Strong unicity criterion in some space of operators

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Abstract. Let X be a finite dimensional Banach space and let $Y \subset X$ be a hyperplane. Let $\mathscr{L}_Y = \{L \in \mathscr{L}(X,Y) : L\,|_Y = 0\}$. In this note, we present sufficient and necessary conditions on $L_0 \in \mathscr{L}_Y$ being a strongly unique best approximation for given $L \in \mathscr{L}(X)$. Next we apply this characterization to the case of $X = l^n_\infty$ and to generalization of Theorem I.1.3 from [12] (see also [13]).

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0. Introduction.

Let W be a normed linear space and let $V \subset W$ be its nonempty subset. An element $v \in V$ is called a best approximation to $w \in W$ iff

$$\|w - v\| = \operatorname{dist}(w, V) = \inf\{\|w - y\|;\ y \in V\}. \tag{0.1}$$

If an element v satisfies additionally

$$\begin{aligned} &\|w - y\| \geq \|w - v\| + r \cdot \|y - v\| \quad \text{with a constant } r > 0 \\ &\text{independent of } y \in V, \end{aligned} \tag{0.2}$$

then v is said to be a strongly unique best approximation (briefly SUBA) to $w \in V$.

The theory of strong uniqueness has its origin in the following result of Newman and Shapiro [11]. Given a compact Hausdorff space T, let $C(T, \mathbb{K})$ denote the Banach space of either complex ($\mathbb{K} = \mathbb{C}$) or real-valued ($\mathbb{K} = \mathbb{R}$) continuous functions on V. If V is a Haar subspace of $C(T, \mathbb{K})$, then for every $w \in C(T, \mathbb{K})$, one can find a constant $r > 0$ such that the best approximation $v \in V$ satisfies one of the following inequalities:

$$\|w - y\| \geq \|w - v\| + r \cdot \|y - v\|, \qquad \text{for } y \in V \tag{0.3}$$

if $\mathbb{K} = \mathbb{R}$, and

$$\|w - y\|^2 \geq \|w - v\|^2 + r \cdot \|y - v\|^2, \qquad \text{for } y \in V \tag{0.4}$$

in the complex case.

The significance of this notion can be illustrated by Cheney's observation that strong unicity of an optimal element yields the continuity of metric projection (see [6]). One can see that the proof of the convergence of the Remez algorithm depends, in fact, on strong unicity (for an extended version see [9]). For more precise information about strong unicity the reader is referred to [3], [4], [8], [11], [14], [15].

In this note we will investigate strong unicity in the case $W = \mathscr{L}(X)$, the space of all linear operators going from a finite dimensional real Banach space X into itself, and $V = \mathscr{L}_Y(X,Y) = \{L \in \mathscr{L}(X,Y) : L\,|_Y = 0\}$ (we will write $\mathscr{L}_Y$ for brevity), where $Y \subset X$ is a hyperplane. We characterize strong unicity in terms of the functionals from X^* or Y^*, which is more convenient for applications. Next we apply this characterization to the case of X being an arbitrary three dimensional Banach space and to $X = l_\infty^n$. In particular, we generalize Theorem I.1.3 from [12] (see also [13]) and Theorem 2.5 b) from [10].

Now we introduce some notations which will be used in this note. By S_X we will denote the unit sphere in a Banach space X. The symbol ext S_X stands for the set of all extremal points of S_X. Given $L \in \mathscr{L}(X)$, we write $\mathscr{P}_Y(L) = \{L_0 \in \mathscr{L}_Y : \|L - L_0\| = \operatorname{dist}(L, \mathscr{L}_Y)\}$. In this note, if nothing special is assumed, X will denote a finite dimensional real Banach space and f a functional from $X^* \setminus \{0\}$. If $Y \subset X$ is a linear subspace and $A \subset X^*$ then $A\,|_Y$ stands for a set of all restrictions of functionals from A. In the sequel we will use the following

Theorem 0.1 (see [10, Theorem 2.3]). *Assume X is a reflexive space and Y is a Banach space both over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Denote by $\mathscr{K}(X,Y)$ the space of all compact operators going from X into Y and let $\mathscr{V} \subset \mathscr{K}(X,Y)$ be a convex set. For given $K \in \mathscr{K}(X,Y)$ and $V \in \mathscr{V}$ put*

$$\operatorname{crit}^*_Y(K - V) = \{h \in \operatorname{ext} S_{Y^*} : \|h \circ (K - V)\| = \|K - V\|\} \tag{0.5}$$

and for every $h \in \operatorname{crit}^*_Y(K - V)$ define

$$A_h = \{x \in \operatorname{ext} S_X : h(K - V)x = \|K - V\|\}. \tag{0.6}$$

Then we have:

