduced and further characterizations of boundedly UC spaces are given.

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1. Introduction.

The notion of boundedly UC space (or boundedly Atsuji space) is introduced in [BDC1]. A metric space X is called boundedly UC space (or boundedly Atsuji) provided each closed and bounded subset of X is UC. A metric space X is a UC space [At], [Be] provided for each metric space Y, each continuous function from X to Y is uniformly continuous.

In the paper [BDC1], many interesting characterizations of boundedly UC spaces are given. One of the most interesting characterizations is an external one by using a relation between the topology of uniform convergence on bounded sets and the Attouch-Wets topology on function spaces.

It is the aim of this paper to give some further characterizations in this direction. In this connection also a new hyperspace topology on function spaces is considered. This topology is called by Beer and Lucchetti the bounded proximal topology [BL1], [BL2].

This topology has good applications in minimization problems and it is a weakening of the Attouch-Wets topology ([AW], [AP], [ALW], [BDC1], [BDC2]). The bounded proximal topology was considered in [BL2] also on epigraphs of functions.

2. Preliminaries.

(X,d) will denote a metrizable space X with a compatible metric d. The open (resp. closed) d-ball with center $x \in X$ and radius $\varepsilon > 0$ will be denoted by $S_d[x,\varepsilon]$ (resp. $B_d[x,\varepsilon]$) and the ε-parallel body $\bigcup\{S_d[a,\varepsilon] : a \in A\}$ for a subset A of X will be denoted by $S_d[A,\varepsilon]$.

Let $CL(X)$ be the family of all nonempty closed subsets of (X,d) and $CLB(X)$ be the family of all nonempty closed and bounded subsets of (X,d). If $A \in CL(X)$, the distance functional $d(\cdot,A) : X \to [0,\infty)$ is described by the familiar formula

$d(x, A) = \inf\{d(x, a) : a \in A\}$. The gap $D_d(A, B)$ between two closed sets A and B is defined by the following formula

$$D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

For $E \subset X$, we specify the following subsets of $CL(X)$:
$E^- = \{F \in CL(X) : F \cap E \neq \emptyset\}$, $E^+ = \{F \in CL(X) : F \subset E\}$, $E^{++} = \{F \in CL(X) :$ there is $\varepsilon > 0$ such that $S_d[F, \varepsilon] \subset E\}$.

It was shown in [BL1] that all of the standard hyperspace topologies arise as weak topologies generated by families of geometric functionals defined on closed sets. The natural example is the Wijsman topology [LL] on $CL(X)$, which is the weakest one such that for each $x \in X$, the function $A \to d(x, A)$ is continuous on $CL(X)$. Also the bounded proximal topology σ_d which will be dealt in this paper can be described in this sense.

Definition ([BL2])**.** Let (X, d) be a metric space. The bounded proximal topology σ_d on $CL(X)$ is the weakest topology τ on $CL(X)$ such that for each $B \in CLB(X)$ the gap functional $A \to D_d(B, A)$ is τ continuous on $CL(X)$. Thus σ_d is completely regular and since σ_d is finer than the Wijsman topology on $CL(X)$, which is Hausdorff, σ_d is also Hausdorff. The topology σ_d is also weaker than the proximal topology, the weak topology determined by $\{D_d(F, \cdot) : F \in CL(X)\}$ [BLLN].

We will mainly use the local presentation of the topology σ_d:

Theorem A ([BL1])**.** *The bounded proximal topology σ_d on $CL(X)$ has as a local base at $A \in CL(X)$ all sets of the form*

$\phi_A[n, a_1, a_2, \dots a_k] = \{F \in CL(X) : F \cap S_d[x_0, n] \subset S_d[A, 1/n]$, and for each $i \leq k$ $d(a_i, F) < 1/n\}$, where x_0 is a fixed but arbitrary point of X, $\{a_1, a_2, \dots a_k\}$ is a finite subset of A and $n \in Z^+$.

