# A uniform boundedness principle of Pták

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*Abstract.* The Antosik-Mikusinski Matrix Theorem is used to give an extension of a uniform boundedness principle due to V. Pták to certain metric linear spaces. An application to bilinear operators is given.

*Keywords:* uniform boundedness, linear operator, bilinear operator

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In [P] V. Pták used the classical sliding hump technique to give an extension of the classical uniform boundedness principle for pointwise bounded families of continuous linear operators on Banach spaces and in [NP] used an abstract sliding hump construction to give a further extension. This generalization of the uniform boundedness principle was used to establish automatic continuity results. A similar generalized uniform boundedness principle was established earlier by Lorentz and MacPhail to give a generalization of the Silvermann-Toeplitz Theorem from summability ([LM]). The Antosik-Mikusinski Matrix Theorem ([AS, 2.2]) can also be considered to be an abstraction of the sliding hump technique and has been used to treat a number of topics in classical functional analysis and measure theory including the uniform boundedness principle (see [AS], especially § 4 for the uniform boundedness principle). In this note we show that the Antosik-Mikusinski Matrix Theorem can be used to extend V. Pták's version of the uniform boundedness principle to metric $\mathcal{K}$-spaces. As an application we use the extended form of the uniform boundedness principle to generalize a result of Pták on bilinear operators.

We fix the notation to be used. Throughout the sequel $X$ will denote a metric linear space and $Y$ a normed linear space. A sequence $\{x_k\}$ is $\mathcal{K}$ convergent if every subsequence of $\{x_k\}$ has a further subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ converges in $X$. A $\mathcal{K}$ convergent sequence converges to 0, but a sequence can converge to 0 and not be $\mathcal{K}$ convergent ([AS, 3.3]). A space in which every sequence which converges to 0 is $\mathcal{K}$ convergent is called a $\mathcal{K}$-space. A complete metric linear space is a $\mathcal{K}$-space ([AS, 3.2]), but there are (normed) $\mathcal{K}$-spaces which are not complete ([K]). The notion of a $\mathcal{K}$-space is a weakened form of completeness which has proven to be a good substitute for completeness in several topics in functional analysis ([AS]).

We first establish a special case of Pták's uniform boundedness principle for metric $\mathcal{K}$-spaces. This result was established for Banach spaces by Lorentz and MacPhail [LM]; see also [M, 4.1].

**Theorem 1.** *Let $X$ be a metric $\mathcal{K}$-space. For each $k \in \mathbb{N}$ let $T_k : X \to Y$ be linear and let $M_k$ be a closed subspace of $X$ such that $T_k$ is continuous on $M_k$ and $M_k \supset M_{k+1}$. If $\{T_k\}$ is pointwise bounded on $X$, then there exists $p$ such that $\{T_k : k \in \mathbb{N}\}$ is equicontinuous on $M_p$.*

PROOF: Let $\{U_k\}$ be a neighborhood base at 0 in $X$ with $U_k \supset U_{k+1}$, $\bigcap_{k=1}^{\infty} U_k = \{0\}$. If the conclusion fails, for each $k$ there exist $n_k$ and $x_k \in U_k \cap M_k$ such that $\|T_{n_k} x_k\| > k$, where we may assume $n_{k+1} > n_k$. For notation convenience assume that $n_k = k$. Consider the matrix $M = [(1/i)T_i x_j]$. Since $\{T_i\}$ is pointwise bounded, the columns of $M$ converge to 0. Since $x_j \to 0$ in $X$, a $\mathcal{K}$-space, given any increasing sequence of positive integers $\{p_j\}$ there is a subsequence $\{q_j\}$ of $\{p_j\}$ such that $\sum_{j=1}^{\infty} x_{q_j} = x \in X$. For each fixed $i$, $\sum_{q_j \geq i} x_{q_j} \in M_i$ since $M_i$ is closed and $M_i \supset M_{k+1}$ so that

$$\sum_{j=1}^{\infty} \frac{1}{i} T_i x_{q_j} = \frac{1}{i} T_i \Big( \sum_{q_j < i} x_{q_j} \Big) + \frac{1}{i} T_i \Big( \sum_{q_j \geq i} x_{q_j} \Big) = \frac{1}{i} T_i x$$

since $T_i$ is continuous on $M_i$. Therefore, since $\{T_i\}$ is pointwise bounded,

$$\lim_i \sum_{j=1}^{\infty} \frac{1}{i} T_i x_{q_j} = \lim_i \frac{1}{i} T_i x = 0.$$

Hence, $M$ is a $\mathcal{K}$-matrix ([AS, § 2]) and the diagonal of $M$ converges to 0 by the Antosik-Mikusinski Matrix Theorem (AS, 2.2). This contradicts $\|T_i x_i\| > 1$ above. □

This result was obtained in [LM] for Banach spaces by a sliding hump technique. It was obtained for the case when $X$ is a Banach space and $M_k$ is the (closed) kernel of $T_k$ by Pták in [P]. It is shown in 2.1.3 of [PB] that the assumption that $M_k$ is a closed kernel cannot be replaced with the assumption that $M_k$ is a Baire space. It is also shown in 2.7. of [PB] that the assumption that $Y$ is a normed space cannot be replaced with $Y$ is a metric linear space.