(a) *$V \in \mathscr{P}_{\mathscr{V}}(K)$ (the set of all best approximants to K in $\mathscr{V}$) if and only if for every $U \in \mathscr{V}$ there exists $h \in \operatorname{crit}^*_Y(K - V)$ with $\inf\{re(h(U - V)x) : x \in A_h\} \le 0$.*

(b) *V is a SUBA to K in $\mathscr{V}$ with a constant $r > 0$ if and only if for every $U \in \mathscr{V}$ there exists $h \in \operatorname{crit}^*(K - V)$ with $\inf\{re(h(U - V)x) : x \in A_h\} \le r \cdot \|U - V\|$.*

I. The main result.

We start with two preliminary remarks.

Remark 1.1. *For $L \in \mathscr{L}(X)$ let us set*

$$\operatorname{crit}(L) = \{x \in S_X : \|Lx\| = \|L\|\}. \tag{1.1}$$

Assume $L_0 \in \mathscr{P}_Y(L)$, $Y = \ker f$, $\|f\| = 1$ and $\|L - L_0\| > \|L\,|_Y\|$. Put

$$C_{L-L_0} = \{x \in \operatorname{crit}(L - L_0) : f(x) > 0\}. \tag{1.2}$$

Then C_{L-L_0} *is a nonempty closed set,* $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$ *and* $C_{L-L_0} \cup -C_{L-L_0} = \mathrm{crit}\,(L-L_0)$.

PROOF: It is clear that the set $A = \{x \in \mathrm{crit}\,(L-L_0) : f(x) \geq 0\}$ is closed. Since $\mathrm{dist}\,(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$ and Y is a hyperplane,

$$A = \{x \in \mathrm{crit}\,(L-L_0);\ f(x) > 0\},$$

which proves that C_{L-L_0} is closed. The fact $\mathrm{crit}\,(L-L_0) \cap Y = \emptyset$ implies immediately that $C_{L-L_0} \cup -C_{L-L_0} = \mathrm{crit}\,(L)$. By (1.2) $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$. □

Remark 1.2. *Let* $L \in \mathscr{L}(X)$, $\mathrm{dist}\,(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$, $L_0 \in \mathscr{L}_Y$. *Define*

$$D_{L-L_0} = \{h \in \mathrm{crit}^*_X(L-L_0) : C_{L-L_0} \cap A_h \neq \emptyset\} \tag{1.3}$$

(*see* (0.6)) *and if* $L \in \mathscr{L}(X,Y)$,

$$D^Y_{L-L_0} = \{h \in \mathrm{crit}^*_Y(L-L_0) : C_{L-L_0} \cap A_h \neq \emptyset\}. \tag{1.4}$$

Then D_{L-L_0} ($D^Y_{L-L_0}$ *resp.*) *is a compact set,* $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$ ($D^Y_{L-L_0} \cap -D^Y_{L-L_0} = \emptyset$ *resp.*).

PROOF: Assume $h \in \mathrm{cl}\,(D_{L-L_0})$ and let $\{h_n\} \subset D_{L-L_0}$, $h_n \to h$. By (1.3), for every $n \in \mathbb{N}$ there exists $x_n \in C_{L-L_0} \cap A_{h_n}$, i.e. $h_n(L-L_0)x_n = \|L-L_0\|$. Passing to the subsequence, if necessary, we may assume $x_n \to x$. By Remark 1.1, $x \in C_{L-L_0}$. Note that $h(L-L_0)x = h_n(L-L_0)x + (h-h_n)(L-L_0)x = h_n(L-L_0)x_n + h_n(L-L_0)(x-x_n) + (h-h_n)(L-L_0)x$. Since the last two terms tend to 0 as $n \to \infty$, $h(L-L_0)x = \|L-L_0\|$ and consequently $x \in A_h$. Since $x \in C_{L-L_0}$, by (1.3) $h \in D_{L-L_0}$. Note that $\mathrm{dist}\,(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$ implies $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$. The proof for the set $D^Y_{L-L_0}$ goes on in the same manner, so we omit it. □

Now we state the main result of this note.