Theorem B ([BL1])**.** *Let (X, d) be a metric space. A subbase for σ_d consists of all sets of the form V^-, where V is open in X, and all sets of the form $(B^c)^{++}$, where $B \in CLB(X)$ and B^c is the complement of B.*

We shall denote by $\tau_{AW}(d)$ the metrizable topology on $CL(X)$ of uniform convergence of distance functionals on bounded subsets of X corresponding to a fixed metric d on X (the Attouch-Wets topology). The topology $\tau_{AW}(d)$ is most naturally presented as a uniform topology, determined by the uniformity Ω_d on $CL(X)$ with the countable base of entourages $\{V_n : n \in Z^+\}$, where for each n

$$V_n = \{(A, B) : \sup\{|d(x, A) - d(x, B)| : x \in S_d[x_0, n]\} < 1/n\}.$$

The point x_0 is a fixed but arbitrary point of X, and the uniformity is independent of its choice [BDC2].

Let us mention also a weaker uniformity Π_d on $CL(X)$ which has a countable base consisting of all sets of the form

$$\Sigma_n = \{(A, B) : A \cap S_d[x_0, n] \subset S_d[B, 1/n] \text{ and } B \cap S_d[x_0, n] \subset S_d[A, 1/n]\}$$

where again x_0 is a fixed but arbitrary point of X and $n \in Z^+$. This uniformity also determines $\tau_{AW}(d)$ [BDC2]. That this second uniformity also gives rise to $\tau_{AW}(d)$ has led some authors to call the topology the bounded Hausdorff topology [Pe] or the topology of the ρ-Hausdorff distance [ALW].

From the presentations of uniformities Ω_d and Π_d we can obtain the following local descriptions of $\tau_{AW}(d)$ topology.

A local base for $\tau_{AW}(d)$ at $A \in CL(X)$ [BDC2] consists of all sets of the form

$$\{F \in CL(X) : \sup_{x \in B} |d(x, F) - d(x, A)| < \varepsilon\},$$

where $B \in CLB(X)$ and $\varepsilon > 0$.

Another local base for $\tau_{AW}(d)$ at $A \in CL(X)$ [BDC2] consists of all sets of the form

$$\Sigma_n[A] = \{F \in CL(X) : F \cap S_d[x_0, n] \subset S_d[A, 1/n] \text{ and } A \cap S_d[x_0, n] \subset S_d[F, 1/n]\},$$

where x_0 is a fixed but arbitrary point from X.

It is very easy to see from the local presentations of $\tau_{AW}(d)$ and σ_d that $\sigma_d \subset \tau_{AW}(d)$ on $CL(X)$.

The Attouch-Wets topology splits into its lower and upper halves [BL1] $\tau^+_{AW}(d)$ and $\tau^-_{AW}(d)$, where a local base for $\tau^+_{AW}(d)$ (resp. $\tau^-_{AW}(d)$) at A consists of all sets of the form $\Theta^+_A[B, \varepsilon] = \{F \in CL(X) :$ for each $x \in B$ $d(x, A) - \varepsilon < d(x, F)\}$ ($\Theta^-_A[B, \varepsilon] = \{F \in CL(X) :$ for each $x \in B$ $d(x, F) < d(x, A) + \varepsilon\}$), where $B \in CLB(X)$, $\varepsilon > 0$.

Clearly $\tau^+_{AW}(d) \subset \sigma_d$ on $CL(X)$ (see [BL1]).

In our paper we will work mainly with boundedly UC spaces.

Definition ([BDC1])**.** A metric space (X, d) is called boundedly UC (or boundedly Atsuji) provided each closed and bounded subset of X is UC.

The following characterizations proved in [BDC1] will be useful further:

Theorem C. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is boundedly* UC*;*
(2) *For each metric space Y and for each continuous function f from X to Y, f is uniformly continuous on bounded subsets of X;*
(3) *Whenever $\{x_n\}$ is a bounded sequence in X such that $\{d(x_n, \{x_n\}^c\}$ converges to* 0*, then $\{x_n\}$ has a cluster point;*
(4) *Whenever B is a closed and bounded subset of X and $\{V_i : i \in I\}$ is a collection of open subsets of X with $B \subset \bigcup V_i$, then there is $\delta > 0$ such that each subset of X of diameter less than δ which meets B lies entirely within some V_i.*

Clearly every boundedly compact metric space is boundedly UC (a metric space (X, d) is called boundedly compact provided each closed and bounded subset of X is compact).