From Theorem 1 we can easily obtain the general form of the uniform boundedness principle given in [NP].

**Theorem 2.** *Let $X$ be a metric $\mathcal{K}$-space. For each $a \in A$ let $T_a : X \to Y$ be linear and $M_a$ be a closed linear subspace of $X$ such that $T_a$ is continuous on $M_a$. If $\mathcal{T} = \{T_a : a \in A\}$ is pointwise bounded on $X$, then there exists a finite subset $F \subset A$ such that $\mathcal{T}$ is equicontinuous on $\bigcap_{a \in F} M_a$.*

PROOF: Let $\{U_k\}$ be as in the proof of Theorem 1. Suppose the conclusion fails. Pick $a_1 \in A$. There exist $a_2 \in A$, $x_2 \in M_{a_1} \cap U_1$ such that $\|T_{a_2} x_2\| > 1$. There exist $a_2 \in A$, $x_3 \in M_{a_1} \cap M_{a_2} \cap U_2$ such that $\|T_{a_3} x_3\| > 1$. Continuing this construction produces sequences $\{a_k\} \subset A$, $x_k \in M_{a_1} \cap \cdots \cap M_{a_k} \cap U_k$ such that $\|T_{a_k} x_k\| > 1$. Since $T_{a_k}$ is continuous on $M_{a_1} \cap \cdots \cap M_{a_k}$, Theorem 1 implies that $\{T_{a_j} : j \in \mathbb{N}\}$ is equicontinuous on some $M_{a_1} \cap \cdots \cap M_{a_p}$. Since $x_j \to 0$, this is impossible by the construction. □

This result was extended to Fréchet spaces $X$ in 4.9.15 of [PB]. It was also shown in 4.9.16 of [PB] that $X$ is a Banach space cannot be replaced in Theorem 1 by $X$ is normed Baire even if each $M_k$ is a Baire space. Since a metric $\mathcal{K}$-space is a Baire space but need not be complete, Theorem 2 gives an improvement of 4.9.15 of [PB]. The methods employed above are much simpler than those employed in [PB].

As an application of Theorem 2 we give an extension of Theorem 3.2 of [P] to families of bilinear maps. Let $Z$ be a normed linear space. If $b : X \times Y \to Z$ is bilinear, for $x \in X$ ($y \in Y$) we let $b(x, \cdot) : Y \to Z$ ($b(\cdot, y) : X \to Z$) be the linear map $b(x, \cdot)(y) = b(x, y)$ ($b(\cdot, y)(x) = b(x, y)$). If $\mathcal{B} = \{b_a : a \in A\}$ is a family of bilinear maps from $X \times Y \to Z$, $\mathcal{B}$ is said to be right equicontinuous if for each $x \in X$ the family $\{b_a(x, \cdot) : a \in A\}$ is an equicontinuous family of linear maps ([AS, § 6]).

**Theorem 3.** *Let $X$ be a metric $\mathcal{K}$-space and $\mathcal{B} = \{b_a : a \in A\}$ a family of bilinear maps from $X \times Y$ to $Z$. Assume*

(1) *$\mathcal{B}$ is right equicontinuous,*

(2) *for each $a \in A$ $y \in Y$ there is a closed subspace $M(a, y)$ of $X$ such that $b_a(\cdot, y)$ is continuous on $M(a, y)$.*

*Then there exist $y_1, \dots, y_k \in Y$, $a_1, \dots, a_k \in A$ such that $\mathcal{B}$ is equicontinuous on $M \times Y$ where $M = \bigcap_{j=1}^k M(a_j, y_j)$.*

Proof: Set $\mathcal{T} = \{b_a(\cdot, y) : a \in A, \ \|y\| \le 1\}$. Since $\mathcal{B}$ is right equicontinuous, $\mathcal{T}$ is a pointwise bounded family of linear maps from $X$ to $Z$. By Theorem 2 there exist $a_1, \dots, a_k \in A$, $y_1, \dots, y_k \in Y$ with $\|y_j\| \le 1$ such that $\mathcal{T}$ is equicontinuous on $M = \bigcap_{j=1}^k M(a_j, y_j)$. Therefore, given $\varepsilon > 0$ there exists a neighborhood $U$ of 0 in $X$ such that $\|b_a(x, y)\| < \varepsilon$ for $x \in U \cap M$, $\|y\| \le 1$. Hence, $\mathcal{B}$ is equicontinuous on $M \times Y$. $\square$

The case when $\mathcal{B}$ consists of a single bilinear map is just Theorem 3.2 of [P]. Theorem 3 gives a generalization of the uniform boundedness principle for bilinear maps given in 6.16 of [AS].

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