Theorem 1.3. *Assume* $L \in \mathscr{L}(X)$ *and let* $Y = \ker f$, $\|f\| = 1$. *Assume furthermore that* $\mathrm{dist}\,(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$ *and let* $L_0 \in \mathscr{L}_Y$. *Then the following conditions are equivalent:*

(a) L_0 *is a* SUBA *to* L *in* $\mathscr{L}_Y$ ($L_0 \in \mathscr{P}_Y$ *resp.*),
(b) $0 \in \mathrm{int\ conv}\ D_{L-L_0}\,|_Y$ ($0 \in \mathrm{conv}\ D_{L-L_0}\,|_Y$ *resp.*).

PROOF: Assume L_0 is a SUBA to L in $\mathscr{L}_Y$ and let $0 \notin \mathrm{int\ conv}\ D_{L-L_0}\,|_Y$. It means that there exists $\psi \in Y^{**}$ with $\psi(h) \geq 0$ for every $h \in D_{L-L_0}\,|_Y$ (we may assume $\|\psi\| = 1$). Since Y is finite dimensional, $\psi = y$ for some $y \in S_Y$. Define $L_1 = f(\cdot) \cdot y$ and note that $L_1 \in \mathscr{L}_Y$. By (1.3) and Remark 1.2, for every $h \in D_{L-L_0}$ we have

$$\begin{aligned}\inf\{h(L_1x) : x \in A_h\} &= \inf\{f(x) \cdot h(y) : x \in A_h\} = \\ &= h(y) \cdot \inf\{f(x) : x \in A_h\} \geq 0 > -r \cdot \|L_1\| \quad \text{for every} \quad r > 0.\end{aligned}$$

By Theorem 0.1 L_0 is not a SUBA to L in $\mathscr{L}\,|_Y$; a contradiction. Since by Remark 1.2 the set D_{L-L_0} is compact and consequently conv D_{L-L_0} is also compact, the proof of the second case goes on the same line.

To prove the converse, let us define a function $g : S_Y \to \mathbb{R}$ by

$$g(y) = \inf\{g_h(y) : h \in D_{L-L_0}\} \quad \text{for} \quad y \in S_Y; \tag{1.5}$$

where $g_h(y) = \inf\{f(x) \cdot h(y) : x \in A_h\}$. Note that the function $S_Y \ni y \to f(x) \cdot h(y)$ is continuous and consequently the functions g_h and g are upper-semicontinuous.

Now assume $0 \in \operatorname{int} \operatorname{conv} D_{L-L_0}\,|_Y$. It means that for every $y \in S_Y$ there exists $h \in D_{L-L_0}$ with $h(y) < 0$. (If no, then $D_{L-L_0}\,|_Y \subset \{h \in Y^* : h(y) \geq 0\}$ for some $y \in S_Y$ and consequently $\operatorname{int} \operatorname{conv} D_{L-L_0}\,|_Y \subset \{h \in Y^* : h(y) > 0\}$. But $0 \in \operatorname{int} \operatorname{conv} D_{L-L_0}$; a contradiction.) Since $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\|$ and Y is a hyperplane, for every $y \in S_Y$, $g(y) < 0$. Since g is upper-semicontinuous, the value $\gamma = \max\{g(y) : y \in S_Y\}$ is attained in some point $y_0 \in S_Y$ and consequently $\gamma < 0$. We show that L_0 is a SUBA to L in $\mathscr{L}_Y$ with $r = -\gamma$. To do this, fix $L_1 \in \mathscr{L}_Y \setminus \{0\}$. It is clear that $L_1 = f(\cdot) \cdot y_1$ for some $y_1 \in Y \setminus \{0\}$. Put $y_2 = y_1/\|y_1\|$, fix $\varepsilon > 0$ and take $h \in D_{L-L_0}$ with $g_h(y_2) < g(y_2) + \varepsilon$. Note that

$$\begin{aligned} g_h(y_2) &= \inf\{f(x) \cdot h(y_2) : x \in A_h\} = \inf\{h(L_1 x)/\|y_1\|;\ x \in A_h\} \leq \\ &\leq g(y_2) + \varepsilon \leq -r + \varepsilon, \quad \text{which gives} \quad \inf\{h(L_1 x) : x \in A_h\} \leq -(r - \varepsilon) \cdot \|L_1\|. \end{aligned}$$

Following Theorem 0.1 and Remark 1.2, L_0 is SUBA to L in $\mathscr{L}_Y$ with $r - \varepsilon$ for every $\varepsilon > 0$ and consequently with r. The proof is complete. □

Remark 1.4. *If $L \in \mathscr{L}(X,Y)$ then the set D_{L-L_0} in Theorem* 1.3 *can be replaced by $D^Y_{L-L_0}$ (see* (1.4)*).*

As an immediate consequence of Theorem 1.3 we get

Corollary 1.5. *Assume $L \in \mathscr{L}(X)$, $L_0 \in \mathscr{P}_Y(L)$, $\|L - L_0\| > \|L\,|_Y\|$. Then the set $D_{L-L_0}\,|_Y$ is linearly dependent. If $L \in \mathscr{L}(X,Y)$, the same holds for $D^Y_{L-L_0}$.*

Reasoning as in [10, Theorem 2.5] we may show

Remark 1.6. *The constant r defined in Theorem* 1.3 *is the best possible.*

Now we will point out when the assumption $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\|$ is fulfilled.