Remark 2.1. Suppose that (X, d) is a boundedly UC space, (Y, e) is a metric space and f is continuous function from X to Y. Let B be a closed and bounded subset of X. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that $e(f(x), f(y)) < \varepsilon$ for every $x, y \in X$ with $d(x, y) < \delta$ and $x \in B$.

Actually, this fact is very easy to see from the previous Theorem, from the implication $(1) \to (2)$.

Clearly, the above mentioned property holds for every compact set B in any metric space (X, d).

Now let (X, d) and (Y, e) be metric spaces and let ρ denote the box metric on $X \times Y$, i.e. $\rho[(x_1, y_1), (x_2, y_2)] = \max\{d(x_1, x_2), e(y_1, y_2)\}$.

If $f : X \to Y$ is a function, denote $G(f) = \{(x, f(x)) : x \in X\}$ the graph of f. Denote $C(X, Y)$ the family of all continuous functions from X to Y. We can identify the members of $C(X, Y)$ with their graphs and consider $C(X, Y)$ as a subspace of $CL(X \times Y)$ with the induced above mentioned topologies.

Denote $\tau_{AW}(\rho)$ the Attouch-Wets topology on $C(X, Y)$, σ_ρ the bounded proximal topology on $C(X, Y)$, $\tau_{AW}^+(\rho)$ the upper Attouch-Wets topology on $C(X, Y)$, τ_{UB} the topology of uniform convergence on bounded subsets of X on $C(X, Y)$, τ_{CO} the compact-open topology on $C(X, Y)$ and τ_P the topology of pointwise convergence on $C(X, Y)$. Mostly we work with the mentioned topologies on some subspaces of $C(X, Y)$. We use the same notation for topologies also in such cases.

Remark 2.2. The inclusions $\tau_{AW}^+(\rho) \subset \sigma_\rho \subset \tau_{AW}(\rho)$ on $C(X, Y)$ are clear and the inclusion $\tau_{AW}(\rho) \subset \tau_{UB}$ on $C(X, Y)$ is implied by the assertion (a) of Theorem 4.1 of [BDC2]. Example 1 in [Ho] shows that τ_P need not be weaker than $\tau_{AW}(\rho)$ on $C(X, Y)$.

In [BDC1] the following characterization of boundedly UC spaces is given:

Theorem D ([BDC1])**.** *Let (X, d) be a metric space. The following are equivalent:*

(1) *(X, d) is a boundedly* UC *space;*
(2) *For all bounded metric spaces Y, $\tau_{UB} = \tau_{AW}(\rho)$ on $C(X, Y)$;*
(3) *$\tau_{UB} = \tau_{AW}(\rho)$ on $C(X, [0, 1])$.*

The previous characterization can be very easily extended in the following way:

Theorem 2.3. *Let (X, d) be a metric space. The following are equivalent:*

(1) *(X, d) is a boundedly* UC *space;*
(2) *For all bounded metric spaces Y, $\tau_{UB} = \tau_{AW}^+(\rho)$ on $C(X, Y)$;*
(3) *For all bounded metric spaces Y, $\tau_{UB} = \sigma_\rho$ on $C(X, Y)$;*
(4) *For all bounded metric spaces Y, $\tau_{UB} = \tau_{AW}(\rho)$ on $C(X, Y)$;*
(5) *$\tau_{UB} = \tau_{AW}(\rho)$ on $C(X, [0, 1])$.*

Proof: It is sufficient to prove $(1) \to (2)$ since other implications are clear from Remark 2.2 and Theorem D.

Since $\tau_{AW}^+(\rho) \subset \tau_{UB}$ on $C(X, Y)$ is always true, we prove $\tau_{UB} \subset \tau_{AW}^+(\rho)$ on $C(X, Y)$.

Let B be a bounded set in X, $\varepsilon > 0$ and $f \in C(X,Y)$. Put $G = \{g \in C(X,Y) : e(f(x), g(x)) < \varepsilon$ for every $x \in B\}$. We show that G is a $\tau^+_{AW}(\rho)$-neighbourhood of f. By Remark 2.1 there is $\delta > 0$ such that $e(f(z), f(x)) < \varepsilon/2$ for every x, z with $d(x,z) < \delta$ and $x \in \operatorname{cl} B$. Put $v = \min\{\delta, \varepsilon/2\}$. Then it is very easy to see that $\Theta^+_{G(f)}[B \times Y, v/2] \cap C(X,Y) \subset G$. □

Remark 2.4. We must say that the boundedness of Y in the previous Theorem is very strong requirement. If Y is a bounded metric space then the behaviour of $\tau^+_{AW}(\rho)$, σ_ρ and $\tau_{AW}(\rho)$ on $C(X,Y)$ is nice. For example these topologies are stronger than the compact-open topology on $C(X,Y)$. The proof of this fact uses the same arguments as the proof of (1) → (2) in Theorem 2.3, where the boundedness of B is replaced by compactness.

It should be remarked also that if Y is a bounded metric space, then the upper Attouch-Wets topology and the bounded proximal topology on $C(X,Y)$ coincide for every metric space X.

3. The main results.

The aim of our paper is to remove the boundedness of Y by a strengthening of requirements on function spaces.

Put $L(X,Y) = \{f \in C(X,Y) : f(B)$ is bounded if B is bounded$\}$.

The following characterization holds:

Theorem 3.1. *Let (X,d) be a metric space. The following are equivalent.*

(1) *(X,d) is a boundedly* UC*;*
(2) *For every metric space (Y,e), $\tau_{UB} = \tau_{AW}(\rho)$ on every pointwise equicontinuous subfamily ϕ of $L(X,Y)$;*
(3) *For every bounded metric space (Y,e), $\tau_{UB} = \tau_{AW}(\rho)$ on $C(X,Y)$.*

Proof: (1) → (2) Let $\phi \subset L(X,Y)$ be a pointwise equicontinuous family, where (X,d) is a boundedly UC space and (Y,e) is a metric space. We show that τ_{UB} on ϕ is weaker than $\tau_{AW}(\rho)$. Let $\{f_n\} \subset \phi$ be a sequence $\tau_{AW}(\rho)$-convergent to $f \in \phi$.

Let B be a bounded set in X and $\varepsilon > 0$. We prove that there is $N_0 \in Z^+$ such that for each $n \geq N_0$ $e(f_n(x), f(x)) < \varepsilon$ for every $x \in B$.

The facts $\operatorname{cl} B$ is an Atsuji space and ϕ is pointwise equicontinuous imply that there is $\delta > 0$ such that for every $g \in \phi$ we have $e(g(x), g(y)) < \varepsilon/2$ provided $x, y \in X$, $d(x,y) < \delta$ and $x \in \operatorname{cl} B$. $(*)$
(From the pointwise equicontinuity of ϕ we have that for every $x \in \operatorname{cl} B$ there is $\delta_x > 0$ such that for every $g \in \phi$ we have $e(g(x), g(y)) < \varepsilon/4$ if $d(x,y) < \delta_x$. The family $\{S_d[x, \delta_x] : x \in \operatorname{cl} B\}$ is an open cover of $\operatorname{cl} B$. By (4) in Theorem C there is $\delta > 0$ such that each subset of X of diameter less than δ which meets $\operatorname{cl} B$ lies entirely within some $S_d[u, \delta_u]$, $u \in \operatorname{cl} B$. Let $x \in \operatorname{cl} B$ and $y \in X$ be such that $d(x,y) < \delta$. Thus there is $u \in \operatorname{cl} B$ such that $x, y \in S_d[u, \delta_u]$, i.e. $e(g(x), g(u)) < \varepsilon/4$ and $e(g(y), g(u)) < \varepsilon/4$ for every $g \in \phi$.)

We fix $(x_0, y_0) \in X \times Y$ to serve as a center for ρ-balls in $X \times Y$. Since $f \in L(X,Y)$, there is $M \in Z^+$ such that $B \times f(B) \subset S_\rho[(x_0, y_0), M]$.