Remark 1.7. *Assume X is a Banach space and let $Y \subset X$ be its complemented subspace. Let $\mathscr{P}(X,Y) = \{P \in \mathscr{L}(X,Y) : P\,|_Y = id\}$. Take $P_0 \in \mathscr{P}(X,Y)$ and note that* $\operatorname{dist}(P_0, \mathscr{L}_Y) = \inf\{\|P\| : P \in \mathscr{P}(X,Y)\} = \lambda(X,Y)$.

In many cases of hyperplanes, $\operatorname{dist}(P_0, \mathscr{L}_Y) > \|P_0\,|_Y\| = 1$ (see e.g. [2], [5]).

It is well known (see e.g. [12]) that if X is not a Hilbert space then there exists a hyperplane Y in X satisfying $\lambda(X,Y) > 1$. If $\operatorname{dist}(P_0, \mathscr{L}_Y) > 1$ then it is easy to show that $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\|$ if $\|L - P_0\| < \operatorname{dist}(P_0, \mathscr{L}_Y) - 1$.

Now we show an estimation from above of the number $\operatorname{dist}(L, \mathscr{L}_Y)$.

Proposition 1.8. *Assume X is a Banach space and let Y be its complemented subspace. Then for every $L \in \mathscr{L}(X,Y)$*

$$\|L\,|_Y\,\| \leq \operatorname{dist}(L, \mathscr{L}_Y) \leq \lambda(X,Y) \cdot \|L\,|_Y\,\|.$$

Proof: Fix $L \in \mathscr{L}(X,Y)$ and $\varepsilon > 0$. Take $P_\varepsilon \in \mathscr{P}(X,Y)$ with $\|P_\varepsilon\| < \lambda(X,Y)+\varepsilon$ and put $L_\varepsilon = L \circ (I - P_\varepsilon)$. It is clear that $L_\varepsilon \in \mathscr{L}_Y$. Compute,

$$\|L - L_\varepsilon\| = \|L - L \circ (I - P_\varepsilon)\| = \|L \circ P_\varepsilon\| \leq \|L\,|_Y\,\| \cdot \|P_\varepsilon\|,$$

which gives the desired result. □

Corollary 1.9. *Assume that $\lambda(X,Y) = 1$. Then*

$$\operatorname{dist}(L, \mathscr{L}_Y) = \|L\,|_Y\,\| \quad \textit{for every} \quad L \in \mathscr{L}(X,Y).$$

In particular if there exists $P_0 \in \mathscr{P}(X,Y)$ with $\|P_0\| = 1$, then the operator $L_0 = L \circ (I - P_0) \in P_Y(L)$.

Since in the case when Y is a hyperplane we have $\lambda(X,Y) \leq 2$ (for more precise results see [1], [7], [12, p. 84], we immediately get

Corollary 1.10. *Assume $Y \subset X$ is a hyperplane. Then*

$$\|L\,|_Y\,\| \leq \operatorname{dist}(L, \mathscr{L}_Y) \leq 2 \cdot \|L\,|_Y\,\| \quad \textit{for every} \quad L \in \mathscr{L}(X,Y).$$

II. Applications.

Now we apply Theorem 1.3 to generalize Theorem I.1.3 from [12] (see also [13]).

Theorem 2.1. *Assume X is a three dimensional Banach space and let $Y \subset X$ be a hyperplane. Assume furthermore $L \in \mathscr{L}(X,Y)$, $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$. Then there exists $L_0 \in \mathscr{L}_Y$ which is a* SUBA *to L in $\mathscr{L}_Y$.*

Proof: Since $\mathscr{L}_Y$ is a finitely dimensional linear space, the set $\mathscr{P}_Y(L)$ is nonempty. Take an arbitrary $L_0 \in \mathscr{P}_Y(L)$. By Theorem 1.3 and Remark 1.4 it is sufficient to show that $0 \in \operatorname{int}\operatorname{conv} D^Y_{L-L_0}$ (see 1.4). Assume on the contrary that it is not true. Following Theorem 1.3, $0 \in \operatorname{conv} D^Y_{L-L_0}$. Since $\dim Y = 2$, $0 = \alpha \cdot h_1 + (1-\alpha) \cdot h_2$, where $h_1, h_2 \in D^Y_{L-L_0}$ and $\alpha \in (0,1)$. Since $\|h_1\| = \|h_2\| = 1$, we easily get $\alpha = 1/2$. Consequently $h_1 = -h_2$ which gives $h_1 \in D^Y_{L-L_0} \cap -D^Y_{L-L_0}$; a contradiction with Remark 1.2.