Let $K \in Z^+$ be such that $K > \max\{M, 1/\delta, 2/\varepsilon\}$. $\tau_{AW}(\rho)$-convergence of $\{f_n\}$ to f implies that there is $N_0 \in Z^+$ such that for every $n \geq N_0$

$$(**) \qquad G(f_n) \in \Sigma_K[G(f)].$$

We claim that for every $n \geq N_0$ and every $x \in B$ $e(f_n(x), f(x)) < \varepsilon$. Thus let $n \geq N_0$ and $x \in B$. Since $(x, f(x)) \in S_\rho[(x_0, y_0), K]$, there is $z \in X$ (by $(**)$) such that $\rho[(x, f(x)), (z, f_n(z))] < 1/K$, i.e. $d(x, z) < \delta$ and $e(f(x), f_n(z)) < \varepsilon/2$. From $(*)$ we have also $e(f_n(z), f_n(x)) < \varepsilon/2$, thus $e(f_n(x), f(x)) \leq e(f_n(x), f_n(z)) + e(f_n(z), f(x)) < \varepsilon$.

(2) $\to$ (3) Let Y be a bounded metric space. Let $\{f_n\}$ be a sequence from $C(X, Y)$ $\tau_{AW}(\rho)$-convergent to $f \in C(X, Y)$. Clearly $C(X, Y) = L(X, Y)$. By Remark 2.4, $\{f_n\}$ converges to f in the compact-open topology. Thus the family $\phi = \{f, f_1, f_2, \dots f_n, \dots\}$ is pointwise equicontinuous. By (2), $\tau_{AW}(\rho) = \tau_{UB}$ on ϕ, i.e. $\{f_n\}$ converges to f in τ_{UB} topology. Thus $\tau_{AW}(\rho) = \tau_{UB}$ on $C(X, Y)$.

(3) $\to$ (1) This implication is clear from Theorem D. $\square$

Put $\Delta = \{d(\cdot, A) : A \in CL(X)\}$. Then $\Delta \subset L(X, R)$ and Δ is a pointwise equicontinuous family. Thus by Theorem 3.1 if (X, d) is a boundedly UC space then a sequence$\{A_n\} \subset CL(X)$ $\tau_{AW}(d)$-converges to $A \in CL(X)$ if and only if the sequence of the distance functionals $\{d(\cdot, A_n)\}$ $\tau_{AW}(\rho)$-converges to $d(\cdot, A)$, where ρ is the box metric of d and the usual metric e on R.

However, this is true in every metric space (X, d). It is sufficient to realize that we used the assumption of boundedly UC-ness in the proof of Theorem 3.1 to guarantee the uniform equicontinuity of ϕ on bounded sets. In our case, this fact holds since Δ is even a uniformly equicontinuous family.

Proposition 3.2. *Let (X, d) be a metric space. A sequence $\{A_n\} \subset CL(X)$ $\tau_{AW}(d)$-converges to $A \in CL(X)$ if and only if the sequence of the distance functionals $\{d(\cdot, A_n)\}$ $\tau_{AW}(\rho)$-converges to $d(\cdot, A)$.*

The following examples show that neither Theorem 3.1 nor Proposition 3.2 hold for bounded proximal topology.

Example 3.3. Put $X = Z^+$ with the zero-one metric d and $Y = Z^+$ with the usual metric e. For every $n \in Z^+$ define f_n as follows: $f_n(n) = n$ and $f_n(z) = 0$ otherwise. Then $\{f_n\}$ σ_ρ-converges to the zero function f, but$\{f_n\}$ does not converge uniformly to f.

Example 3.4. Put $X = Z^+$ with the zero-one metric d. For every $n \in Z^+$ define $A_n = X - \{n\}$ and $A = X$. Clearly the sequence $\{A_n\}$ σ_d-converges to A. For every $n \in Z^+$ $d(\cdot, A_n)$ is the following function: $d(n, A_n) = 1$ and $d(x, A_n) = 0$ otherwise. So $\{d(\cdot, A_n)\}$ does not σ_ρ-converges to $d(\cdot, A)$, where ρ is the box metric of d and the usual metric e on R.

However, we have an analog of Theorem 1 in [Ho] for σ_ρ topology.