□

Remark 2.2. *The assumption* $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$ *in Theorem* 2.1 *is essential. Take e.g. $X = l^3_\infty$, $Y = \ker f$, $f = (1/2, 1/2, 0)$. It is easy to check that the operators $P_1 = Id - f(\cdot)\cdot(2,0,0)$ and $P_2 = Id - f(\cdot)\cdot(0,2,0) \in \mathscr{P}(X,Y)$, $P_1 \neq P_2$, $\|P_1\| = \|P_2\| = 1$. Consequently the set $P_Y(P_1) \supset \{0, P_1 - P_2\}$ and strong unicity does not hold.*

Remark 2.3. *The assumption* $\dim X = 3$ *in Theorem* 2.1 *is essential. Take e.g.* $X = l^4_\infty$, $Y = \ker f$, $f = (1/3, 1/3, 1/3, 0)$. *It is well known* (*see* [5] *or* [12]) *that* $\lambda(X,Y) = 4/3$. *Take* $P_0 \in \mathscr{P}(X,Y)$ *with* $\|P_0\| = 4/3$ (*the formula for such a projection is given for example in* [12, p. 104]). *Then* $\operatorname{dist}(P_0, \mathscr{L}_Y) = \lambda(X,Y) = 4/3 > 1 = \|P_0\,|_Y\,\|$. *By Theorem* 2.5 b) *of* [10], 0 *is not a* SUBA *to* P_0 *in* $\mathscr{L}_Y$.

Now we use Theorem 1.3 to extend Theorem 2.5 b) from [10].

Theorem 2.4. *Assume* $X = l^n_\infty$, $Y = \ker f$, $\|f\|_1 = 1$. *Let* $L \in \mathscr{L}(X)$ *and let* $\operatorname{dist}(L, \mathscr{L}_Y) > \|L\,|_Y\,\|$ (*by Theorem* 1 *from* [5] *and Remark* 1.7 *such operators exist if and only if* $|f_i| < 1/2$ *for* $i = 1, \ldots, n$). *If* $|f_i| > 0$ *for* $i = 1, \ldots, n$ *then there exists* $L_0 \in \mathscr{L}_Y$ *which is a* SUBA *to* L *in* $\mathscr{L}_Y$.

Proof: Since $\mathscr{L}_Y$ is a finitely dimensional space the set $\mathscr{P}_Y(L)$ is nonempty. Fix $L_0 \in \mathscr{P}_Y(L)$ and let $D_{L-L_0} = \{\phi_1, \ldots, \phi_k\}$ where $\phi_i = \mp e_{j(i)}$ for $i = 1, \ldots, k$. By Theorem 1.3, $0 \in \operatorname{conv} D_{L-L_0}\,|_Y$. Hence $0 = \sum_{i=1}^{l} \lambda_i \cdot \phi_i$, $l \leq k$, $\lambda_i > 0$, $\sum_{i=1}^{l} \lambda_i = 1$. Since $\dim Y = n-1$, by Carathéodory's Theorem we can assume $l \leq n$. We will show that $l = n$. To do this, by Corollary 1.5, it is sufficient to show that for each $i \in \{1, \ldots, n\}$ the set $E_i = \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, , e_n\}$ is total over Y. So assume $\sum_{i=1}^{n} f_i \cdot y_i = 0$ and $e_j(y) = 0$ for $j \neq i$. It means that $y_j = 0$ for $j \neq i$ and $f_i \cdot y_i = 0$. Since $f_i \neq 0$, $y_i = 0$. Consequently $l = n$ and $0 \in \operatorname{int} \operatorname{conv} D_{L-L_0}\,|_Y$. By Theorem 1.3, L_0 is a SUBA to L in $\mathscr{L}_Y$. The proof is complete. □

Note that Remark 2.3 shows that the assumption $f_i \neq 0$ for $i = 1, \ldots, n$ in Theorem 2.4 is essential.

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