Theorem 3.5. *Let (X,d) be a locally connected metric space and (Y,e) be a metric space. Then the compact-open topology on $C(X,Y)$ is weaker than σ_ρ on $C(X,Y)$.*

PROOF: Suppose τ_{CO} is not weaker than σ_ρ on $C(X,Y)$. There is $U \in \tau_{CO}$ such that $U \notin \sigma_\rho$. Thus there is $f \in U$ with the following property: for every σ_ρ-neighbourhood V of f, $V \not\subset U$. Thus there is a net $\{f_t : t \in T\}$ in $C(X,Y)$ which σ_ρ-converges to f such that $f_t \notin U$ for every $t \in T$. There is $\varepsilon > 0$ ($\varepsilon < 1$) and a compact set K in X such that $\{g \in C(X,Y) : e(g(x), f(x)) < \varepsilon$ for every $x \in K\} \subset U$. For every $t \in T$ there is a point $x_t \in K$ such that $e(f_t(x_t), f(x_t)) \geq \varepsilon$. Let x be a cluster point of $\{x_t : t \in T\}$. We can suppose that $\{x_t : t \in T\}$ converges to x, otherwise we will work with a subnet of $\{x_t : t \in T\}$ and corresponding subnet of $\{f_t : t \in T\}$. Let $1 > \alpha > 0$ be such that for every $z \in S_d[x,\alpha]$ $e(f(z), f(x)) < \varepsilon/4$. Let O be a connected neighbourhood of x for which $O \subset S_d[x, \alpha/2]$ and $\delta > 0$ be such that $\delta < \alpha$ and $S_d[x,\delta] \subset O$.

We fix (a,b) in $X \times Y$ to serve as center for ρ-balls in $X \times Y$. There is $M \in Z^+$ such that $S_\rho[(x,f(x)),2] \subset S_\rho[(a,b),M]$ and $M > \max\{2/\delta, 4/\varepsilon\}$. Let $i \in T$ be such that for every $j \geq i$ $x_j \in S_d[x,\delta/2]$ and $G(f_j) \in \phi_{G(f)}[M,(x,f(x))]$. Thus for each $j \geq i$ $\rho[(x,f(x)), G(f_j)] < 1/M$. Put $A = \{y \in Y : e(f(x),y) = 3\varepsilon/4\}$. The connectedness of O and the continuity of functions imply that for each $j \geq i$ there is $y_j \in 0$ such that $f_j(y_j) \in A$. Let $j \geq i$. The inclusion $G(f_j) \in \phi_{G(f)}[M,(x,f(x))]$ guarantees the existence of a point $y \in X$ for which $\rho[(y,f(y)),(y_j,f_j(y_j))] < 1/M$, i.e. $y \in S_d[x,\alpha]$ and $e(f_j(y_j), f(y)) < \varepsilon/4$. Then $3\varepsilon/4 = e(f_j(y_j), f(x)) \leq e(f_j(y_j), f(y)) + e(f(y), f(x))$. Thus $e(f(y), f(x)) \geq \varepsilon/2$ and that is a contradiction. $\square$

Example 1 in [Ho] shows that the assumption of local connectedness in Theorem 3.5 is essential.

If ρ is a metric on $X \times Y$ such that every bounded set in $X \times Y$ is totally bounded set then we know by Theorem 3.4 in [BL2] that $\tau_{AW}(\rho) = \sigma_\rho$ on $CL(X \times Y)$, i.e. $\tau_{AW}(\rho) = \sigma_\rho$ also on $C(X,Y)$. The coincidence of these two topologies occurs also in other case as the following easy proposition shows.

Proposition 3.6. *Let (X,d) and (Y,e) be metric spaces and X be boundedly compact. Then $\tau_{AW}(\rho) = \sigma_\rho$ on $C(X,Y)$.*

PROOF: Since the inclusion $\sigma_\rho \subset \tau_{AW}(\rho)$ is clear, we prove the inclusion $\tau_{AW}(\rho) \subset \sigma_\rho$. We fix (a,b) in $X \times Y$ to serve as center for ρ-balls in $X \times Y$. Let $f \in C(X,Y)$ and $n \in Z^+$. Since $G(f) \cap B_\rho[(a,b),n]$ is compact, we choose $\{a_1, a_2, \dots a_k\} \in G(f)$ such that $G(f) \cap B_\rho[(a,b),n] \subset S_\rho[\{a_1,a_2,\dots a_k\}, 1/2n]$. It is very easy to see that $\phi_{G(f)}[2n, a_1, a_2, \dots a_k] \subset \Sigma_n[G(f)]$. $\square$

Example 3.3 shows that the assumption of boundedly compactness is essential.

The following theorem gives further sufficient conditions under which $\tau_{UB} = \tau_{AW}(\rho)$ on $L(X,Y)$.

Theorem 3.7. *Let (X,d) be a locally connected boundedly* UC *space and (Y,e) be metric space. Then $\tau_{UB} = \tau_{AW}(\rho)$ on $L(X,Y)$.*

Proof: It is sufficient to prove that $\tau_{UB} \subset \tau_{AW}(\rho)$ on $L(X,Y)$. Suppose $\tau_{UB} \not\subset \tau_{AW}(\rho)$ on $L(X,Y)$. There is $V \in \tau_{UB}$ such that $V \notin \tau_{AW}(\rho)$. Thus there is $f \in V \cap L(X,Y)$ such that every $\tau_{AW}(\rho)$-neighbourhood of f is not contained in V. There is a sequence $\{f_n\} \subset L(X,Y) - V$ such that $\{f_n\}$ $\tau_{AW}(\rho)$-converges to f. Let B be a bounded subset of X and $\varepsilon > 0$ such that the set $G = \{g \in L(X,Y) : e(g(x), f(x)) < \varepsilon$ for every $x \in B\}$ is contained in V. Since for every $n \in Z^+$ $f_n \notin G$, there is a sequence $\{x_n\} \subset B$ with the following property: $e(f_n(x_n), f(x_n)) \geq \varepsilon$ for every $n \in Z^+$. We have two possibilities:

(1) $\{x_n\}$ has a cluster point in X;
(2) $\{x_n\}$ has no cluster point in X.

In the case (1) let x be a cluster point of $\{x_n\}$. Now the local connectedness of X applied on x leads to a contradiction by the same argument as in Theorem 3.5.

In the case (2) the sequence $\{d(x_n, \{x_n\}^c\}$ does not converge to 0, otherwise by Theorem C (3), $\{x_n\}$ has a cluster point in X. Thus there is $\delta > 0$ such that $S_d[x_n, \delta] = \{x_n\}$ for every $n \in J$, where J is an infinite subset of Z^+.

We fix $(a,b) \in X \times Y$ to serve as a center for ρ-balls in $X \times Y$. Since $f \in L(X,Y)$ there is $M \in Z^+$ such that $B \times f(B) \subset S_\rho[(a,b), M]$. Let $K \in Z^+$ be such that $K > \max\{M, 1/\delta, 1/\varepsilon\}$. $\tau_{AW}(\rho)$-convergence of $\{f_n\}$ to f implies that there is $m \in Z^+$ such that for every $n \geq m$

$$G(f_n) \in \Sigma_K[G(f)].$$

Let $n \in J$ and $n \geq m$. By the previous inclusion, $G(f_n) \in \Sigma_K[G(f)]$, thus there is $z_n \in X$ such that $\rho[(x_n, f(x_n)), (z_n, f_n(z_n))] < 1/K < \delta$, i.e. $d(x_n, z_n) < \delta$ thus $x_n = z_n$ and $e(f(x_n), f_n(x_n)) < \varepsilon$. This is a contradiction. □

Corollary 3.8 (Theorem 2 [Ho]). *Let (X,d) be a locally connected boundedly compact space and (Y,e) be a metric space. Then the compact-open topology and the Attouch-Wets topology on $C(X,Y)$ coincide.*

Proof: Clearly (X,d) is a boundedly UC space, $C(X,Y) = L(X,Y)$ and $\tau_{UB} = \tau_{CO}$ on $C(X,Y)$. □

From Corollary 3.8 we know that if (X,d) is a locally connected boundedly compact space and (Y,e) is a complete metric space, then $(C(X,Y), \tau_{AW}(\rho))$ is a completely metrizable space.

To study complete subspaces of $C(X,Y)$ with respect to the uniformity Ω_ρ can be interesting. Clearly $(C(X,Y), \Omega_\rho)$ need not be complete. If $X = Y = R$ with the usual metric and $f_n(x) = nx$ for all x, then $\{f_n\}$ is Ω_ρ-cauchy sequence without a limit point in $C(X,Y)$.

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